A STUDY ON LAMBERT SERIES ASSOCIATED WITH CUSP FORMS AND RANKIN–COHEN BRACKETS ON HERMITIAN JACOBI FORMS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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Dedicated to

My Ajji(Grandmother) Divangata(Late) Shrimati Rajeshwari

and

My Ajja(Grandfather) Divangata(Late) Shriman Neerugaru Subbabhatta

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **A STUDY ON LAMBERT SERIES ASSOCIATED WITH CUSP FORMS AND RANKIN–COHEN BRACKETS ON HERMITIAN JACOBI FORMS** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfilment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to *certify* that the Research Thesis entitled **A STUDY ON LAMBERT SERIES ASSOCIATED WITH CUSP FORMS AND RANKIN–COHEN BRACKETS ON HERMITIAN JACOBI FORMS** submitted by **Mr. SUMUKHA S**, (Register Number: 177036MA005) as the record of the research work carried out by him is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

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ब्रह्मार्पणमस्तु ।

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ABSTRACT

A Lambert series is a series of the form $L(q) = \sum_{n=1}^{\infty} a(n)\frac{q^n}{1-q^n}$, where $a(n)$ is an arithmetic function and $q \in \mathbb{C}$. By setting $b(n) = \sum_{d|n} a(d)$ and $q = e^{-y}$, the series will take the form $L(y) = \sum_{n=1}^{\infty} b(n)e^{-ny}$. In 1981, Zagier, conjectured that the Lambert series $y^{12}\sum_{n=1}^{\infty} \tau^2(n)e^{-4\pi ny}$, which is the constant term of the automorphic form $y^{12}|\Delta(z)|^2$, where $\Delta(z)$ is the Ramanujan cusp form of weight 12, has an asymptotic expansion when $y \to 0^+$, and it can be expressed in terms of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. In 2000, Hafner and Stopple under the assumption of the Riemann Hypothesis proved this conjecture. In this thesis, we consider a Lambert series associated to a cusp form and the Möbius function. Using the functional equation of the L-function associated to the cusp form and the functional equation of the Riemann zeta function, we prove an exact formula for the Lambert series. As a consequence, we also derive an asymptotic expansion for the same. We extend our work to higher level cusp forms by considering a more general twisted Lambert series. We also establish an exact formula and asymptotic expansion for a Lambert series associated with the Symmetric square L-function.

Rankin–Cohen brackets are bilinear differential operators defined on the space of modular forms. In 2015, Herrero constructed the adjoint map of some linear maps defined by using the Rankin–Cohen brackets. In this thesis, we generalize the work of Herrero to the case of Hermitian Jacobi forms over $\mathbb{Q}(i)$. Given a fixed Hermitian Jacobi cusp form, we define a family of linear operators between spaces of Hermitian Jacobi cusp forms using Rankin–Cohen brackets. We compute the adjoint maps of such family with respect to the Petersson scalar product. The Fourier coefficients of the Hermitian Jacobi cusp forms constructed using this method involve special values of certain Dirichlet series associated to Hermitian Jacobi cusp forms.

Keywords: *Lambert series, Riemann zeta function, non-trivial zeros, modular form, cusp form, Dirichlet L-function, Symmetric square L-function, Rankin-Selberg L-function, Hermitian Jacobi forms, Rankin–Cohen bracket, Adjoint map.*

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CHAPTER 1

INTRODUCTION

A Lambert series is a series of the form

$$L(q) = \sum_{n=1}^{\infty} a(n) \frac{q^n}{1-q^n},$$

where $a(n)$ is an arithmetic function and $q \in \mathbb{C}$. If the series $\sum_{n=1}^{\infty} a(n)$ converges, then $L(q)$ converges for all q with $|q| \neq 1$. Otherwise, it converges whenever the series $\sum_{n=1}^{\infty} a(n)q^n$ converges. For $|q| < 1$, by setting $b(n) = \sum_{d|n} a(d)$, we can show that

$$L(q) = \sum_{n=1}^{\infty} b(n)q^n.$$

If we take $a(n) = 1$ for all n, then we get $b(n) = \sum_{d|n} 1$, which is the divisor function of n. If we take $q = e^{-z}$, then the series will take the form

$$L(z) = \sum_{n=1}^{\infty} b(n)e^{-nz}.$$

Lambert considered this series in relation with the convergence of power series. Over the years, this family of series has been studied by various mathematicians. Readers may refer to Berndt (1999) for more details. Hardy and Ramanujan (1918) used the behaviour of a certain Lambert series as $q \to 1$ to derive an asymptotic expansion for the general partition function.

Ramanujan, during his stay at Trinity College, communicated an identity involving the Möbius function to Hardy and Littewood. For any positive real number r,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-r/n^2} = \sqrt{\frac{\pi}{r}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\pi^2/n^2 r}.$$

Unfortunately, this formula is not true. A nice explanation about the incorrectness of the above formula was given by Berndt (1998). Later, Hardy and Littlewood (1916) established the following corrected version of the above formula:

Let α_1 and α_2 be two positive real numbers such that $\alpha_1\alpha_2 = \pi$. Assume that all the non-trivial zeros of the Riemann zeta function $\zeta(s)$ are simple. Then

$$\sqrt{\alpha_1}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\alpha_1^2/n^2} - \sqrt{\alpha_2}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\alpha_2{}^2/n^2} = -\frac{1}{2\sqrt{\alpha_2}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)\alpha_2{}^{\rho}}{\zeta'(\rho)}, \quad (1.0.1)$$

where the sum on the right hand side runs over all the non-trivial zeros of $\zeta(s)$. This infinite sum is convergent under the assumption of bracketing the terms such that the non-trivial zeros ρ_1 and ρ_2 are included in the same bracket if they satisfy

$$|\Im(\rho_1) - \Im(\rho_2)| < e^{-\frac{C_0\Im(\rho_1)}{\log(\Im(\rho_1))}} + e^{-\frac{C_0\Im(\rho_2)}{\log(\Im(\rho_2))}},$$

where C_0 is some positive constant. Hardy and Littlewood also mentioned that it is quite possible that this series may converge without the assumption of bracketing the terms, but they were unable to prove it even after assuming the Riemann Hypothesis. Over the years, formula (1.0.1) has attracted many mathematicians. Readers can also see detailed discussions on this corrected formula in (Berndt, 1998, p. 470), (Paris and Kaminski, 2001, p. 143), and (Titchmarsh, 1986, p. 219). As an application of the above formula (1.0.1), Hardy and Littlewood showed that the following condition

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\frac{x}{n^2}} = \sum_{n=1}^{\infty}\frac{(-x)^n}{n!\,\zeta(2n+1)} = O\left(x^{-\frac{1}{4}+\varepsilon}\right), \quad \text{as } x\to\infty, \quad (1.0.2)$$

is actually equivalent to the Riemann Hypothesis (Hardy and Littlewood, 1916, p. 161). Bhaskaran (1997) connected the formula (1.0.1) with Wiener's Tauberian theory and the Fourier reciprocity. Dixit in (Dixit, 2012, Theorem 1.9) obtained a character analogue of the identity (1.0.1) and later he also established (Dixit, 2013, Theorem 1.7) a one variable generalization of (1.0.1) under the assumption that the series

$$\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}{}_1F_1\left(\frac{1-\rho}{2};\frac{1}{2};-\frac{z^2}{4}\right)\pi^{\rho/2}\alpha^{\rho},$$

where ${}_1F_1(a;b;c)$ defined in (1.5.1) below, is convergent. Again, Dixit, Roy, and Zaharescu (Dixit et al. (2015)) gave a generalization of (1.0.1) in the setting of Hecke eigenform. In the same paper, they also obtained a Riesz-type criterion for the Riemann Hypothesis similar to (1.0.2). Recently, Roy et al. (2016) studied an identity similar to (1.0.1), corresponding to an arithmetic function, which is the convolution of Dirichlet characters and the Möbius function.

In 1981, Zagier, conjectured that the Lambert series $y^{12}\sum_{n=1}^{\infty}\tau^2(n)e^{-4\pi ny}$, which is the constant term of the automorphic form $y^{12}|\Delta(z)|^2$, where $\Delta(z)$ is the Ramanujan cusp form of weight 12, has an asymptotic expansion when $y \to 0^+$, and it can be expressed in terms of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. In thesis, inspired by the conjecture of Zagier (1981), we study some interesting Lambert series associated with cusp forms.

Let $k_1, k_2 > 0$ and $\nu \geq 0$. Let f and g be modular forms of weight k_1, k_2 on the full modular group $SL_2(\mathbb{Z})$. The ν-th Rankin–Cohen bracket $[\ ,\ ]_\nu$ is a bilinear differential operator defined by

$$[f,g]_\nu := \sum_{l=0}^{\nu}(-1)^l\binom{k_1+\nu-1}{\nu-l}\binom{k_2+\nu-1}{l}f^{(l)}g^{(\nu-l)}.$$

Then $[f,g]_\nu$ is a modular form of weight $k_1+k_2+2\nu$. Works of Rankin (1956) and explicit examples given by Cohen (1975) led to the development of Rankin–Cohen brackets. Over the years, Rankin–Cohen brackets have been defined and studied for various automorphic forms such as Jacobi forms, Siegel forms etc. In the case of Hermitian Jacobi forms, the Rankin–Cohen brackets were introduced by Kim (2002). However her results are incorrect if the underlying field is the Gaussian number field. Later, Martin and Senadheera (2017) corrected the results of Kim for the Gaussian number field. Also, Martin (2016), in his thesis, introduced a different kind of Rankin–Cohen bracket on Hermitian Jacobi forms for the Gaussian number field. This definition of Rankin–Cohen bracket by Martin is similar to the definition of Rankin–Cohen bracket introduced by Choie (1997, 1998) in the case of Jacobi forms. In this thesis, we define a family of linear operators between spaces of Hermitian Jacobi cusp forms using Rankin–Cohen brackets defined by Martin (2016). We compute the adjoint maps of such family with respect to the Petersson scalar product.

We start with some basic definitions and results. This is to facilitate the readability of the thesis. For details regarding analytic number theory and modular forms, the readers are referred to Iwaniec and Kowalski (2004), Diamond and Shurman (2005) and Murty et al. (2015).

1.1 SOME ARITHMETIC FUNCTIONS

1.1.1 DIRICHLET CHARACTERS

Let M be a natural number. A Dirichlet character χ of modulus M is a homomorphism from the multiplicative group $(\mathbb{Z}/M\mathbb{Z})^*$ into $\mathbb{C}^*$. It can be extended to $\mathbb{N}$ by defining

$$\chi(n) = \begin{cases} \chi(n \pmod M) & \text{if } \gcd(n,M) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For a divisor d of M, if there exists a character χ' of modulus d and $\chi(n) = \chi'(n)$ for all n with $\gcd(n,M) = 1$, then we say χ is induced by χ'. A character which is not induced by any other character is called a primitive character. If the homomorphism χ is trivial, that is, if $\chi(n) = 1$ for all n with $\gcd(n,M) = 1$, then χ is called the principal character of modulus M.

1.1.2 THE MÖBIUS FUNCTION

The Möbius function $\mu(n)$ is an arithmetic function defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is a square-free positive integer with} \\ & \text{an even number of prime factors,} \\ -1 & \text{if } n \text{ is a square-free positive integer with} \\ & \text{an odd number of prime factors,} \\ 0 & \text{if } n \text{ has a squared prime factor.} \end{cases}$$

1.2 MODULAR FORMS

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ be the complex upper half-plane in the Euclidean topology. The group $GL_2^+(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q}) \,\middle|\, ad - bc > 0 \right\}$ acts on $\mathbb{H}$ by fractional linear transformation, that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}, \quad \forall\, z \in \mathbb{H} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}).$$

1.2.1 THE MODULAR GROUP AND THE CONGRUENCE SUBGROUPS

The subgroup $SL_2(\mathbb{Z})$ of $GL_2^+(\mathbb{Q})$, defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \,\middle|\, ad - bc = 1 \right\}$$

is called the full modular group, or simply the modular group. For $N \in \mathbb{N}$ the principal congruence subgroup of level N, $\Gamma(N)$, is defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Note that $\Gamma(1) = SL_2(\mathbb{Z})$.
A subgroup Γ of $SL_2(\mathbb{Z})$ is called a congruence subgroup if there exists $N \in \mathbb{N}$ such that $\Gamma(N) \subset \Gamma$. The least such N is called the level of Γ. Note that, since $\Gamma(N)$ is a subgroup of finite index in $SL_2(\mathbb{Z})$, every congruence subgroup is also of finite index in $SL_2(\mathbb{Z})$.

Example 1.2.1. *For $N \in \mathbb{N}$, let*

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\}.$$

The group $\Gamma_0(N)$ is a congruence subgroup, called the Hecke subgroup.

Example 1.2.2. *Consider the surjective homomorphism $\Gamma_0(N) \to \left(\mathbb{Z}/N\mathbb{Z}\right)^*$ defined by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to d \pmod{N}.$$

The kernel of this map, denoted by $\Gamma_1(N)$, is given by

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,\middle|\, d \equiv 1 \pmod{N} \right\},$$

and is also a congruence subgroup.

1.2.2 THE SPACE OF MODULAR FORMS

The extended upper half-plane is defined by $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$. By identifying ∞ with $\frac{1}{0}$, the action of a congruence subgroup can be extended to $\mathbb{H}^*$ in a natural way by defining

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{r}{s} = \frac{ar+bs}{cr+ds}, \text{ for } \frac{r}{s} \in \mathbb{Q} \cup \{i\infty\}.$$

The equivalence classes in $\mathbb{Q} \cup \{i\infty\}$ under the action of a congruence subgroup Γ are called the cusps of Γ.

This group action extends to a family of actions on the set of complex valued functions on $\mathbb{H}$. Let k be a positive integer. For a complex function f defined on $\mathbb{H}$, the action of the congruence subgroup Γ is given by the k-slash operator as follows:

$$f \mid_k \gamma(z) = (cz+d)^{-k} f(\gamma z), \quad \forall\, z \in \mathbb{H},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k on a congruence subgroup Γ if it satisfies the following conditions:

(1) For all $\gamma \in \Gamma$ and $z \in \mathbb{H}$,

$$f \mid_k \gamma(z) = f(z).$$

(2) For $\gamma \in SL_2(\mathbb{Z})$, $f|_k\gamma$ has a Fourier expansion of the form

$$f|_k\gamma(z) = \sum_{n=0}^{\infty} a_{f|_k\gamma}(n) q_h^n,$$

where $q_h = e^{2\pi i z/h}$, $z \in \mathbb{H}$, and h is the smallest positive integer such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$.

If $a_{f|_k\gamma}(0) = 0$ for all $\gamma \in SL_2(\mathbb{Z})$ then f is called a cusp form. We say that the cusp form is normalised if $a_f(1) = 1$. The finite dimensional complex vector space of all modular forms of weight k on a congruence subgroup Γ is denoted by $M_k(\Gamma)$. The subspace of $M_k(\Gamma)$ consisting of all cusp forms is denoted by $S_k(\Gamma)$.

Let χ be a Dirichlet character modulo N. A function f is called a modular form of weight k, level N and character χ (or Nebentypus χ) if $f \in M_k(\Gamma_1(N))$ and satisfies

$$f(\gamma z) = \chi(d)(cz+d)^k f(z),$$

for all $\gamma \in \Gamma_0(N),\ z \in \mathbb{H}$.

The space of all such modular forms is denoted by $M_k(\Gamma_0(N), \chi)$ and the corresponding space of cusp forms is denoted by $S_k(\Gamma_0(N), \chi)$. In particular, if χ is a principal character then we get $M_k(\Gamma_0(N), \chi) = M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N), \chi) = S_k(\Gamma_0(N))$.

1.2.3 THE RAMANUJAN CUSP FORM

Let $q = e^{2\pi iz}$, $z \in \mathbb{H}$. The Ramanujan cusp form Δ is given by the infinite product $q\prod_{n=1}^{\infty}(1-q^n)^{24}$. Then Δ is indeed a modular form of weight 12 on the full modular group $SL_2(\mathbb{Z})$. Moreover, it is a cusp form with the Fourier series expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n,$$

where $\tau(n)$ is famously known as the Ramanujan tau function. Ramanujan (2000) studied this infinite series Δ. His conjectures on its Fourier coefficients $\tau(n)$ resulted in development of Hecke theory for modular forms.

1.2.4 EISENSTEIN SERIES OF LEVEL ONE

Let $k \geq 4,\ z \in \mathbb{H}$. The Eisenstein series of weight k for $SL_2(\mathbb{Z})$ is defined by

$$G_k(z) = \sum_{\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^k}.$$

Then the series $G_k(z)$ converges absolutely. Moreover, $G_k(z) \in M_k(SL_2(\mathbb{Z}))$. The normalized Eisenstein series is given by

$$E_k(z) = \frac{1}{2\zeta(k)} G_k(z),$$

where $\zeta(s)$ is the Riemann zeta function (see 1.4.1).

1.2.5 THE PETERSSON SCALAR PRODUCT

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ with index $[SL_2(\mathbb{Z}) : \Gamma]$. The Petersson scalar product $\langle\ ,\ \rangle_\Gamma$ is defined by

$$\langle f, g\rangle_\Gamma = \frac{1}{[SL_2(\mathbb{Z}) : \Gamma]} \int_{\Gamma\setminus\mathbb{H}} y^k f(z)\overline{g(z)} \frac{\mathrm{d}x\mathrm{d}y}{y^2},$$

where $\Gamma\setminus\mathbb{H}$ is any fundamental domain for the action of Γ on $\mathbb{H}$, and $z = x+iy$. We see that this integral is convergent and its evaluation is independent of the choice of the fundamental domain. With respect to this scalar product, $S_k(\Gamma)$ becomes a finite dimensional Hilbert space.

1.2.6 THE HECKE OPERATORS AND HECKE EIGENFORMS

Let m be a positive integer. The Hecke operators T_m are linear operators on the space $M_k(\Gamma_0(N),\chi)$ which are defined as follows:
If $f(z)=\sum_{n=0}^{\infty} a_f(n)q^n \in M_k(\Gamma_0(N),\chi),\ q=e^{2\pi iz}$, then

$$T_m(f(z))=\sum_{n=0}^{\infty}\left(\sum_{d|gcd(m,n)}\chi(d)d^{k-1}a_f(\frac{mn}{d^2})\right)q^n,$$

where $a_f(\frac{mn}{d^2})$ is taken to be zero unless $\frac{mn}{d^2}$ is a non-negative integer.
If $f \in S_k(\Gamma_0(N),\chi)$ is an eigenfunction for all Hecke operators T_m, then it is called a Hecke eigenform. If $f(z)=\sum_{n=0}^{\infty} a_f(n)q^n \in M_k(\Gamma_0(N),\chi)$ is an eigenform, then it is known that, $a_f(n)\in\mathbb{R}$, for every n.

1.3 HERMITIAN JACOBI FORMS

Eichler and Zagier (1985), in their monograph, studied a class of functions called Jacobi forms. They are holomorphic functions on $\mathbb{H}\times\mathbb{C}$. These functions appear as Fourier-Jacobi coefficients of Siegel modular forms. They play a very important role in establishing Saito-Kurokawa conjecture which gives a connection between the space of elliptic modular forms and the space of Siegel modular forms.

Hermitian Jacobi forms are holomorphic functions on $\mathbb{H}\times\mathbb{C}\times\mathbb{C}$. They appear as Fourier-Jacobi coefficients of Hermitian modular forms which are generalisation of Siegel modular forms. Haverkamp (1995, 1996) systematically studied Hermitian Jacobi forms. Later Das (2010a,b) and Sasaki (2007) contributed further to the theory of Hermitian Jacobi forms over $\mathbb{Q}(i)$. In the case of the classical Jacobi forms, the action of the heat operator can be “corrected" so that Jacobi forms of weight k are mapped to Jacobi forms of weight $k+2$ (see Richter (2009)). In the case of the Hermitian Jacobi forms, the heat operator L_m for any integer m is defined by

$$L_m := \frac{1}{(2\pi i)^2}\left(8\pi im\frac{\partial}{\partial\tau}-4\frac{\partial^2}{\partial w\partial z}\right).$$

Richter and Senadheera (2015) showed that with the original definition of Hermitian Jacobi forms over $\mathbb{Q}(i)$, the action of the heat operator cannot be “corrected" as in the case of the classical Jacobi forms. The authors introduced the concept of parity for Hermitian Jacobi forms over $\mathbb{Q}(i)$ such that the action of the heat operator can be "corrected".

1.3.1 BASIC DEFINITIONS

Let $\mathscr{O} := \mathbb{Z}[i]$, the ring of integers of $\mathbb{Q}(i)$. Let $\mathscr{O}^{\#} := \frac{i}{2}\mathscr{O}$ be the inverse different of $\mathbb{Q}(i)$ over $\mathbb{Q}$. That is, $\mathscr{O}^{\#} = \{x \in \mathbb{Q}(i) \mid tr(xy) \in \mathbb{Z}[i], \text{for all } y \in \mathbb{Z}(i)\}$. Let $\mathscr{O}^{\times} = \{\pm 1, \pm i\}$ be the set of units in $\mathscr{O}$. Let $\Gamma(\mathscr{O}) := \{\varepsilon M \mid \varepsilon \in \mathscr{O}^{\times},\ M \in SL_2(\mathbb{Z})\}$. The Hermitian Jacobi group is defined by

$$\Gamma^J(\mathscr{O}) := \Gamma(\mathscr{O}) \ltimes \mathscr{O}^2 = \{\gamma = (\varepsilon M, X) \mid \varepsilon M \in \Gamma(\mathscr{O}),\ X \in \mathscr{O}^2\}.$$

The action of the Hermitian Jacobi group $\Gamma^J(\mathscr{O})$ on $\mathbb{H} \times \mathbb{C}^2$ is defined by

$$\gamma \cdot (\tau, z, w) = \left(\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}, [\lambda, \mu]\right) \cdot (\tau, z, w) = \left(\frac{a\tau + b}{c\tau + d}, \varepsilon \frac{z + \lambda\tau + \mu}{c\tau + d}, \bar{\varepsilon} \frac{w + \bar{\lambda}\tau + \bar{\mu}}{c\tau + d}\right),$$

where $\varepsilon \in \mathscr{O}^{\times}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\lambda, \mu \in \mathscr{O}$. This action extends to a family of actions on the set of functions from $\mathbb{H} \times \mathbb{C}^2$ to $\mathbb{C}$. Let $\phi : \mathbb{H} \times \mathbb{C}^2 \to \mathbb{C}$. Let $\delta \in \{+, -\}$, k be a positive integer and m be a non-negative integer. We define the action of $\Gamma^J(\mathscr{O})$ on ϕ by

$$\left(\phi|_{k,m,\delta}\gamma\right)(\tau, z, w) = \sigma(\varepsilon)\varepsilon^{-k}(cz + d)^{-k} e^{2\pi i m(\lambda\bar{\lambda}\tau + \bar{\lambda}z + \lambda w) - \frac{c(z+\lambda\tau+\mu)(w+\bar{\lambda}\tau+\bar{\mu})}{c\tau+d}} \phi(\gamma \cdot (\tau, z, w)) \tag{1.3.1}$$

where

$$\sigma(\varepsilon) = \begin{cases} 1 & \text{if} \quad \delta = +, \\ \varepsilon^2 & \text{if} \quad \delta = -, \end{cases}$$

and $\gamma = \left(\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}, [\lambda, \mu]\right) \in \Gamma^J(\mathscr{O})$.

Definition 1.3.1. *(Richter and Senadheera (2015)) A holomorphic function $\phi : \mathbb{H} \times \mathbb{C}^2 \to \mathbb{C}$ is a Hermitian Jacobi form of weight k, index m and parity δ on $\Gamma^J(\mathscr{O})$ if for each $\gamma = \left(\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}, [\lambda, \mu]\right) \in \Gamma^J(\mathscr{O})$ we have*

$$\phi \mid_{k,m,\delta} \gamma = \phi,$$

and ϕ has a Fourier expansion of the form

$$\phi(\tau, z, w) = \sum_{\substack{n \in \mathbb{Z}, r \in \mathscr{O}^{\#} \\ nm - |r|^2 \geq 0}} c(n, r) e^{2\pi i (n\tau + rz + \bar{r}w)}. \tag{1.3.2}$$

A Hermitian Jacobi form is called a Hermitian Jacobi cusp form if $c(n,r) = 0$ *whenever* $nm - |r|^2 = 0$ *in the Fourier expansion given in* (1.3.2).

We denote by $J^{\delta}_{k,m}(\Gamma^J(\mathscr{O}))$ the complex vetor space of Hermitian Jacobi forms of weight k, index m and parity δ on $\Gamma^J(\mathscr{O})$. We denote by $J^{\delta,cusp}_{k,m}(\Gamma^J(\mathscr{O}))$ the vector space of Hermitian Jacobi cusp forms of weight k, index m and parity δ on $\Gamma^J(\mathscr{O})$.

1.3.2 POINCARÉ SERIES FOR HERMITIAN JACOBI FORMS

It is easy to check that for any positive integer k, m and $\delta \in \{+,-\}$, we have

$$e^{2\pi i(n\tau+rz+\bar{r}w)} |_{k,m,\delta}\, \gamma = e^{2\pi i(n\tau+rz+\bar{r}w)}$$

for $n \in \mathbb{Z}$ and $r \in \mathscr{O}^{\#}$, if and only if

$$\gamma \in \Gamma^J_{\infty}(\mathscr{O}) := \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, [0,\mu] \right) \mid t \in \mathbb{Z}, \mu \in \mathscr{O} \right\}.$$

For fixed m,n,r with $nm - |r|^2 > 0$, the (n,r)-th Poincaré series of weight $k > 4$, index m, and parity δ on $\Gamma^J(\mathscr{O})$ is defined by

$$P^{k,m,\delta}_{n,r}(\tau,z,w) = \sum_{\gamma \in \Gamma^J_{\infty}(\mathscr{O}) \backslash \Gamma^J(\mathscr{O})} \left(e^{2\pi i(n\tau+rz+\bar{r}w)} |_{k,m,\delta}\, \gamma \right)(\tau,z,w).$$

It is known that $P^{k,m,\delta}_{n,r} \in J^{\delta,cusp}_{k,m}(\Gamma^J(\mathscr{O}))$.

1.3.3 THE PETERSSON SCALAR PRODUCT ON HERMITIAN JACOBI FORMS

Let $\tau = u + iv \in \mathbb{H}, z = x_1 + iy_1 \in \mathbb{C}, w = x_2 + iy_2 \in \mathbb{C}$. The measure

$$\mathrm{d}V = v^{-4} \mathrm{d}u \mathrm{d}v \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2,$$

is invariant for the action of $\Gamma^J(\mathscr{O})$ on $\mathbb{H} \times \mathbb{C}^2$.

Definition 1.3.2. *(Haverkamp (1995)) Let* $\phi, \psi \in J^{\delta}_{k,m}(\Gamma^J(\mathscr{O}))$ *such that at least one among them is a cusp form. The Petersson scalar product of* ϕ, ψ *is defined by*

$$\langle \phi, \psi \rangle = \int\limits_{\Gamma^J(\mathscr{O}) \backslash \mathbb{H} \times \mathbb{C}^2} \phi(\tau,z,w) \overline{\psi(\tau,z,w)} e^{\frac{-\pi m}{v}|w-\bar{z}|^2} v^k \mathrm{d}V.$$

With respect to the above scalar product $J^{\delta}_{k,m}(\Gamma^J(\mathscr{O}))$ becomes a finite dimensional Hilbert space.

We have the following lemma (Kumar and Ramakrishnan, 2018, Lemma 2.2).

Lemma 1.3.3. *Let $\phi \in J^{\delta}_{k,m}(\Gamma^J(\mathscr{O}))$ with Fourier coefficients $c(n,r)$. Let n,r be such that $nm-|r|^2>0$. Then*

$$\langle \phi, P^{k,m,\delta}_{n,r}\rangle = c(n,r)\frac{m^{k-3}\Gamma(k-2)}{\pi^{k-2}(4(nm-|r|^2))^{k-2}}.$$

1.3.4 RANKIN–COHEN BRACKETS FOR HERMITIAN JACOBI FORMS

Let $\phi(\tau,w,z),\psi(\tau,w,z)$ be holomorphic functions on $\mathbb{H}\times\mathbb{C}^2$. Let k_1,k_2,m_1 and m_2 be positive integers and $\nu \geq 0$ be an integer. Martin (2016) defined the ν-th Rankin–Cohen bracket as

$$[[\phi,\psi]]_\nu := \sum_{l=0}^{\nu}(-1)^l\binom{k_1+\nu-2}{\nu-l}\binom{k_2+\nu-2}{l}m_1^{\nu-l}m_2^l L^l_{m_1}(\phi)L^{\nu-l}_{m_2}(\psi), \quad (1.3.3)$$

where $L_m = \frac{1}{(2\pi i)^2}\left(8\pi i m\frac{\partial}{\partial\tau}-4\frac{\partial^2}{\partial w\partial z}\right)$ is the heat operator.

Martin proved that if $\phi \in J^{\delta_1}_{k_1,m_1}(\Gamma^J(\mathscr{O}))$ and $\psi \in J^{\delta_2}_{k_2,m_2}(\Gamma^J(\mathscr{O}))$ then $[[\phi,\psi]]_\nu \in J^{\delta_1\delta_2(-1)^\nu}_{k_1+k_2+2\nu,m_1+m_2}(\Gamma^J(\mathscr{O}))$. If one of these is a cusp form then the resultant Rankin–Cohen bracket is also a cusp form.

1.4 DIRICHLET SERIES AND L-FUNCTIONS

1.4.1 THE REIMANN ZETA FUNCTION

Let $s\in\mathbb{C}$ with $\Re(s)>1$. The Riemann zeta function is given by the series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The series converges absolutely, and hence defines an analytic function in the region $\Re(s)>1$. Euler proved that $\zeta(s)$ has a product expansion:

$$\zeta(s)=\prod_{p:\,\text{prime}}(1-p^{-s})^{-1}.$$

Riemann established the meromorphic continuation of $\zeta(s)$ to the whole complex plane with a simple pole at $s=1$. He also established the following functional equation:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1.4.1}$$

From the functional equation we can see that $\zeta(s)$ has zeros at $s=-2,-4,\ldots$. These are called the trivial zeros of $\zeta(s)$. Any other zero of $\zeta(s)$ is called a non-trivial zero. From the functional equation it can be observed that all the non-trivial zeros lie in the open strip $0<\Re(s)<1$. Riemann conjectured that all the non-trivial zeros lie on the the line $\Re(s)=1/2$, called the critical line. The conjecture is famously known as the Riemann hypothesis and is unsolved till date.

1.4.2 DIRICHLET *L*-FUNCTIONS

Let χ be a Dirichlet character of modulus M. Dirichlet introduced the following series associated to χ:

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

This converges absolutely in $\Re(s)>1$, and can be continued analytically to the entire complex plane, unless it is a principal character, in which case the meromorphic continuation of $L(s,\chi)$ will have a simple pole at $s=1$. The generalized Riemann hypothesis states that all the non-trivial zeros of $L(s,\chi)$ lie on the the line $\Re(s)=1/2$.

1.4.3 THE *L*-FUNCTION ATTACHED TO MODULAR FORMS

Let $f(z)\in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform with the Fourier series expansion

$$f(z)=\sum_{n=1}^{\infty}a_f(n)e^{2\pi inz},\quad \forall z\in\mathbb{H}. \tag{1.4.2}$$

The L-function associated to the cusp form f is defined as

$$L(f,s):=\sum_{n=1}^{\infty}a_f(n)n^{-s}.$$

That this series is absolutely convergent for $\Re(s)>\frac{k+1}{2}$ immediately follows from Deligne's bound $a_f(n)=O(n^{(k-1)/2}+\varepsilon)$ for any $\varepsilon>0$ (see Deligne (1974)). It is also well-known that $L(f,s)$ has an Euler product representation

$$L(f,s) = \prod_{p:\,\text{prime}} (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}, \quad \Re(s) > \frac{k+1}{2},$$

where α_p and β_p are complex conjugates with $\alpha_p + \beta_p = a_f(p)$ and $\alpha_p\beta_p = p^{k-1}$. Hecke proved that $L(f,s)$ can be analytically continued to an entire function and it satisfies the following functional equation:

$$(2\pi)^{-s}\Gamma(s)L(f,s) = i^k(2\pi)^{-(k-s)}\Gamma(k-s)L(f,k-s). \tag{1.4.3}$$

This functional equation is equivalent to the following transformation formula:

$$\sum_{n=1}^{\infty} a_f(n)e^{-ny} = \left(\frac{2\pi}{y}\right)^k \sum_{n=1}^{\infty} a_f(n)e^{\left(\frac{-4\pi^2 n}{y}\right)},$$

for $\Re(y) > 0$. For more details on this equivalence, the reader is referred works of Bochner (1951) andChandrasekharan and Narasimhan (1961).

1.5 SOME SPECIAL FUNCTIONS

1.5.1 THE GAMMA FUNCTION

Let z be a complex number with $\Im(z) > 0$. The Gamma function is defined by

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}\mathrm{d}t.$$

This extends meromorphically to the whole complex plane with simple poles at $z = -n$ of residue $(-1)^n/n!$. The function doesn't vanish anywhere in the complex plane.

1.5.2 THE GENERALIZED HYPERGEOMETRIC FUNCTION

Let $a_1,\cdots,a_p$ and $b_1,\cdots,b_q$ be complex numbers. We denote ${}_pF_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;z\right)$ as the generalized hypergeometric series (Olver et al., 2010, p. 404, Equation 16.2.1) defined by

$${}_pF_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \tag{1.5.1}$$

where $(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}$. If $p \leq q$, this series converges for all complex values of z. When $p = q+1$ it converges for $|z| < 1$, but it can be analytically continued to the whole complex plane if we introduce a branch cut from 1 to $+\infty$.

1.5.3 THE MEIJER G-FUNCTION

Definition 1.5.1. *(Olver et al., 2010, p. 415, Definition 16.17)*
Let m,n,p,q be four non-negative integers such that $0 \leq m \leq q$, $0 \leq n \leq p$. Let $a_1,\cdots,a_p$ and $b_1,\cdots,b_q$ be $p+q$ complex numbers such that $a_i - b_j \notin \mathbb{N}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Then the Meijer G-function is defined by the following line integral:

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix} \middle| z\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s)\prod_{j=1}^n \Gamma(1-a_j+s)z^s}{\prod_{j=m+1}^q \Gamma(1-b_j+s)\prod_{j=n+1}^p \Gamma(a_j-s)}\mathrm{d}s, \quad (1.5.2)$$

where the line of integration L separates the poles of the factors $\Gamma(b_j - s)$ from those of the factors $\Gamma(1-a_j+s)$. We consider the line of integration L going from $-i\infty$ to $+i\infty$. Note that the integral converges if $p+q < 2(m+n)$ and $|\arg(z)| < (m+n-\frac{p+q}{2})\pi$.

Now we shall state Slater's theorem (Olver et al., 2010, p. 415, Equation 16.17.2), which will enable us to write the Meijer G-function in terms of generalized hypergeometric functions. If $p \leq q$ and $b_j - b_k \notin \mathbb{Z}$ for $j \neq k$, $1 \leq j,k \leq m$, then

$$\begin{aligned} &G_{p,q}^{m,n}\left(\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix} \middle| z\right) \\ &= \sum_{k=1}^m A_{p,q,k}^{m,n}(z)\, {}_pF_{q-1}\big(1+b_k-a_1,\cdots,1+b_k-a_p;1+b_k-b_1,\cdots, \\ &\qquad *,\cdots,1+b_k-b_q;(-1)^{p-m-n}z\big), \end{aligned} \quad (1.5.3)$$

where $*$ indicates that the entry $1+b_k-b_k$ is omitted and

$$A_{p,q,k}^{m,n}(z) := \frac{z^{b_k}\prod_{j=1,j\neq k}^m \Gamma(b_j-b_k)\prod_{j=1}^n \Gamma(1+b_k-a_j)}{\prod_{j=m+1}^q \Gamma(1+b_k-b_j)\prod_{j=n+1}^p \Gamma(a_j-b_k)}.$$

1.6 ORGANIZATION OF THE THESIS

This thesis consists of six chapters. In Chapter 1 we have given the mathematical background which is necessary to understand the subsequent chapters. In Chapter 2, we have investigated a Lambert series associated to the Fourier coefficients of a cusp form on the full modular group and the Möbius function $\mu(n)$. We derive an exact formula for the

Lambert series in terms of the non-trivial zeros of the Riemann zeta function, and the error term is expressed as an infinite series involving generalized hypergeometric series ${}_2F_1(a,b;c;z)$ using the functional equations of the L-function associated to the cusp form and the Riemann zeta function. In Chapter 3, we have generalized the works of Chapter 2 to higher level modular forms, and also have obtained a character analogue. By continuing our investigation of Lambert series, in Chapter 4 we have investigated a Lambert series associated with the Symmetric square L-function. In Chapter 5, we have defined a family of linear operators between spaces of Hermitian Jacobi cusp forms using Rankin–Cohen brackets. We have computed the adjoint maps of such family with respect to the Petersson scalar product. The conclusion and scope for future work is given in the final chapter.

CHAPTER 2

LAMBERT SERIES ASSOCIATED TO A CUSP FORM AND THE MÖBIUS FUNCTION

Zagier conjectured (Zagier, 1981, p. 417) that the constant term of the automorphic form $y^{12}|\Delta(z)|^2$, where $\Delta(z)$ is the Ramanujan cusp form of weight 12, that is, the Lambert series

$$a_0(y) := y^{12} \sum_{n=1}^{\infty} \tau^2(n) e^{-4\pi n y}, \tag{2.0.1}$$

has an asymptotic expansion when $y \to 0^+$, and it can be expressed in terms of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Interestingly, he also observed that the graph of $a_0(y)$ has an oscillatory behaviour when $y \to 0^+$. His main prediction was that $a_0(y)$ will have the following asymptotic expansion:

$$a_0(y) \sim C + \sum_{\rho} y^{1-\frac{\rho}{2}} A_\rho,$$

where C is some constant, and the sum over ρ runs through all the non-trivial zeros of $\zeta(s)$, and A_ρ are some complex numbers. Assuming the Riemann Hypothesis, that is, writing $\rho = \frac{1}{2} \pm i t_n$, above expression becomes

$$a_0(y) \sim C + y^{3/4} \sum_{n=1}^{\infty} a_n \cos\left(\phi_n + \frac{t_n \log(y)}{2}\right), \quad \text{as } y \to 0^+,$$

where a_n and ϕ_n are some constants. Due to the presence of cosine functions in the above asymptotic expansion, Zagier mentioned that $a_0(y)$ will have an oscillatory behaviour as $y \to 0^+$. In 2000, Hafner and Stopple (2000), under the assumption of the Riemann Hypothesis, proved both the asymptotic expansion as well as oscillatory behaviour of the Lambert series (2.0.1). Recently, Chakraborty et al. (2017) proved that, under the assumption of the Riemann Hypothesis, the following series

$$b_0(y) := y^k \sum_{n=1}^{\infty} |a_f^2(n)| e^{-ny},$$

also has an asymptotic expansion when $y \to 0^+$ and that it can be expressed in terms of the non-trivial zeros of $\zeta(s)$, where $a_f(n)$ is the nth Fourier coefficient of a Hecke eigenform f of weight k over $\mathrm{SL}_2(\mathbb{Z})$. In Chakraborty et al. (2018), authors observed that the same phenomenon also occurs for any cusp form over the congruence subgroup $\Gamma_0(N)$ and derived an asymptotic expansion of the corresponding Lambert series. Recently, Banerjee and Chakraborty (2019) also studied the asymptotic behaviour of a Lambert series associated to Maass cusp forms.

2.1 AN ASYMPTOTIC RESULT FOR LAMBERT SERIES ASSOCIATED TO A CUSP FORM AND THE MÖBIUS FUNCTION

Define, $a_f^*(n) := (a_f * \mu)(n) = \sum_{d|n} a_f(d)\mu(\frac{n}{d})$. It is easy to observe that the Dirichlet series $\sum_{n=1}^{\infty} a_f^*(n) n^{-s}$ is absolutely convergent for $\Re(s) > \frac{k+1}{2}$. In the present chapter, we study the Lambert series $\sum_{n=1}^{\infty} a_f^*(n) e^{-ny}$ for $y > 0$. Chakraborty et al. (2018) stated that the asymptotic expansion of this Lambert series can also be expressed in terms of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Here we find that their prediction is correct. Not only that, we also establish the oscillatory behaviour of the Lambert series $y^{1/2} \sum_{n=1}^{\infty} a_f^*(n) e^{-ny}$ as $y \to 0^+$.

Let us define the arithmetic function $A_f^*(n)$ associated to $a_f(n)$ and $\mu(n)$ by

$$A_f^*(n) := \left(a_f * \mu_k\right)(n), \quad \text{where} \quad \mu_k(n) = \mu(n) n^{k-1}. \tag{2.1.1}$$

Note that the Dirichlet series $\sum_{n=1}^{\infty} A_f^*(n) n^{-s}$ is absolutely convergent for $\Re(s) > k$.

The following theorem gives an exact formula for the Lambert series $\sum_{n=1}^{\infty} a_f^*(n) e^{-ny}$, which eventually yields an asymptotic expansion of the Lambert series.

Theorem 2.1.1. *(Juyal et al. (2022b)) Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a cusp form with the n-th Fourier coefficient $a_f(n)$. Assume that all the non-trivial zeros of $\zeta(s)$ are simple. Then* *for any positive y,*

$$\sum_{n=1}^{\infty} a_f^*(n) e^{-ny} = \mathscr{P}(y) + 2\Gamma(k) \left(\frac{i}{2\pi}\right)^k \sum_{n=1}^{\infty} \frac{A_f^*(n)}{n^k} \left[{}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; \frac{1}{2}; -\frac{y^2}{4n^2\pi^2}\right) - 1 \right],$$

where

$$\mathscr{P}(y) = \sum_{\rho} \frac{L(f,\rho)\Gamma(\rho)}{\zeta'(\rho)} \frac{1}{y^{\rho}}, \tag{2.1.2}$$

and that the sum over ρ which runs through all the non-trivial zeros of $\zeta(s)$, involves bracketing the terms so that the terms corresponding to ρ_1 and ρ_2 are included in the same bracket if they satisfy

$$|\Im(\rho_1) - \Im(\rho_2)| < e^{-C\frac{|\Im(\rho_1)|}{\log(|\Im(\rho_1)|)}} + e^{-C\frac{|\Im(\rho_2)|}{\log(|\Im(\rho_2)|)}},$$

where C is some positive constant.

An immediate consequence of the above theorem under the Riemann Hypothesis is the following asymptotic result:

Corollary 2.1.2. *Let M be a positive integer and $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform with the n-th Fourier coefficient $a_f(n)$. Assume the Riemann Hypothesis and all the non-trivial zeros of $\zeta(s)$ are simple. Then for $y \to 0^+$, we have*

$$y^{\frac{1}{2}} \sum_{n=1}^{\infty} a_f^*(n) e^{-ny} = 2\sum_{n=1}^{\infty} r_n \cos(\theta_n - t_n \log(y)) + \sum_{m=1}^{M-1} C_m y^{2m+\frac{1}{2}} + O_{f,k}(y^{2M+\frac{1}{2}}),$$

where C_m are absolute constants depending only on f, and $r_n e^{i\theta_n}$ is the polar representation of $L(f,\rho_n)\Gamma(\rho_n)(\zeta'(\rho_n))^{-1}$ with $\rho_n = \frac{1}{2} + it_n$ denoting the nth non-trivial zero of $\zeta(s)$ in the upper critical line.

Remark 2.1.3. *Theorem* 2.1.1 *can be extended by analytic continuation for $\Re(y) > 0$, and also note that Theorem* 2.1.1 *is true for any cusp form, whereas Corollary* 2.1.2 *is true for any normalized Hecke eigenform. In Table 2.1, we have numerically verified Theorem 2.1.1 for the Ramanujan cusp form.*

Remark 2.1.4. *Due to the presence of the cosine functions in Corollary* 2.1.2*, one can observe that the Lambert series $y^{1/2}\sum_{n=1}^{\infty} a_f^*(n)e^{-ny}$ also has an oscillatory behaviour as $y \to 0^+$.*

In the next section, we state a few well-known results which will be useful throughout the chapter.

2.2 WELL-KNOWN RESULTS

Lemma 2.2.1. *Suppose there exists a sequence of arbitrarily large positive numbers* T *with* $|T - \Im(\rho)| > e^{-A|\Im(\rho)|/\log(|\Im(\rho)|)}$ *for every non-trivial zero* ρ *of* $\zeta(s)$*, where* A *is some suitable positive constant. Then,*

$$\frac{1}{|\zeta(\sigma+iT)|} < e^{BT},$$

for some suitable constant $0 < B < \pi/4$.

Proof. A proof of this lemma can be found in (Titchmarsh, 1986, p. 219). □

Lemma 2.2.2. *In any vertical strip* $\sigma_0 \le \sigma \le b$*, there exists a constant* $C(\sigma_0)$*, such that*

$$|L(f,\sigma+iT)| \ll |T|^{C(\sigma_0)}$$

as $|T| \to \infty$.

Proof. One can see this result in (Iwaniec and Kowalski, 2004, p. 97, Lemma 5.2). □

Lemma 2.2.3 (Stirling's formula for the Gamma function). *For* $s = \sigma + iT$ *in a vertical strip* $\alpha \le \sigma \le \beta$,

$$|\Gamma(\sigma+iT)| = \sqrt{2\pi}|T|^{\sigma-1/2}e^{-\frac{1}{2}\pi|T|}\left(1+O\left(\frac{1}{|T|}\right)\right) \quad \text{as} \quad |T| \to \infty. \tag{2.2.1}$$

Proof. One can see a proof of this result in (Iwaniec and Kowalski, 2004, p. 151). □

Lemma 2.2.4 (Duplication formula for the Gamma function). *For any complex number* z*, we have*

$$\Gamma(2z) = \frac{\Gamma(z)\Gamma(z+\frac{1}{2})\,2^{2z}}{2\sqrt{\pi}}. \tag{2.2.2}$$

Lemma 2.2.5 (Inverse Mellin transform for the Gamma function). *Let* y *and* c *be two positive real numbers. Then*

$$e^{-y} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)y^{-s}\mathrm{ds}.$$

Now we are ready to give the proof of Theorem 2.1.1.

2.3 PROOF OF THEOREM 2.1.1 AND COROLLARY 2.1.2

Proof of Theorem 2.1.1. Using inverse Mellin transform for the Gamma function, we write

$$\sum_{n=1}^{\infty} a_f^*(n)e^{-ny} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)L(f,s)}{\zeta(s)} y^{-s}\mathrm{d}s, \tag{2.3.1}$$

for any $c > \frac{k+1}{2}$. The functional equation (1.4.3) of $L(f,s)$ implies that $\Gamma(s)L(f,s)$ is an entire function, and thus poles of the integrand function will be at the zeros of the Riemann zeta function. To simplify (2.3.1), we consider the contour $\mathscr{C}$ determined by the line segments $[c-iT, c+iT], [c+iT, \lambda+iT], [\lambda+iT, \lambda-iT]$, and $[\lambda-iT, c-iT]$, where T is some large positive real number and $-1 < \lambda < 0$. Now appealing to the Cauchy residue theorem, we get

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\Gamma(s)L(f,s)}{\zeta(s)} y^{-s}\mathrm{d}s = \mathscr{P}_T(y), \tag{2.3.2}$$

where $\mathscr{P}_T(y)$ denotes the residual function consisting of finitely many terms contributed by the non-trivial zeros ρ of $\zeta(s)$ with $|\Im(\rho)| \le T$. Our first goal is to prove that the horizontal integrals

$$H_1 := \frac{1}{2\pi i}\int_{c+iT}^{\lambda+iT} \frac{\Gamma(s)L(f,s)}{\zeta(s)} y^{-s}\mathrm{d}s, \quad H_2 := \frac{1}{2\pi i}\int_{\lambda-iT}^{c-iT} \frac{\Gamma(s)L(f,s)}{\zeta(s)} y^{-s}\mathrm{d}s,$$

tend to zero as $T \to \infty$. One can write

$$H_1 = \frac{1}{2\pi i}\int_{c}^{\lambda} \frac{\Gamma(\sigma+iT)L(f,\sigma+iT)}{\zeta(\sigma+iT)y^{\sigma+iT}}\mathrm{d}\sigma.$$

Thus

$$|H_1| \ll \frac{1}{2\pi}\int_{c}^{\lambda} \frac{|\Gamma(\sigma+iT)||L(f,\sigma+iT)|}{|\zeta(\sigma+iT)|y^{\sigma}}\mathrm{d}\sigma.$$

Now invoking Lemmas 2.2.1, 2.2.2, and 2.2.3, one can show that

$$|H_1| \ll |T|^A e^{BT-\frac{\pi}{2}|T|},$$

where $0 < B < \pi/4$. This implies H_1 vanishes as $T \to \infty$. Similarly one can show that H_2 also vanishes as $T \to \infty$. Therefore, letting $T \to \infty$ in (2.3.2) and using (2.3.1), we have

$$\sum_{n=1}^{\infty} a_f^*(n)e^{-ny} = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma(s)L(f,s)}{\zeta(s)} y^{-s}\mathrm{d}s + \mathscr{P}(y). \tag{2.3.3}$$

Now if we assume all the non-trivial zeros of $\zeta(s)$ are simple, we can evaluate $\mathscr{P}(y)$ as

$$\mathscr{P}(y) = \sum_{\rho} \lim_{s\to\rho} \frac{(s-\rho)\Gamma(s)L(f,s)}{\zeta(s)} y^{-s} = \sum_{\rho} \frac{\Gamma(\rho)L(f,\rho)}{\zeta'(\rho)y^{\rho}}, \tag{2.3.4}$$

where ρ runs through all the non-trivial zeros of $\zeta(s)$ in the critical strip. In general, if m_ρ is the multiplicity of ρ, then

$$\mathscr{P}(y) = \sum_{\rho} \frac{1}{(m_\rho - 1)!} \lim_{s\to\rho} \frac{\mathrm{d}^{m_\rho-1}}{\mathrm{d}s^{m_\rho-1}} \left\{ \frac{(s-\rho)^{m_\rho}\Gamma(s)L(f,s)}{\zeta(s)y^s} \right\}.$$

Now we shall concentrate on the following integral:

$$J := \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma(s)L(f,s)}{\zeta(s)} y^{-s}\mathrm{d}s. \tag{2.3.5}$$

Use functional equation (1.4.3) of $L(f,s)$ to get

$$J = \left(\frac{i}{2\pi}\right)^k \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma(k-s)L(f,k-s)}{\zeta(s)} \left(\frac{y}{4\pi^2}\right)^{-s} \mathrm{d}s,$$

change the variable $s \leftrightarrow k-s$, to obtain

$$J = \left(\frac{i}{2\pi}\right)^k \frac{1}{2\pi i}\int_{k-\lambda-i\infty}^{k-\lambda+i\infty} \frac{\Gamma(s)L(f,s)}{\zeta(k-s)} \left(\frac{y}{4\pi^2}\right)^{s-k} \mathrm{d}s. \tag{2.3.6}$$

Here we make use of the functional equation of $\zeta(s)$, that is, (1.4.1) and replace s by $k-s$ to see

$$\frac{1}{\zeta(k-s)} = \frac{\pi^{s-k+\frac{1}{2}}\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{s-k+1}{2}\right)\zeta(s-k+1)}. \tag{2.3.7}$$

Substitute (2.3.7) in (2.3.6) and simplify to obtain

$$J = \sqrt{\pi}\left(\frac{2i}{y}\right)^k \frac{1}{2\pi i}\int_{k-\lambda-i\infty}^{k-\lambda+i\infty} \frac{\Gamma(s)\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{s-k+1}{2}\right)} \frac{L(f,s)}{\zeta(s-k+1)} \left(\frac{y}{4\pi}\right)^s \mathrm{d}s. \tag{2.3.8}$$

Note that, $k < k-\lambda < k+1$ as $-1 < \lambda < 0$, so $L(f,s)$ and $\frac{1}{\zeta(s-k+1)}$ both are absolutely convergent on the line $\Re(s) = k-\lambda$. Therefore, using the definition (2.1.1) of $A_f^*(n)$, we can see that

$$\frac{L(f,s)}{\zeta(s-k+1)}=\sum_{n=1}^{\infty}\frac{a_f(n)}{n^s}\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s}=\sum_{n=1}^{\infty}\frac{A_f^*(n)}{n^s}, \tag{2.3.9}$$

where $\mu_k(n)$ and $A_f^*(n)$ are as defined in (2.1.1).

Using (2.3.9) in (2.3.8) and interchanging the order of summation and integration, we get

$$J=\sqrt{\pi}\left(\frac{2i}{y}\right)^k\sum_{n=1}^{\infty}A_f^*(n)\frac{1}{2\pi i}\int_{k-\lambda-i\infty}^{k-\lambda+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{s-k+1}{2}\right)}\left(\frac{y}{4n\pi}\right)^s\mathrm{d}s.$$

Next, replace s by $2s$, and then employ duplication formula (2.2.2) for the Gamma function $\Gamma(2s)$, to deduce

$$J=\left(\frac{2i}{y}\right)^k\sum_{n=1}^{\infty}A_f^*(n)\frac{1}{2\pi i}\int_{\frac{k-\lambda}{2}-i\infty}^{\frac{k-\lambda}{2}+i\infty}\frac{\Gamma(s)\Gamma(s+\frac{1}{2})\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(s+\frac{1-k}{2}\right)}\left(\frac{y^2}{4n^2\pi^2}\right)^s\mathrm{d}s. \tag{2.3.10}$$

At this juncture, our main aim is to simplify the integral

$$I:=\frac{1}{2\pi i}\int_{\frac{k-\lambda}{2}-i\infty}^{\frac{k-\lambda}{2}+i\infty}\frac{\Gamma(s)\Gamma(s+\frac{1}{2})\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(s+\frac{1-k}{2}\right)}z^s\mathrm{d}s,$$

where $z=\frac{y^2}{4n^2\pi^2}$. Observing the poles of the integrand, one can verify that the line of integration $\Re(s)=(k-\lambda)/2$ does not separate all the poles of the factors $\Gamma(s)\Gamma(s+\frac{1}{2})$ from those of the factor $\Gamma(\frac{k}{2}-s)$. Thus we have to choose a new line of integration such that it separates the poles of $\Gamma(s)\Gamma(s+\frac{1}{2})$ from that of $\Gamma(\frac{k}{2}-s)$. Consider the contour $\mathscr{C}'$ determined by the line segments $[\frac{k-\lambda}{2}-iT,\frac{k-\lambda}{2}+iT],[\frac{k-\lambda}{2}+iT,d+iT],[d+iT,d-iT]$, and $[d-iT,\frac{k-\lambda}{2}-iT]$, where T is some large positive real number and $0<d<\frac{k}{2}$. Now again utilizing the Cauchy residue theorem, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}'}\frac{\Gamma(s)\Gamma(s+\frac{1}{2})\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(s+\frac{1-k}{2}\right)}z^s\mathrm{d}s=\operatorname*{Res}_{s=\frac{k}{2}}\frac{\Gamma(s)\Gamma(s+\frac{1}{2})\Gamma(\frac{k}{2}-s)}{\Gamma(s+\frac{1-k}{2})}z^s.$$

Letting $T\to\infty$ and using Stirling's formula (2.2.1) for the Gamma function, one can show that horizontal integrals vanish. Finally, calculating residue at $k/2$, we have

$$I=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{\Gamma(s)\Gamma(s+\frac{1}{2})\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(s+\frac{1-k}{2}\right)}z^s\mathrm{d}s-\frac{\Gamma(\frac{k}{2})\Gamma(\frac{1+k}{2})}{\Gamma(\frac{1}{2})}z^{\frac{k}{2}}. \tag{2.3.11}$$

Now we make use of the definition (1.5.2) of the Meijer G-function. Considering

$m=1, n=2, p=2, q=2$, and $a_1=1, a_2=\frac{1}{2}, b_1=\frac{k}{2}, b_2=\frac{k+1}{2}$, and verifying all the conditions for the Meijer G-function, we will have

$$\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{\Gamma(s)\Gamma(s+\frac{1}{2})\Gamma\left(\frac{k}{2}-s\right)}{\Gamma\left(s+\frac{1-k}{2}\right)}z^s\mathrm{d}s = G_{2,2}^{1,2}\left(\begin{matrix}1, \frac{1}{2}\\ \frac{k}{2}, \frac{1+k}{2}\end{matrix}\bigg| z\right). \tag{2.3.12}$$

Employing Slater's theorem (1.5.3), one can derive

$$G_{2,2}^{1,2}\left(\begin{matrix}1, \frac{1}{2}\\ \frac{k}{2}, \frac{1+k}{2}\end{matrix}\bigg| z\right) = \frac{z^{\frac{k}{2}}\Gamma(\frac{k}{2})\Gamma(\frac{1+k}{2})}{\Gamma(\frac{1}{2})}{}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; \frac{1}{2}; -z\right). \tag{2.3.13}$$

Combine (2.3.11), (2.3.12), and (2.3.13) to deduce that

$$I = \frac{z^{\frac{k}{2}}\Gamma(\frac{k}{2})\Gamma(\frac{1+k}{2})}{\Gamma(\frac{1}{2})}\left[{}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; \frac{1}{2}; -z\right)-1\right].$$

Now substitute the above representation of I in (2.3.10) to obtain

$$J = \left(\frac{2i}{y}\right)^k\sum_{n=1}^{\infty}A_f^*(n)\frac{z^{\frac{k}{2}}\Gamma(\frac{k}{2})\Gamma(\frac{1+k}{2})}{\Gamma(\frac{1}{2})}\left[{}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; \frac{1}{2}; -z\right)-1\right]. \tag{2.3.14}$$

Finally, substituting $z=\frac{y^2}{4n^2\pi^2}$, and combining (2.3.14), (2.3.5), and (2.3.4) in (2.3.3), and using duplication formula, we can complete the proof of Theorem 2.1.1. □

Remark 2.3.1. *Here we point out that the series involving ${}_2F_1$ in Theorem* 2.1.1*, that is,*

$$\sum_{n=1}^{\infty}\frac{A_f^*(n)}{n^k}\left[{}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; \frac{1}{2}; -\frac{y^2}{4n^2\pi^2}\right)-1\right] \tag{2.3.15}$$

is absolutely convergent for any $y>0$. For any fixed positive real number y, we can always find a large natural number N such that $\frac{y^2}{4n^2\pi^2}<1$ for all $n\geq N$. Therefore, using the definition of the generalized hypergeometric series, for $n\geq N$, we write

$$\begin{aligned}\left|{}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; \frac{1}{2}; -\frac{y^2}{4n^2\pi^2}\right)-1\right| &\leq \frac{y^2}{4n^2\pi^2}\sum_{m=1}^{\infty}\frac{\left(\frac{k}{2}\right)_m\left(\frac{k+1}{2}\right)_m}{\left(\frac{1}{2}\right)_m m!}\left(\frac{y^2}{4n^2\pi^2}\right)^{m-1}\\ &\leq \frac{y^2}{4n^2\pi^2}\sum_{m=1}^{\infty}\frac{\left(\frac{k}{2}\right)_m\left(\frac{k+1}{2}\right)_m}{\left(\frac{1}{2}\right)_m m!}\left(\frac{y^2}{4N^2\pi^2}\right)^{m-1}\\ &\leq \frac{N^2}{n^2}\sum_{m=1}^{\infty}\frac{\left(\frac{k}{2}\right)_m\left(\frac{k+1}{2}\right)_m}{\left(\frac{1}{2}\right)_m m!}\left(\frac{y^2}{4N^2\pi^2}\right)^{m}.\end{aligned}$$

Hence, we have

$$\left|{}_2F_1\left(\frac{k}{2},\frac{k+1}{2};\frac{1}{2};-\frac{y^2}{4n^2\pi^2}\right)-1\right|\leq\frac{N^2}{n^2}\left({}_2F_1\left(\frac{k}{2},\frac{k+1}{2};\frac{1}{2};\frac{y^2}{4N^2\pi^2}\right)-1\right),$$

for all $n\geq N$. Now separate the first $N-1$ terms in the series (2.3.15) *and then use the above bound for remaining infinitely many terms to see the absolute convergence of the series* (2.3.15)*. Note that we have to use the fact that the Dirichlet series* (2.3.9) *is absolutely convergent for $\Re(s)>k$.*

Proof of Corollary 2.1.2. Using the definition of the generalized hypergeometric series (Olver et al., 2010, p. 404, Equation 16.2.1) for $y\to 0^+$, one can write following asymptotic expansion of ${}_2F_1$:

$$ {}_2F_1\left(\frac{k}{2},\frac{k+1}{2};\frac{1}{2};-\frac{y^2}{4n^2\pi^2}\right)-1=\sum_{m=1}^{M-1}B_m\left(\frac{y}{n}\right)^{2m}+O_k\left(\left(\frac{y}{n}\right)^{2M}\right), \tag{2.3.16}$$

where $B_m=\frac{(-1)^m\left(\frac{k}{2}\right)_m\left(\frac{k+1}{2}\right)_m}{\left(\frac{1}{2}\right)_m(m)!(4\pi^2)^m}$, and M is any positive integer. Now use (2.3.16) in Theorem 2.1.1 to see that

$$\begin{aligned}\sum_{n=1}^{\infty}a_f^*(n)e^{-ny}&=2\Gamma(k)\left(\frac{i}{2\pi}\right)^k\sum_{m=1}^{M-1}\sum_{n=1}^{\infty}\frac{A_f^*(n)}{n^{k+2m}}B_my^{2m}+O_k\left(\sum_{n=1}^{\infty}\frac{|A_f^*(n)|}{n^{k+2M}}y^{2M}\right)+\mathscr{P}(y)\\&=\sum_{m=1}^{M-1}C_my^{2m}+O_{f,k}(y^{2M})+\mathscr{P}(y),\end{aligned} \tag{2.3.17}$$

where $C_m=2\Gamma(k)\left(\frac{i}{2\pi}\right)^kB_m\sum_{n=1}^{\infty}\frac{A_f^*(n)}{n^{k+2m}}$. Note that C_m's are finite quantities since the Dirichlet series $\sum_{n=1}^{\infty}A_f^*(n)n^{-s}$ is absolutely convergent for $\Re(s)>k$. From the functional equation of the Riemann zeta function it is immediate that if $\rho_n=\frac{1}{2}+it_n$ denotes the nth non-trivial zero, then $\frac{1}{2}-it_n$ is also a non-trivial zero. Therefore, one can write the infinite series expression (2.1.2) of $\mathscr{P}(y)$ as

$$\mathscr{P}(y)=\sum_{\substack{\rho_n=\frac{1}{2}+it_n,\\ t_n>0}}2\Re\left(\frac{L(f,\rho_n)\Gamma(\rho_n)}{y^{\rho_n}\zeta'(\rho_n)}\right).$$

Here we have used the fact that f is a normalized Hecke eigenform. To simplify $\mathscr{P}(y)$ even more, we write $r_ne^{i\theta_n}=L(f,\rho_n)\Gamma(\rho_n)(\zeta'(\rho_n))^{-1}$ to see

$$\sqrt{y}\mathscr{P}(y)=\sum_{\substack{\rho_n=\frac{1}{2}+it_n,\\ t_n>0}}2r_n\cos(\theta_n-t_n\log(y)). \tag{2.3.18}$$

Now combine (2.3.18) and (2.3.17) to complete the proof of Corollary 2.1.2. $\square$

Table 2.1 : Verification of Theorem 2.1.1: We took $f(z) = \Delta(z)$, the Ramanujan cusp form. Here, in the left-hand side and right-hand side in the sum over n we considered only the first 5000 terms. In the sum over ρ for $\mathscr{P}(y)$, we considered only 20 terms.

y	Left-hand side	Right-hand side
0.123	$-0.0004629993871...$	$-0.0004629912383...$
1	$-0.0204523567610...$	$-0.0204523567622...$
1.1234	$-0.0223212278858...$	$-0.0223212278873...$
$\sqrt{3}$	$-0.0185761774446...$	$-0.0185761774481...$
2.543	$+0.0035666059027...$	$+0.0035666058953...$
π	$+0.0124169011322...$	$+0.0124169011209...$
$2^{\sqrt{2}}$	$+0.0062298234741...$	$+0.0062298234660...$
$10+\sqrt{5}$	$4.8516617384 \times 10^{-6}$	$4.8514902993 \times 10^{-6}$

We note that it was enough to take only the first 20 zeros in order for the first eight digits on both sides of the Theorem 2.1.1 to coincide. This indicates that the residual term $\mathscr{P}(y)$ is indeed rapidly convergent.

CHAPTER 3

TWISTED LAMBERT SERIES ASSOCIATED TO A CUSP FORM AND THE MÖBIUS FUNCTION

In this chapter we continue our study of Lambert series and generalize the results of the previous chapter. Here, we investigate the following Lambert series:

$$A_f(y) := \sum_{n=1}^{\infty} [a_f(n)\psi(n) * \mu(n)\psi'(n)]e^{-ny}, \tag{3.0.1}$$

where $a_f(n)$ is the nth Fourier coefficient of a cusp form f of weight k, level Q and Nebentypus χ, and ψ, ψ' are primitive Dirichlet characters. We derive an exact formula for the above Lambert series (3.0.1) involving the non-trivial zeros of $L(s, \psi')$ and a generalized hypergeometric function, and thereby generalize Theorem 2.1.1 in two different directions. On the one hand, our work generalizes Theorem 2.1.1 for congruence subgroups and on the other hand, we also get a character analogue. As an application, we also derive an asymptotic expansion and establish an oscillatory behavior of $y^{1/2}A_f(y)$ as $y \to 0^+$. Let k and Q be two positive integers. Let χ be a Dirichlet character modulo Q and the Gauss sum ε_χ is defined by $\varepsilon_\chi := \sum_{j=1}^{Q} \chi(j)e^{\frac{2\pi i j}{Q}}$. Consider $f(z) \in S_k(\Gamma_0(Q), \chi)$ with the Fourier series expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi i n z}, \quad \forall z \in \mathbb{H}. \tag{3.0.2}$$

It is known that for a positive integer r such that $(Q, r) = 1$ and a primitive Dirichlet character ψ modulo r, the ψ-twist of f is defined by

$$f_\psi(z) = \sum_{n=1}^{\infty} a_f(n)\psi(n)e^{2\pi i n z},$$

is an element of $S_k(\Gamma_0(N), \chi\psi^2)$, where $N = Qr^2$. Hence it is natural to consider the following Dirichlet series:

$$L_f(s,\psi) := \sum_{n=1}^{\infty} \frac{a_f(n)\psi(n)}{n^s}.$$

The above series converges and hence defines an analytic function in $\Re(s) > \frac{k+1}{2}$. It is known as the ψ-twist of $L_f(s)$, where $L_f(s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}$, the L-function attached to f. We can analytically extend $L_f(s,\psi)$ into an entire function and the completed L-function $\Lambda_f(s,\psi) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_f(s,\psi)$ satisfies the following functional equation (Murty et al., 2015, p. 131)

$$\Lambda_f(s,\psi) = \frac{i^k \varepsilon_\chi^2}{r} \chi(r)\psi(Q)\Lambda_g(k-s,\overline{\psi}), \tag{3.0.3}$$

where

$$g(z) = Q^{k/2}(Qz)^{-k} f\left(-\frac{1}{Qz}\right) := \sum_{n=1}^{\infty} a_g(n) e^{2\pi i n z} \in S_k(\Gamma_0(Q), \overline{\chi})).$$

3.1 THE MAIN IDENTITY

For a fixed natural number k, let $\mu_k(n) = \mu(n)n^{k-1}$. Let ψ' be a primitive Dirichlet character modulo M, and

$$a := \begin{cases} 0, & \psi'(-1) = 1, \\ 1, & \psi'(-1) = -1. \end{cases}$$

The following result gives an exact formula for the Lambert series 3.0.1.

Theorem 3.1.1. *(Maji et al. (2022)) Let $f \in S_k(\Gamma_0(Q),\chi)$ be a cusp form with the n-th Fourier coefficient $a_f(n)$. Let ψ and ψ' be primitive Dirichlet characters of modulus r and M respectively. Assume that all the non-trivial zeros of $L(s,\psi')$ are simple. For any positive y, we have*

$$\sum_{n=1}^{\infty} [a_f(n)\psi(n) * \mu(n)\psi'(n)] e^{-ny} = 2N^{k/2}\Gamma(k+a)\left(\frac{yN}{M}\right)^a \left(\frac{i}{2\pi}\right)^{k+a} \frac{\chi(r)\psi(Q)\varepsilon_\psi^2}{r\varepsilon_{\psi'}}$$
$$\times \sum_{n=1}^{\infty} \frac{[a_g(n)\overline{\psi}(n) * \mu_k(n)\overline{\psi'}(n)]}{n^{k+a}} \left[{}_2F_1\left(\frac{k+a}{2}, \frac{k+1+a}{2}; \frac{1+2a}{2}; -\frac{N^2y^2}{4M^2n^2\pi^2}\right) - (1-a)\right]$$
$$+ \mathscr{R}(y) + R_0,$$

where $N = Qr^2$, and the terms R_0 and $\mathscr{R}(y)$ are defined as

$$R_0 = \begin{cases} \frac{L_f'(0,\psi)}{L'(0,\psi')}, & \text{if } a = 0 \text{ and } M > 1, \\ 0, & \textit{otherwise} \end{cases} \quad \text{and} \quad \mathscr{R}(y) = \sum_{\rho} \frac{L_f(\rho,\psi)\Gamma(\rho)}{L'(\rho,\psi')} \frac{1}{y^{\rho}}, \quad (3.1.1)$$

where the sum over ρ in $\mathscr{R}(y)$, running through all the non-trivial zeros of $L(s,\psi')$, involves bracketing the terms so that the terms corresponding to ρ_1 and ρ_2 are included in the same bracket if they satisfy

$$|\Im(\rho_1) - \Im(\rho_2)| < e^{-\frac{C|\Im(\rho_1)|}{\log(|\Im(\rho_1)|+3)}} + e^{-\frac{C|\Im(\rho_2)|}{\log(|\Im(\rho_2)|+3)}}, \quad (3.1.2)$$

where C is some positive constant.

First we collect some preliminary results which we use in the proof of the above theorem. The following lemma gives an important bound for the inverse of the Dirichlet L-function.

Lemma 3.1.2. *Assume there exists a sequence of arbitrarily large positive numbers T satisfying $|T - \Im(\rho)| > e^{-A|\Im(\rho)|/\log(|\Im(\rho)|+3)}$ for every non-trivial zero ρ of $L(s,\psi')$, where A is some suitable positive constant. Then,*

$$\frac{1}{|L(\sigma + iT, \psi')|} < e^{BT},$$

for some suitable constant $0 < B < \pi/4$.

Proof. A proof of this lemma can be given along similar lines to that in (Titchmarsh, 1986, p. 219). □

The next result says that any L-function associated to a cusp form can be bounded by a suitable polynomial in a vertical strip.

Lemma 3.1.3. *In any vertical strip $\sigma_0 \leq \sigma \leq b$, there exists a constant $C(\sigma_0)$, such that*

$$|L_f(\sigma + iT, \psi)| \ll |T|^{C(\sigma_0)}$$

as $|T| \to \infty$.

Proof. One can see this result in (Iwaniec and Kowalski, 2004, p. 97, Lemma 5.2). □

Next we state the functional equation for the Dirichlet L-function.

Lemma 3.1.4. *Let ψ' be a Dirichlet character Modulo M. Then the Dirichlet L-function $L(s,\psi') = \sum_{n=1}^{\infty} \frac{\psi'(n)}{n^s}$ analytically extends to the whole complex plane and satisfies the functional equation:*

$$\left(\frac{\pi}{M}\right)^{-\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\psi') = \frac{\varepsilon'_{\psi}}{i^a\sqrt{M}} \left(\frac{\pi}{M}\right)^{-\frac{1-s+a}{2}} \Gamma\left(\frac{1-s+a}{2}\right) L(1-s,\overline{\psi'}).$$

Now we are ready to give the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. First, we note that the Lambert series

$$\sum_{n=1}^{\infty} [a_f(n)\psi(n) * \mu(n)\psi'(n)]e^{-ny}$$

converges absolutely and uniformly for any $y > 0$. Now using inverse Mellin transform for $\Gamma(s)$, one can write

$$\begin{aligned}\sum_{n=1}^{\infty} [a_f(n)\psi(n) * \mu(n)\psi'(n)]e^{-ny} &= \sum_{n=1}^{\infty} [a_f(n)\psi(n) * \mu(n)\psi'(n)] \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{(ny)^s} \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)L_f(s,\psi)}{L(s,\psi')} y^{-s} \mathrm{d}s. \qquad (3.1.3)\end{aligned}$$

Here the interchange of summation and integration is possible only for $\Re(s) = c > \frac{k+1}{2}$. Next, to simplify this line integral we shall take help of contour integration and use Cauchy's residue theorem. Consider the contour $\mathscr{C}_T$ determined by the line segments $[c-iT, c+iT], [c+iT, \lambda+iT], [\lambda+iT, \lambda-iT]$, and $[\lambda-iT, c-iT]$, where T is some large positive real number and $-1 < \lambda < 0$. Before using Cauchy's residue theorem, let us identify the poles of the integrand function. From (3.0.3), it follows that $\Gamma(s)L_f(s,\psi)$ has no poles since $\Lambda_f(s,\psi)$ is an entire function. Hence poles of the integrand are only due to the zeros of $L(s,\psi')$. Note that, if ψ' is an even character of modulus $M > 1$, then $L(s,\psi')$ has trivial zeros at $0, -2, -4, \cdots$. And if ψ' is an odd character, then $L(s,\psi')$ has trivial zeros at $-1, -3, -5, \cdots$. Again, we know that the non-trivial zeros of $L(s,\psi')$ lie in the strip $0 < \Re(s) < 1$. Therefore, applying Cauchy's residue theorem, we have

$$\frac{1}{2\pi i} \int_{\mathscr{C}_T} \frac{\Gamma(s)L_f(s,\psi)}{L(s,\psi')} y^{-s} \mathrm{d}s = \mathscr{R}_T(y) + R_0, \qquad (3.1.4)$$

where $\mathscr{R}_T(y)$ denotes the residual function, which includes finitely many terms contributed by the non-trivial zeros ρ of $L(s,\psi')$ with $|\Im(\rho)| < T$ and R_0 is the residue at

$s=0$. Now, we can write

$$\int_{\mathscr{C}_T} \frac{\Gamma(s)L_f(s,\psi)}{L(s,\psi')}y^{-s}\mathrm{d}s = \left(\int_{c-iT}^{c+iT} + \int_{c+iT}^{\lambda+iT} + \int_{\lambda+iT}^{\lambda-iT} + \int_{\lambda-iT}^{c-iT}\right)\frac{\Gamma(s)L_f(s,\psi)}{L(s,\psi')}y^{-s}\mathrm{d}s. \tag{3.1.5}$$

Next, utilizing Lemmas 2.2.3, 3.1.2 and 3.1.3, one can show that both of the horizontal integrals

$$H_1(T,y) := \frac{1}{2\pi i}\int_{c+iT}^{\lambda+iT} \frac{\Gamma(s)L_f(s,\psi)}{L(s,\psi')}y^{-s}\mathrm{d}s,\ H_2(T,y) := \frac{1}{2\pi i}\int_{\lambda-iT}^{c-iT} \frac{\Gamma(s)L_f(s,\psi)}{L(s,\psi')}y^{-s}\mathrm{d}s,$$

tend to zero as $T\to\infty$ through those values of T which satisfy

$$|T-\Im(\rho)| > e^{-A|\Im(\rho)|/\log(|\Im(\rho)|+3)}.$$

Therefore, letting $T\to\infty$ in (3.1.4) and in view of (3.1.3) and (3.1.5), we have

$$\sum_{n=1}^{\infty}[a_f(n)\psi(n)*\mu(n)\psi'(n)]e^{-ny} = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(s)L_f(s,\psi)}{L(s,\psi')}y^{-s}\mathrm{d}s + \mathscr{R}(y) + R_0, \tag{3.1.6}$$

where the contribution of the residual term R_0 will be taken into account only when ψ' is an even character with modulus $M>1$. Therefore, we have

$$R_0 = \lim_{s\to 0} s\Gamma(s)\frac{L_f(s,\psi)}{L(s,\psi')}y^{-s} = \begin{cases} \frac{L_f'(0,\psi)}{L'(0,\psi')}, & \text{if } a=0,\ M>1, \\ 0, & \text{otherwise.}\end{cases} \tag{3.1.7}$$

The function $\mathscr{R}(y)$ is the sum of the residual terms coming from the non-trivial zeros ρ of $L(s,\psi')$. This term can be evaluated in the following way:

$$\mathscr{R}(y) = \sum_{\rho}\lim_{s\to\rho}(s-\rho)\frac{\Gamma(s)L_f(s,\psi)}{L(s,\psi')}y^{-s} = \sum_{\rho}\frac{\Gamma(\rho)L_f(\rho,\psi)}{L'(\rho,\psi')y^{-\rho}}, \tag{3.1.8}$$

where the summation runs over the non-trivial zeros ρ of $L(s,\psi')$. Here we note that we have used the assumption that all the non-trivial zeros of $L(s,\psi')$ are simple. Even if we do not assume the simplicity of zeros, then also we can figure out this residual term. Now we shall try to evaluate the left vertical integral

$$V(y) := \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(s)L_f(s,\psi)}{L(s,\psi')}y^{-s}\mathrm{d}s, \tag{3.1.9}$$

where $-1 < \lambda < 0$. The functional equation (3.0.3) of $L_f(s, \psi)$ suggests that

$$\Gamma(s)L_f(s,\psi) = i^k \left(\frac{\sqrt{N}}{2\pi}\right)^{k-2s} \frac{\varepsilon_\psi^2}{r}\chi(r)\psi(Q)\Gamma(k-s)L_g(k-s,\overline{\psi}). \tag{3.1.10}$$

Again, Lemma 3.1.4 yields that

$$\frac{1}{L(s,\psi')} = \frac{i^a\sqrt{M}}{\varepsilon_{\psi'}}\left(\frac{\pi}{M}\right)^{\frac{1-2s}{2}} \frac{\Gamma(\frac{s+a}{2})}{\Gamma(\frac{1-s+a}{2})}\frac{1}{L(1-s,\overline{\psi'})}. \tag{3.1.11}$$

Now substituting (3.1.10) and (3.1.11) in (3.1.9) and simplifying, we get

$$V(y) = \frac{\sqrt{\pi}i^{k+a}N^{\frac{k}{2}}\varepsilon_\psi^2\chi(r)\psi(Q)}{2^k\pi^k r\varepsilon_{\psi'}}\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma(\frac{s+a}{2})\Gamma(k-s)}{\Gamma(\frac{1-s+a}{2})}\frac{L_g(k-s,\overline{\psi})}{L(1-s,\overline{\psi'})}\left(\frac{Ny}{4\pi M}\right)^{-s}\mathrm{d}s.$$

After a change of variable from s to $k-s$, $V(y)$ takes the following form:

$$V(y) = C_{k,\chi,\psi,\psi'}\frac{1}{2\pi i}\int_{k-\lambda-i\infty}^{k-\lambda+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k-s+a}{2}\right)}{\Gamma\left(\frac{s-k+1+a}{2}\right)}\frac{L_g(s,\overline{\psi})}{L(s-k+1,\overline{\psi'})}\left(\frac{Ny}{4\pi M}\right)^{s}\mathrm{d}s, \tag{3.1.12}$$

where

$$C_{k,\chi,\psi,\psi'} := \frac{\sqrt{\pi}i^{k+a}\varepsilon_\psi^2\chi(r)\psi(Q)}{r\varepsilon_{\psi'}}\left(\frac{2M}{y\sqrt{N}}\right)^k. \tag{3.1.13}$$

Note that the Dirichlet series expansions of $L_g(s,\overline{\psi})$ and $L(s-k+1,\overline{\psi'})$ are absolutely convergent on the line $\Re(s) = k-\lambda$ since $k-\lambda > k$ as $-1 < \lambda < 0$. Therefore, on the line $\Re(s) = k-\lambda$, with the help of the Dirichlet series expansion, one can write

$$\frac{L_g(s,\overline{\psi})}{L(s-k+1,\overline{\psi'})} = \sum_{n=1}^{\infty}\frac{a_g(n)\overline{\psi}(n)*\mu_k(n)\overline{\psi'}(n)}{n^s}. \tag{3.1.14}$$

Now, substituting (3.1.14) in (3.1.12) and then taking summation outside of integration, we get

$$V(y) = C_{k,\chi,\psi,\psi'}\sum_{n=1}^{\infty}[a_g(n)\overline{\psi}(n)*\mu_k(n)\overline{\psi'}(n)]\frac{1}{2\pi i}\int_{k-\lambda-i\infty}^{k-\lambda+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k-s+a}{2}\right)}{\Gamma\left(\frac{s-k+1+a}{2}\right)}\left(\frac{Ny}{4\pi Mn}\right)^{s}\mathrm{d}s. \tag{3.1.15}$$

To simplify $V(y)$ further, we shall concentrate on the following integral:

$$U_{n,a}(y) := \frac{1}{2\pi i}\int_{k-\lambda-i\infty}^{k-\lambda+i\infty}\frac{\Gamma(s)\Gamma\left(\frac{k-s+a}{2}\right)}{\Gamma\left(\frac{s-k+1+a}{2}\right)}\left(\frac{Ny}{4\pi Mn}\right)^{s}\mathrm{d}s. \tag{3.1.16}$$

By a change of variable from s to $2s$ and invoking (2.2.2), we get

$$U_{n,a}(y)=\frac{1}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{\frac{k-\lambda}{2}-i\infty}^{\frac{k-\lambda}{2}+i\infty}\frac{\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(s+\frac{1-k+a}{2}\right)}z^s\mathrm{d}s, \tag{3.1.17}$$

where $z=\left(\frac{Ny}{2\pi Mn}\right)^2$. To get a more comprehensible form for $U_{n,a}(y)$, we must employ the definition of the Meijer G-function. Unfortunately, by analysing the poles of the integrand of $U_{n,a}(y)$ we can verify that the line of integration $\Re(s)=\frac{k-\lambda}{2}$ does not distinguish all the poles of $\Gamma\left(\frac{k+a}{2}-s\right)$ from the poles of $\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)$. Note that $\frac{k}{2}<\frac{k-\lambda}{2}<\frac{k+1}{2}$ as $-1<\lambda<0$. Thus, we shift the line of integration $\Re(s)=\frac{k-\lambda}{2}$ to the line $\Re(s)=c'$ where $\frac{k}{2}-1<c'<\frac{k}{2}$. Now we can see that the line of integration $\Re(s)=c'$ does separate the poles. At this moment, we construct a new rectangular contour $\mathfrak{C}$ joining the line segments $[c'-iT,\frac{k-\lambda}{2}-iT]$, $\left[\frac{k-\lambda}{2}-iT,\frac{k-\lambda}{2}+iT\right]$, $\left[\frac{k-\lambda}{2}+iT,c'+iT\right]$, and $[c'+iT,c'-iT]$ and employing Cacuchy's residue theorem, we have

$$\frac{1}{2\pi i}\int_{\mathfrak{C}}\frac{\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(s+\frac{1-k+a}{2}\right)}z^s\mathrm{d}s=R_{\frac{k}{2}}, \tag{3.1.18}$$

where $R_{\frac{k}{2}}$ is the residue at $s=\frac{k}{2}$, and it can be evaluated as

$$R_{\frac{k}{2}}=\begin{cases}-\frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}+a\right)}z^{\frac{k}{2}} & \text{if } a=0,\\ 0, & \text{if } a=1.\end{cases} \tag{3.1.19}$$

Now using Stirling's formula for the Gamma function, one can show that the contribution of the horizontal integrals vanish as $T\to\infty$. Therefore, letting $T\to\infty$ in (3.1.18), we have

$$\frac{1}{2\pi i}\int_{\frac{k-\lambda}{2}-i\infty}^{\frac{k-\lambda}{2}+i\infty}\frac{\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(s+\frac{1-k+a}{2}\right)}z^s\mathrm{d}s=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(s+\frac{1-k+a}{2}\right)}z^s\mathrm{d}s+R_{\frac{k}{2}}. \tag{3.1.20}$$

Now utilizing the definition (1.5.2) of the Meijer G-function and verifying all the necessary conditions, one can show that

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k+a}{2}-s\right)}{\Gamma\left(s+\frac{1-k+a}{2}\right)}z^s\mathrm{d}s=G_{2,2}^{1,2}\left(\begin{matrix}1,\frac{1}{2}\\ \frac{k+a}{2},\frac{1+k-a}{2}\end{matrix}\,\middle|\,z\right). \tag{3.1.21}$$

Next, we shall invoke Slater's theorem (1.5.3) to simplify Meijer G-function in terms of hypergeometric functions. Thus after a significant simplification, we obtain

$$G_{2,2}^{1,2}\left(\begin{matrix} 1, \frac{1}{2} \\ \frac{k+a}{2}, \frac{1+k-a}{2} \end{matrix} \Big| z\right) = z^{\frac{k+a}{2}} \frac{\Gamma\left(\frac{k+a}{2}\right)\Gamma\left(\frac{1+k+a}{2}\right)}{\Gamma\left(\frac{1}{2}+a\right)} {}_2F_1\left(\frac{k+a}{2}, \frac{k+1+a}{2}; \frac{1+2a}{2}; -z\right) \tag{3.1.22}$$

Now substituting $z = \left(\frac{Ny}{2\pi Mn}\right)^2$ in (3.1.22) and in view of (3.1.19), (3.1.20) and (3.1.21), the integral $U_{n,a}(y)$ becomes

$$U_{n,a}(y) = \frac{2}{\sqrt{\pi}}\left(\frac{Ny}{2\pi Mn}\right)^{k+a} \frac{\Gamma(k+a)}{2^k}\left[{}_2F_1\left(\frac{k+a}{2}, \frac{k+1+a}{2}; \frac{1+2a}{2}; -z\right) - (1-a)\right]. \tag{3.1.23}$$

Substituting the above expression of $U_{n,a}(y)$ in (3.1.15), the final expression for the left vertical integral $V(y)$ reduces to

$$\begin{aligned} V(y) = 2N^{\frac{k}{2}+a}\Gamma(k+a)\left(\frac{y}{M}\right)^a\left(\frac{i}{2\pi}\right)^{k+a} \frac{\chi(r)\psi(Q)\varepsilon_\psi^2}{r\varepsilon_{\psi'}} \sum_{n=1}^{\infty} \frac{[a_g(n)\overline{\psi}(n) * \mu_k(n)\overline{\psi'}(n)]}{n^{k+a}} \\ \times\left[{}_2F_1\left(\frac{k+a}{2}, \frac{k+1+a}{2}; \frac{1+2a}{2}; -\frac{N^2y^2}{4M^2n^2\pi^2}\right) - (1-a)\right]. \end{aligned} \tag{3.1.24}$$

Finally, combining (3.1.6), (3.1.7), (3.1.8) and (3.1.24), we finish the proof of Theorem 3.1.1. □

Remark 3.1.5. *The identity in Theorem* 3.1.1 *can be extended analytically for* $\Re(y) > 0$. *Also, by substituting* $Q = r = M = 1$ *in Theorem* 3.1.1, *one can immediately recover Theorem 2.1.1.*

Remark 3.1.6. *Note that the series in Theorem* 3.1.1*involving generalized hypergeometric function is indeed convergent. To see this, first we take* $a = 0$. *Then using series definition of* ${}_2F_1$ *for large n, we have* ${}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; \frac{1}{2}; -\frac{N^2y^2}{2^2M^2n^2\pi^2}\right) - 1 = O_{f,\psi,\psi'}\left(\frac{1}{n^2}\right)$. *Hence we get a natural number L such that,*

$$\sum_{n=L}^{\infty} \frac{[a_g(n)\overline{\psi}(n) * \mu_k(n)\overline{\psi'}(n)]}{n^k}\left[{}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; \frac{1}{2}; -\frac{N^2y^2}{2^2M^2n^2\pi^2}\right) - 1\right]$$

$$\ll \sum_{n=L}^{\infty} \frac{[a_g(n)\overline{\psi}(n) * \mu_k(n)\overline{\psi'}(n)]}{n^{k+2}},$$

which is a convergent series. On the other hand, when $a = 1$ *for large n, we have*

${}_2F_1\left(\frac{k+1}{2},\frac{k+2}{2};\frac{3}{2};-\frac{N^2y^2}{2^2M^2n^2\pi^2}\right)=O_{f,\psi,\psi'}(1)$. *Hence we get a natural number L' such that,*

$$\sum_{n=L'}^{\infty}\frac{[a_g(n)\overline{\psi}(n)*\mu_k(n)\overline{\psi'}(n)]}{n^{k+1}}\left[{}_2F_1\left(\frac{k+1}{2},\frac{k+2}{2};\frac{3}{2};-\frac{N^2y^2}{2^2M^2n^2\pi^2}\right)\right]$$

$$\ll\sum_{n=L'}^{\infty}\frac{[a_g(n)\overline{\psi}(n)*\mu_k(n)\overline{\psi'}(n)]}{n^{k+1}},$$

which is also convergent.

3.2 SOME SPECIAL CASES

As a special case of Theorem 3.1.1, by taking $M=r=1$, we get a higher level analogue of Theorem 2.1.1. We note this special case as a corollary.

Corollary 3.2.1. *Let $f\in S_k(\Gamma_0(Q),\chi)$ be a cusp form. Assume that all the non-trivial zeros of $\zeta(s)$ are simple. Then for $y>0$, we have*

$$\sum_{n=1}^{\infty}[a_f(n)*\mu(n)]e^{-ny}=\frac{i^k\Gamma(k)Q^{\frac{k}{2}}}{2^{k-1}\pi^k}\sum_{n=1}^{\infty}\frac{[a_g(n)*\mu_k(n)]}{n^k}\left[{}_2F_1\left(\frac{k}{2},\frac{k+1}{2};\frac{1}{2};-\frac{Q^2y^2}{4n^2\pi^2}\right)-1\right]$$
$$+\mathscr{R}(y),$$

where $\mathscr{R}(y)=\sum_\rho\frac{L_f(\rho)\Gamma(\rho)}{\zeta'(\rho)}\frac{1}{y^\rho}$, the sum over ρ runs through all the non-trivial zeros of $\zeta(s)$ involving bracketing as in (3.1.2).

On the other hand, if we let $Q=1, M=r$, and $\psi=\psi'$ in Theorem 3.1.1, we get a character analogue of Theorem 2.1.1, as given below:

Corollary 3.2.2. *Let $f\in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a cusp form. Let ψ be a primitive Dirichlet character modulo r. Assume that all the non-trivial zeros of $L(s,\psi)$ are simple. For $y>0$, we have*

$$\sum_{n=1}^{\infty}\psi(n)[a_f(n)*\mu(n)]e^{-ny}=R_0+\mathscr{R}(y)+2y^ar^{k+a-1}\varepsilon_\psi\left(\frac{i}{2\pi}\right)^{k+a}$$
$$\times\sum_{n=1}^{\infty}\frac{\overline{\psi}(n)[a_f(n)*\mu_k(n)]}{n^{k+a}}\left[{}_2F_1\left(\frac{k+a}{2},\frac{k+1+a}{2};\frac{1+2a}{2};-\frac{r^2y^2}{4n^2\pi^2}\right)-(1-a)\right],$$

where R_0 is defined as in Theorem 3.1.1 *and $\mathscr{R}(y)=\sum_\rho\frac{L_f(\rho,\psi)\Gamma(\rho)}{L'(\rho,\psi)}\frac{1}{y^\rho}$, where the sum over ρ involves bracketing as in* (3.1.2).

Now letting $Q=r=1$ in Theorem 3.1.1, we obtain the following result.

Corollary 3.2.3. *Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a cusp form. Let ψ' be a primitive Dirichlet character modulo M. Assume that all the non-trivial zeros of $L(s,\psi')$ are simple. Then for any positive y, we have*

$$\sum_{n=1}^{\infty}[a_f(n)*\mu(n)\psi'(n)]e^{-ny} = R_0 + \mathscr{R}(y) + \frac{2}{\varepsilon'_{\psi}}\Gamma(k+a)\left(\frac{y}{M}\right)^{a}\left(\frac{i}{2\pi}\right)^{k+a}$$
$$\times \sum_{n=1}^{\infty}\frac{[a_f(n)*\mu_k(n)\overline{\psi'}(n)]}{n^{k+a}}\left[{}_2F_1\left(\frac{k+a}{2},\frac{k+1+a}{2};\frac{1+2a}{2};-\frac{y^2}{4M^2n^2\pi^2}\right)-(1-a)\right],$$

where R_0 and $\mathscr{R}(y)$ are defined as in Theorem 3.1.1.

At the end we have given a Table 3.1, which includes numerical evidences for this corollary.

3.3 AN ASYMPTOTIC RESULT INVOLVING THE NON-TRIVIAL ZEROS OF $L(s,\psi')$

Now we state an asymptotic expansion for the Lambert series (3.0.1) as an application of Theorem 3.1.1.

Corollary 3.3.1. *With notations as in Theorem 3.1.1, we have, for $y \to 0^+$,*

$$\sum_{n=1}^{\infty}[a_f(n)\psi(n)*\mu(n)\psi'(n)]e^{-ny} = R_0 + \mathscr{R}(y) + \sum_{m=0}^{M'-1}B_{m,a}y^{2m+a} + O_{f,\psi,\psi'}(y^{2M'+a})$$

where M' is any large positive integer and $B_{m,a}$'s are some explicit constants. Further, if f is a normalized Hecke eigenform and χ,ψ and ψ' are real, then under the assumption of simplicity of the non-trivial zeros of $L(s,\psi')$ and the Generalized Riemann Hypothesis we have

$$y^{\frac{1}{2}}\sum_{n=1}^{\infty}[a_f(n)\psi(n)*\mu(n)\psi'(n)]e^{-ny} = y^{\frac{1}{2}}R_0 + \sum_{n=1}^{\infty}r_n cos(\theta_n - t_n\log y)$$
$$+\sum_{m=0}^{M'-1}B_{m,a}y^{2m+a+\frac{1}{2}} + O_{f,\psi,\psi'}(y^{2M'+a+\frac{1}{2}}).$$

Here $r_ne^{i\theta_n}$ denotes the polar representation of $2L_f(\rho_n,\psi)\Gamma(\rho_n)(L'(\rho_n,\psi'))^{-1}$, and the n-th non-trivial zero of $L(s,\psi')$ in the upper critical line is given by $\rho_n = s_n + it_n$.

Proof of Corollary 3.3.1. Making use of the asymptotic expansion of ${}_2F_1$, for $y \to 0^+$ we have

$$ {}_2F_1\left(\frac{k+a}{2}, \frac{k+1+a}{2}; \frac{1+2a}{2}; -\frac{N^2y^2}{2^2M^2n^2\pi^2}\right) = \sum_{m=0}^{M'-1} \frac{\left(\frac{k+a}{2}\right)_m \left(\frac{k+1+a}{2}\right)_m}{\left(\frac{1+2a}{2}\right)_m m!} \left(\frac{-Ny}{2Mn\pi}\right)^{2m} + O_{f,\psi,\psi'}\left(\left(\frac{y}{n}\right)^{2M'}\right), \tag{3.3.1} $$

where M' is any large positive integer. Now employing (3.3.1) in Theorem 3.1.1, we get

$$ \sum_{n=1}^{\infty} [a_f(n)\psi(n) * \mu(n)\psi'(n)]e^{-ny} = R_0 + \mathscr{R}(y) + \sum_{m=0}^{M'-1} B_{m,a} y^{2m+a} + O_{f,\psi,\psi'}(y^{2M'+a}), \tag{3.3.2} $$

where the constants $B_{m,a}$ can be evaluated by the following formula:
For $m = 0$,

$$ B_{m,a} = 2aN^{k/2}\Gamma(k+a)\left(\frac{yN}{M}\right)^a \left(\frac{i}{2\pi}\right)^{k+a} \frac{\chi(r)\psi(Q)\varepsilon_\psi^2}{r\varepsilon_{\psi'}} \sum_{n=1}^{\infty} \frac{[a_g(n)\overline{\psi}(n) * \mu_k(n)\overline{\psi'}(n)]}{n^{k+a+2m}}, $$

and for $m \geq 1$,

$$ B_{m,a} = \frac{2N^{k/2}\Gamma(k+a)}{(2\pi)^2 m}\left(\frac{yN}{M}\right)^{a+2m} \left(\frac{i}{2\pi}\right)^{k+a} \frac{\chi(r)\psi(Q)\varepsilon_\psi^2}{r\varepsilon_{\psi'}} \frac{(\frac{k+a}{2})_m(\frac{k+1+a}{2})_m}{(\frac{1+2a}{2})_m m!} \times \sum_{n=1}^{\infty} \frac{a_g(n)\overline{\psi}(n) * \mu_k(n)\overline{\psi'}(n)}{n^{k+a+2m}}. \tag{3.3.3} $$

Here we note that both of the above infinite series are absolutely convergent. Now we assume f is a normalized Hecke eigenform and χ, ψ and ψ' are real characters. Then for any complex number s, we get

$$ \frac{L_f(\bar{s},\psi)\Gamma(\bar{s})}{L'(\bar{s},\psi')}\frac{1}{y^{\bar{s}}} = \overline{\left(\frac{L_f(s,\psi)\Gamma(s)}{L'(s,\psi')}\frac{1}{y^s}\right)}. $$

Another important observation is that if $\frac{1}{2} + it_n$ is a non-trivial zero of $L(s,\psi')$, then $\frac{1}{2} - it_n$ is also a non-trivial zero of $L(s,\psi')$ since ψ' is a real character. Therefore, assuming the Generalized Riemann Hypothesis and the simplicity of the non-trivial zeros of $L(s,\psi')$, we can write

$$ \mathscr{R}(y) = \sum_{\rho_n = \frac{1}{2}+it_n, t_n>0} 2\Re\left(\frac{L_f(\rho_n,\psi)\Gamma(\rho_n)}{L'(\rho_n,\psi')} y^{-\rho_n}\right), $$

where the sum is running over all the non-trivial zeros of $L(s,\psi')$ in the upper critical line. Finally, representing $2L_f(\rho_n,\psi)\Gamma(\rho_n)L'(\rho_n,\psi')^{-1}$ in the polar form by $r_n e^{i\theta_n}$ and simplifying we complete the proof of Corollary 3.3.1. $\square$

Remark 3.3.2. *The cosine functions in Corollary* 3.3.1 *suggests the oscillatory behavior of the Lambert series* $y^{\frac{1}{2}}\sum_{n=1}^{\infty}[a_f(n)\psi(n)*\mu(n)\psi'(n)]e^{-ny}$ *as* $y\to 0^+$, *which is also consistent with the observation of Zagier (1981) for* $a_0(y)$.

Table 3.1 : Verification of Corollary 3.2.3. Let ψ' be a Dirichlet character modulo 5 with $\psi'(\bar{1})=\psi'(\bar{4})=1$ and $\psi'(\bar{2})=\psi'(\bar{3})=-1$. We took $f(z)=\Delta(z)$ as Ramanujan delta function, and the left-hand side and right-hand side series over n with only first 2000 terms, and the sum over ρ for $\mathscr{R}(y)$ is taken over only 22 terms.

y	Left-hand side	Right-hand side
1.589	0.02160533841	0.02160532545
$1+\sqrt{5}$	0.01599519746	0.01599520708
0.0749	0.03507904537	0.03507917507
$4-\pi$	0.01767636417	0.01767636262
$\pi^{\sqrt{3}}$	0.00069009521	0.00069009799
5.7395	0.00298669912	0.00298669847

CHAPTER 4

LAMBERT SERIES ASSOCIATED TO THE SYMMETRIC SQUARE L-FUNCTION

Let $f(z) \in S_k(\Gamma_0(N), \chi)$ be a normalized Hecke eigenform with the Fourier series expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}, \quad \forall z \in \mathbb{H}. \tag{4.0.1}$$

The L-function associated to $f(z)$ satisfies the following Euler product representation:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p:\,\text{prime}} \left(1 - a_f(p)p^{-s} + \chi(p)p^{k-1-2s}\right)^{-1}$$
$$= \prod_{p:\,\text{prime}} (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}, \quad \Re(s) > \frac{k+1}{2},$$

where the complex conjugates α_p and β_p satisfy the relations $\alpha_p + \beta_p = a_f(p)$ and $\alpha_p\beta_p = \chi(p)p^{k-1}$. Shimura, with the help of these complex numbers (Shimura, 1975, Equation (0.2)), defined a new L-function associated to a Hecke eigenform $f(z)$, namely the symmetric square L-function, which is given below:

$$L(s, \mathrm{Sym}^2(f) \otimes \psi) := \prod_{p:\,\text{prime}} \left(1 - \psi(p)\alpha_p^2 p^{-s}\right)^{-1} \left(1 - \psi(p)\beta_p^2 p^{-s}\right)^{-1} \left(1 - \psi(p)\alpha_p\beta_p p^{-s}\right)^{-1}.$$

This is one of the important examples of an L-function associated to a $\mathrm{GL}(3)$-automorphic form and its analytic continuation and functional equation has been studied by Shimura. For $\Re(s) > k$, $L(s, \mathrm{Sym}^2 f)$ has a absolutely convergent series representation of the form $\sum_{n=1}^{\infty} a_{\mathrm{Sym}^2(f)}(n) n^{-s}$. More generally, we can define the symmetric power L-function associated to $f(z)$ as follows:

$$L(s, \mathrm{Sym}^n(f) \otimes \psi) := \prod_{p:\,\text{prime}} \prod_{i=0}^{n} \left(1 - \psi(p)\alpha_p^i \beta_p^{n-i}\right)^{-1}.$$

Interested readers can see Murty's lecture notes (Murty (2004)) for more information on the symmetric power L-function. Upon simplification of the Euler product of the symmetric square L-function, Shimura observed that

$$L(s,\mathrm{Sym}^2(f)\otimes\psi)=L(2s-2k+2,\chi^2\psi^2)\sum_{n=1}^{\infty}\frac{a_f(n^2)\psi(n)}{n^s}, \tag{4.0.2}$$

where $L(s,\chi)$ is the usual Dirichlet L-function. In the same paper, Shimura established following important result:

Theorem 4.0.1. *Let us define*

$$L^*(s,\mathrm{Sym}^2(f)\otimes\psi):=N^s\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s-k+2-\lambda_0}{2}\right)L(s,\mathrm{Sym}^2(f)\otimes\psi),$$

where $\lambda_0=\begin{cases}0, & \text{if } \chi\psi(-1)=1,\\ 1. & \text{if } \chi\psi(-1)=-1.\end{cases}$

Then $L^*(s,\mathrm{Sym}^2(f)\otimes\psi)$ can be analytically continued to the complex plane except for simple poles at $s=k$ and at $s=k-1$.

Rankin (1939) and Selberg (1940) independently studied following interesting Dirichlet series associated to the cusp form $f(z)$, namely,

$$RS(s,f\otimes\bar{f}):=\sum_{n=1}^{\infty}a_f^2(n)n^{-s},\quad \Re(s)>k.$$

This Dirichlet series is known as Rankin-Selberg L-function associated to $f(z)$. For a general construction of the Rankin-Selberg L-function, readers can see the paper of Li (1979). The Rankin-Selberg L-function and the symmetric square L-function are intimately connected with each other. This connection was established by Shimura. He observed that the following relation holds:

$$L(s,\mathrm{Sym}^2(f)\otimes\psi)L(s-k+1,\chi\psi)=RS(s,f\otimes\bar{f}\otimes\psi)L(2s-2k+2,\chi^2\psi^2).$$

For simplicity, now onwards we assume χ and ψ both are trivial characters. Thus, the above relation becomes

$$L(s,\mathrm{Sym}^2(f))\zeta(s-k+1)=RS(s,f\otimes\bar{f})\zeta(2s-2k+2).$$

Since χ and ψ both are trivial, one can see that $\lambda_0=0$. In this case, Shimura showed that the completed symmetric square L-function $L^*(s,\mathrm{Sym}^2(f))$ is entire and satisfies the following beautiful functional equation:

$$L^*(s, \mathrm{Sym}^2(f)) = L^*(2k-1-s, \mathrm{Sym}^2(f)). \tag{4.0.3}$$

The normalized version of the above functional equation can be found in Iwaniec and Michel (2001).
The following lemma gives an important bound for symmetric square L-function on a vertical strip.

Lemma 4.0.2. *In a vertical strip $\sigma_0 \leq \sigma \leq d$, we have*

$$|L(\sigma + iT, \mathrm{Sym}^2(f))| = O(|T|^{A(\sigma_0)}), \quad \text{as } |T| \to \infty,$$

where $A(\sigma_0)$ is some constant that depends on σ_0.

Proof. One can find the proof of this lemma in (Iwaniec and Kowalski, 2004, p. 97). □

In this chapter, we investigate an asymptotic expansion of the Lambert series

$$y^k \sum_{n=1}^{\infty} a_f(n^2) e^{-ny}$$

as $y \to 0^+$. Interestingly, we observe that the asymptotic expansion of this Lambert series can also be written in terms of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. The functional equation (4.0.3) will play a crucial role to obtain our main result.

4.1 AN IDENTITY INVOLVING NON-TRIVIAL ZEROS OF $\zeta(s)$ AND GENERALIZED HYPERGEOMETRIC FUNCTIONS

We define an arithmetic function, $B_f(n)$, connected with the symmetric square L-function by the relation:

$$B_f(n) := (a_{\mathrm{Sym}^2(f)} * b)(n), \text{ where } b(n) = \begin{cases} m^{2k-1}, & \text{if } \quad n = m^2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.1}$$

One can show that the Dirichlet series associated to $B_f(n)$ is absolutely convergent for $\Re(s) > k$. Now we are ready to state the Main Theorem of this section.

Theorem 4.1.1. *(Juyal et al. (2022a) Let $f(z) \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a Hecke eigenform with the n-th Fourier coefficient $a_f(n)$. Assume all the non-trivial zeros of $\zeta(s)$ are simple. Then for any positive real number y, we have*

$$\sum_{n=1}^{\infty} a_f(n^2)e^{-ny} = \frac{\Gamma(k)y^{1-k}}{2\pi^2}\sum_{n=1}^{\infty}\frac{B_f(n)}{n^k}\left[{}_3F_2\left(\frac{k}{2},\frac{k+1}{2},1;\frac{1}{4},\frac{3}{4};-\left(\frac{y}{8n\pi}\right)^2\right)-1\right] + \mathscr{Q}(y),$$

where

$$\mathscr{Q}(y) = \frac{1}{2y^{k-1}}\sum_{\rho}\frac{\Gamma\left(\frac{\rho}{2}+k-1\right)L(\frac{\rho}{2}+k-1,\mathrm{Sym}^2(f))}{y^{\frac{\rho}{2}}\zeta'(\rho)}, \tag{4.1.2}$$

and the sum over ρ runs through all the non-trivial zeros of $\zeta(s)$ and bracketing the terms so that the terms corresponding to ρ_1 and ρ_2 are included in the same bracket if they satisfy

$$|\Im(\rho_1)-\Im(\rho_2)| < e^{-\frac{C|\Im(\rho_1)|}{\log(|\Im(\rho_1)|)}} + e^{-\frac{C|\Im(\rho_2)|}{\log(|\Im(\rho_2)|)}},$$

where C is some positive constant.

Proof of Theorem 4.1.1. First, we show that the Mellin transform of the Lambert series $\sum_{n=1}^{\infty}\psi(n)a_f(n^2)e^{-ny}$ is equal to

$$\frac{\Gamma(s)L(s,\mathrm{Sym}^2(f)\otimes\psi)}{L(2s-2k+2,\chi^2\psi^2)} \quad \text{for } \Re(s) > k.$$

That is, for $\Re(s) > k$, we write

$$\begin{aligned}\int_0^{\infty}\sum_{n=1}^{\infty}\psi(n)a_f(n^2)e^{-ny}y^{s-1}dy &= \sum_{n=1}^{\infty}\psi(n)a_f(n^2)\int_0^{\infty}e^{-ny}y^{s-1}\mathrm{d}y\\ &= \Gamma(s)\sum_{n=1}^{\infty}\psi(n)a_f(n^2)n^{-s}\\ &= \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f)\otimes\psi)}{L(2s-2k+2,\chi^2\psi^2)}.\end{aligned}$$

In the last step we have used the identity (4.0.2). By inverse Mellin transform, we can see that for $y > 0$,

$$\sum_{n=1}^{\infty}\psi(n)a_f(n^2)e^{-ny} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)L(s,\mathrm{Sym}^2(f)\otimes\psi)}{L(2s-2k+2,\chi^2\psi^2)}y^{-s}\mathrm{d}s, \tag{4.1.3}$$

where $\Re(s) = c > k$. As mentioned before, for simplicity of calculation, we assume that

χ and ψ are trivial characters. Thus, the above equation (4.1.3) becomes

$$\sum_{n=1}^{\infty} a_f(n^2)e^{-ny} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f))}{\zeta(2s-2k+2)} y^{-s}\mathrm{d}s. \tag{4.1.4}$$

Now we shall analyze the poles of the integrand function. Note that $\Gamma(s)L(s,\mathrm{Sym}^2(f))$ is an entire function since $L^*(s,\mathrm{Sym}^2(f))$ is entire as we are dealing with trivial character χ. In general, $L^*(s,\mathrm{Sym}^2(f))$ may not be an entire function. Assuming the Riemann Hypothesis, one can see that the integrand function has infinitely many poles on $\Re(s) = k - \frac{3}{4}$. Furthermore, the integrand function has simple poles at $k-n$ for $n \geq 2$ due to the trivial zeros of $\zeta(2s-2k+2)$. Consider the following rectangular contour $\mathscr{C} : [c-iT, c+iT], [c+iT, d+iT], [d+iT, d-iT]$, and $[d-iT, c-iT]$, where $k-2 < d < k-1$ and T is a large positive real number. We can observe that the integrand function has finitely many poles inside this contour $\mathscr{C}$ due to the non-trivial zeros ρ of $\zeta(2s-2k+2)$ with $|\Im(\rho)| < T$ and the poles at $k-n$, for $n \geq 2$, are lying outside the contour. Therefore, employing Cauchy residue theorem, we have

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f))}{\zeta(2s-2k+2)} y^{-s}\mathrm{d}s = \mathscr{Q}_T(y), \tag{4.1.5}$$

where $\mathscr{Q}_T(y)$ denotes the residual term that includes finitely many terms that are supplied by the non-trivial zeros ρ of $\zeta(2s-2k+2)$ with $|\Im(\rho)| < T$. We denote the two vertical integrals as

$$\begin{aligned}
V_1(T,y) &:= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f))}{\zeta(2s-2k+2)} y^{-s}\mathrm{d}s,\\
V_2(T,y) &:= \frac{1}{2\pi i}\int_{d-iT}^{d+iT} \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f))}{\zeta(2s-2k+2)} y^{-s}\mathrm{d}s,
\end{aligned}$$

and the horizontal integrals are denoted as

$$\begin{aligned}
H_1(T,y) &:= \frac{1}{2\pi i}\int_{c+iT}^{d+iT} \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f))}{\zeta(2s-2k+2)} y^{-s}\mathrm{d}s,\\
H_2(T,y) &:= \frac{1}{2\pi i}\int_{d-iT}^{c-iT} \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f))}{\zeta(2s-2k+2)} y^{-s}\mathrm{d}s.
\end{aligned}$$

We show that the contribution of the horizontal integrals vanish as $T \to \infty$. One can write

$$H_1(T,y) = \frac{1}{2\pi i}\int_{c}^{d} \frac{\Gamma(\sigma+iT)L(\sigma+iT,\mathrm{Sym}^2(f))}{\zeta(2\sigma-2k+2+2iT)} y^{-\sigma-iT}\mathrm{d}\sigma.$$

Thus,

$$|H_1(T,y)| \ll \int_c^d \frac{|\Gamma(\sigma+iT)||L(\sigma+iT,\mathrm{Sym}^2(f))|}{|\zeta(2\sigma-2k+2+2iT)|} y^{-\sigma} \mathrm{d}\sigma.$$

Use Lemmas 2.2.3, 2.2.1 and 4.0.2, to derive that

$$|H_1(T,y)| \ll |T|^C e^{C_2 T - \frac{\pi}{4}|T|},$$

where C and C_2 are some constants with $0 < C_2 < \pi/4$. This immediately implies that $H_1(T,y)$ goes to zero as $T \to \infty$. Similarly we can show that $H_2(T,y)$ also vanishes as $T \to \infty$. Now allowing $T \to \infty$ in (4.1.5) and using (4.1.4), we have

$$\sum_{n=1}^{\infty} a_f(n^2) e^{-ny} = \mathscr{Q}(y) + \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f))}{\zeta(2s-2k+2)} y^{-s} \mathrm{d}s, \tag{4.1.6}$$

where $\mathscr{Q}(y) = \lim_{T\to\infty} \mathscr{Q}_T(y)$ is the residual function consisting of infinitely many terms. Assuming the simplicity hypothesis, that is, all the non-trivial zeros of $\zeta(s)$ are simple, one can show that

$$\begin{aligned} \mathscr{Q}(y) &= \sum_{\rho} \lim_{s \to \frac{\rho}{2}+k-1} \left(s - \frac{\rho}{2} - k + 1\right) \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f))}{\zeta(2s-2k+2)} y^{-s} \\ &= \frac{1}{2y^{k-1}} \sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2}+k-1\right) L(\frac{\rho}{2}+k-1,\mathrm{Sym}^2(f))}{\zeta'(\rho)}, \end{aligned} \tag{4.1.7}$$

where the sum over ρ runs through all the non-trivial zeros of $\zeta(s)$, bracketing the terms as before.

Now we shall try to simplify the left vertical integral:

$$V_2(y) = \lim_{T\to\infty} V_2(T,y) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)L(s,\mathrm{Sym}^2(f))}{\zeta(2s-2k+2)} y^{-s} \mathrm{d}s. \tag{4.1.8}$$

First we shall make use of the functional equation of the symmetric square L-function (4.0.3) and with the help of the duplication formula for the Gamma function (2.2.2), one can obtain

$$\begin{aligned} V_2(y) = \frac{1}{2\pi^{3k-1}} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} & \frac{\Gamma\left(\frac{2k-1}{2} - \frac{s}{2}\right) \Gamma\left(k - \frac{s}{2}\right) \Gamma\left(\frac{k+1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{2-k}{2} + \frac{s}{2}\right) \zeta(2s-2k+2)} \\ & \times L(2k-1-s,\mathrm{Sym}^2(f)) \left(\frac{yN^2}{2\pi^3}\right)^{-s} \mathrm{d}s. \end{aligned} \tag{4.1.9}$$

Replace s by $2s-2k+2$ in (1.4.1) to see

$$\zeta(2s-2k+2) = \frac{\pi^{2s-2k+2}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{2k-2s-1}{2}\right)}{\Gamma(1-k+s)} \zeta(2k-2s-1). \tag{4.1.10}$$

Substituting (4.1.10) in (4.1.9) and simplifying, we have

$$V_2(y) = \frac{1}{2\pi^{k+\frac{1}{2}}} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma\left(\frac{2k-1}{2}-\frac{s}{2}\right)\Gamma\left(k-\frac{s}{2}\right)\Gamma\left(\frac{k+1}{2}-\frac{s}{2}\right)\Gamma(1-k+s)}{\Gamma\left(\frac{2-k}{2}+\frac{s}{2}\right)\Gamma\left(\frac{2k-1}{2}-s\right)\zeta(2k-2s-1)} \times L(2k-1-s, \mathrm{Sym}^2(f)) \left(\frac{y}{2\pi}\right)^{-s} \mathrm{d}s.$$

At this juncture, we would like to shift the line of integration. To do that we change the variable, namely, $2k-1-s=w$. We obtain

$$V_2(y) = \frac{1}{2\pi^{k+\frac{1}{2}}} \frac{1}{2\pi i} \int_{d'-i\infty}^{d'+i\infty} \frac{\Gamma\left(\frac{w}{2}\right)\Gamma\left(\frac{w+1}{2}\right)\Gamma\left(\frac{w}{2}+\frac{2-k}{2}\right)\Gamma(k-w)}{\Gamma\left(\frac{1+k}{2}-\frac{w}{2}\right)\Gamma\left(w+\frac{1-2k}{2}\right)} \times \frac{L(w, \mathrm{Sym}^2(f))}{\zeta(2w-2k+1)} \left(\frac{y}{2\pi}\right)^{w-2k+1} \mathrm{d}w, \tag{4.1.11}$$

where $k < d' = \Re(w) < k+1$ as $k-2 < d = \Re(s) < k-1$. One can easily check that the symmetric square L-function $L(w, \mathrm{Sym}^2(f))$ and $\zeta(2w-2k+1)$ are both absolutely convergent on the line $\Re(w) = d'$. Therefore, we write

$$\begin{aligned} \frac{L(w, \mathrm{Sym}^2(f))}{\zeta(2w-2k+1)} &= \sum_{n=1}^{\infty} \frac{a_{Sym^2(f)}(n)}{n^w} \sum_{n=1}^{\infty} \frac{n^{2k-1}}{n^{2w}} \\ &= \sum_{n=1}^{\infty} \frac{B_f(n)}{n^w}, \end{aligned} \tag{4.1.12}$$

where $B_f(n)$ is defined as in (4.1.1). Implement (4.1.12) in (4.1.11) and interchange the order of integration and summation to derive

$$V_2(y) = \frac{1}{2\pi^{k+\frac{1}{2}}} \left(\frac{y}{2\pi}\right)^{1-2k} \sum_{n=1}^{\infty} B_f(n) I_{k,y}(n), \tag{4.1.13}$$

where

$$I_{k,y}(n) := \frac{1}{2\pi i} \int_{d'-i\infty}^{d'+i\infty} \frac{\Gamma\left(\frac{w}{2}\right)\Gamma\left(\frac{w+1}{2}\right)\Gamma\left(\frac{w}{2}+\frac{2-k}{2}\right)\Gamma(k-w)}{\Gamma\left(\frac{1+k}{2}-\frac{w}{2}\right)\Gamma\left(w+\frac{1-2k}{2}\right)} \left(\frac{y}{2n\pi}\right)^{w} \mathrm{d}w.$$

Now one of our main goals shall be to evaluate this line integral explicitly. First replace

$w \to 2w$,

$$I_{k,y}(n) := \frac{1}{2\pi i}\int_{\frac{d'}{2}-i\infty}^{\frac{d'}{2}+i\infty} \frac{\Gamma(w)\Gamma\left(w+\frac{1}{2}\right)\Gamma\left(w+\frac{2-k}{2}\right)\Gamma(k-2w)}{\Gamma\left(\frac{1+k}{2}-w\right)\Gamma\left(2w+\frac{1-2k}{2}\right)}\left(\frac{y}{2n\pi}\right)^{2w} 2\,\mathrm{d}w. \quad (4.1.14)$$

To simplify more we use the duplication formula for the Gamma function. We use the following two identities:

$$\Gamma(k-2w) = \frac{2^{k-2w}}{2\sqrt{\pi}}\Gamma\left(\frac{k}{2}-w\right)\Gamma\left(\frac{1+k}{2}-w\right), \quad (4.1.15)$$

$$\Gamma\left(2w+\frac{1-2k}{2}\right) = \frac{2^{2w+\frac{1-2k}{2}}}{2\sqrt{\pi}}\Gamma\left(w+\frac{1-2k}{4}\right)\Gamma\left(w+\frac{3-2k}{4}\right). \quad (4.1.16)$$

Invoking (4.1.15) and (4.1.16) in (4.1.14) we have

$$\begin{aligned} I_{k,y}(n) &:= \frac{1}{2\pi i}\int_{\frac{d'}{2}-i\infty}^{\frac{d'}{2}+i\infty} \frac{\Gamma(w)\Gamma\left(w+\frac{1}{2}\right)\Gamma\left(w+\frac{2-k}{2}\right)\Gamma\left(\frac{k}{2}-w\right)2^{2k-4w-\frac{1}{2}}}{\Gamma\left(w+\frac{1-2k}{4}\right)\Gamma\left(w+\frac{3-2k}{4}\right)}\left(\frac{y}{2n\pi}\right)^{2w} 2\,\mathrm{d}w \\ &= \frac{2^{2k+\frac{1}{2}}}{2\pi i}\int_{\frac{d'}{2}-i\infty}^{\frac{d'}{2}+i\infty} \frac{\Gamma(w)\Gamma\left(w+\frac{1}{2}\right)\Gamma\left(w+\frac{2-k}{2}\right)\Gamma\left(\frac{k}{2}-w\right)}{\Gamma\left(w+\frac{1-2k}{4}\right)\Gamma\left(w+\frac{3-2k}{4}\right)}\left(\frac{y}{8n\pi}\right)^{2w}\mathrm{d}w. \quad (4.1.17) \end{aligned}$$

To write this integral in terms of the Meijer G-function, we shall analyze the poles of the integrand function. We know that the poles of $\Gamma(w)$ are at $0,-1,-2,\cdots$; poles of $\Gamma(w+1/2)$ are at $-1/2,-3/2,-5/2,\cdots$; and the poles of $\Gamma\left(w+\frac{2-k}{2}\right)$ are at $k/2-1, k/2-2, k/2-3,\cdots$; whereas the poles of $\Gamma\left(\frac{k}{2}-w\right)$ are at $k/2, k/2+1, k/2+2,\cdots$. So, we can not write the integral (4.1.17) in terms of the Meijer G-function since the line of integration does not separate the poles of the Gamma factors $\Gamma(w)\Gamma\left(w+\frac{1}{2}\right)\Gamma\left(w+\frac{2-k}{2}\right)$ from the poles of the Gamma factor $\Gamma\left(\frac{k}{2}-w\right)$. Hence, we construct a new line of integration so that it separates the poles of the Gamma factors $\Gamma(w)\Gamma\left(w+\frac{1}{2}\right)\Gamma\left(w+\frac{2-k}{2}\right)$ from the poles of the Gamma factor $\Gamma\left(\frac{k}{2}-w\right)$. Now consider the contour $\mathscr{C}'$ consisting of the line segments $[d'-iT, d'+iT], [d'+iT, d''+iT], [d''+iT, d''-iT,]$, and $[d''-iT, d'-iT]$, where $d'' \in (k/2-1, k/2)$, T is some large positive real number. Use Cauchy residue theorem to obtain

$$\frac{1}{2\pi i}\int_{\mathscr{C}'} F_k(w)\mathrm{d}w = \operatorname*{Res}_{s=\frac{k}{2}} F_k(w), \quad (4.1.18)$$

where

$$F_k(w) = \frac{\Gamma(w)\Gamma\left(w+\frac{1}{2}\right)\Gamma\left(w+\frac{2-k}{2}\right)\Gamma\left(\frac{k}{2}-w\right)}{\Gamma\left(w+\frac{1-2k}{4}\right)\Gamma\left(w+\frac{3-2k}{4}\right)}\left(\frac{y}{8n\pi}\right)^{2w}.$$

Again, with the help of Stirling's formula for the Gamma function, one can show that the horizontal integrals tend to zero as T tends to infinity. Therefore, letting $T \to \infty$ in (4.1.18) and calculating the residual term and substituting it in (4.1.17), we get

$$I_{k,y}(n) = \frac{2^{2k+\frac{1}{2}}}{2\pi i}\int_{d''-i\infty}^{d''+i\infty} F_k(w)\mathrm{d}w - \frac{2^{2k+\frac{1}{2}}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}\left(\frac{y}{8n\pi}\right)^k. \tag{4.1.19}$$

Now we shall try to write the line integral along (d'') in terms of the Meijer G-function and to do that we reminisce the definition of the Meijer G-function (1.5.2). We consider $m = 1,\ n = 3,\ p = 3,\ q = 3$ with $a_1 = 1,\ a_2 = 1/2,\ a_3 = k/2$; and $b_1 = k/2,\ b_2 = (1+2k)/4,\ b_3 = (3+2k)/4$. One can easily check that $a_i - b_j \notin \mathbb{N}$ for $1 \le i \le n,\ 1 \le j \le m$ and the inequality $p+q < 2(m+n)$ is also satisfied. Hence, one can write

$$\frac{1}{2\pi i}\int_{d''-i\infty}^{d''+i\infty} F_k(w)\mathrm{d}w = G_{3,3}^{1,3}\left(\begin{matrix} 1,\ \frac{1}{2},\ \frac{k}{2} \\ \frac{k}{2},\ \frac{1+2k}{4},\ \frac{3+2k}{4}\end{matrix}\ \Big|\ \left(\frac{y}{8n\pi}\right)^2\right). \tag{4.1.20}$$

Utilize Slater's theorem (1.5.3) to write the above Meijer G-function in terms of the hypergeometric function:

$$G_{3,3}^{1,3}\left(\begin{matrix} 1,\ \frac{1}{2},\ \frac{k}{2} \\ \frac{k}{2},\ \frac{1+2k}{4},\ \frac{3+2k}{4}\end{matrix}\ \Big|\ z\right) = \frac{z^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}\,{}_3F_2\left(\frac{k}{2},\frac{k+1}{2},1;\frac{1}{4},\frac{3}{4};-z\right). \tag{4.1.21}$$

Substituting $z = \left(\frac{y}{8n\pi}\right)^2$ in (4.1.21) and together with (4.1.20) and (4.1.19), we achieve

$$I_{k,y}(n) = \frac{2^{2k+\frac{1}{2}}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}\left(\frac{y}{8n\pi}\right)^k\left[{}_3F_2\left(\frac{k}{2},\frac{k+1}{2},1;\frac{1}{4},\frac{3}{4};-\left(\frac{y}{8n\pi}\right)^2\right)-1\right]. \tag{4.1.22}$$

Finally, substituting (4.1.22) in (4.1.13) and together with (4.1.6), (4.1.7) and (4.1.8), we complete the proof of Theorem 4.1.1. □

Remark 4.1.2. *Using the definition* (1.5.1) *of the hypergeometric series one can show that the above infinite series is indeed convergent with similar arguments as in Remarks 2.3.1 and 3.1.6.*

4.2 AN ASYMPTOTIC EXPANSION

The asymptotic result given below is an immediate application of this theorem.

Corollary 4.2.1. *Let N be a positive integer and $f(z)$ be a normalized Hecke eigenform defined as in Theorem 4.1.1. Assume the Riemann Hypothesis and simplicity of the non-trivial zeros of $\zeta(s)$. For $y \to 0^+$, we have*

$$y^k \sum_{n=1}^{\infty} a_f(n^2) e^{-ny} = y^{3/4} \sum_{n=1}^{\infty} b_n \cos\left(\delta_n - \frac{t_n}{2}\log(y)\right) + \sum_{j=1}^{N-1} A_j y^{2j+1} + O_{f,k}(y^{2N+1}),$$

where the absolute constants A_j depend only on f and the polar representation of $\Gamma\left(\frac{\rho_n}{2}+k-1\right) L(\frac{\rho_n}{2}+k-1, \mathrm{Sym}^2(f))\,(\zeta'(\rho_n))^{-1}$ is considered as $b_n e^{i\delta_n}$, where $\rho_n = \frac{1}{2}+it_n$ denotes the nth non-trivial zero of $\zeta(s)$.

Proof. With the help of definition (1.5.1) of the hypergeometric series, for any positive integer N, we have

$$ {}_3F_2\left(\frac{k}{2}, \frac{k+1}{2}, 1; \frac{1}{4}, \frac{3}{4}; -\left(\frac{y}{8n\pi}\right)^2\right) - 1 = \sum_{j=1}^{N-1} C_j \left(\frac{y}{n}\right)^{2j} + O_k\left(\left(\frac{y}{n}\right)^{2N}\right) \text{ as } y \to 0^+, \tag{4.2.1}$$

where $C_j = (-1)^j \frac{\left(\frac{k}{2}\right)_j \left(\frac{k+1}{2}\right)_j}{\left(\frac{1}{4}\right)_j \left(\frac{3}{4}\right)_j (8\pi)^{2j}}$. Now invoke (4.2.1) in Theorem 4.1.1 to derive that

$$\begin{aligned} y^k \sum_{n=1}^{\infty} a_f(n^2) e^{-ny} &= \frac{\Gamma(k)}{2\pi^2} \sum_{j=1}^{N-1} C_j y^{2j+1} \sum_{n=1}^{\infty} \frac{B_f(n)}{n^{k+2j}} + O_k\left(y^{2N+1} \sum_{n=1}^{\infty} \frac{B_f(n)}{n^{k+2N}}\right) + y^k \mathscr{Q}(y) \\ &= \sum_{j=1}^{N-1} A_j y^{2j+1} + O_{f,k}\left(y^{2N+1}\right) + y^k \mathscr{Q}(y), \end{aligned} \tag{4.2.2}$$

where $A_j = \frac{\Gamma(k)}{2\pi^2} C_j \sum_{n=1}^{\infty} \frac{B_f(n)}{n^{k+2j}}$ are computable finite constants, since the Dirichlet series associated to $B_f(n)$ is absolutely convergent for $\Re(s) > k$. Assuming the Riemann hypothesis and using the fact that the non-trivial zeros appear in conjugate pairs to write the residual term as

$$\begin{aligned} y^k \mathscr{Q}(y) &= \frac{y}{2} \sum_{\substack{\rho_n = \frac{1}{2}+it_n, \\ t_n > 0}} 2\Re\left(\frac{\Gamma\left(\frac{\rho_n}{2}+k-1\right) L(\frac{\rho_n}{2}+k-1, \mathrm{Sym}^2(f))}{y^{\frac{\rho_n}{2}} \zeta'(\rho_n)}\right) \\ &= y^{3/4} \sum_{\substack{\rho_n = \frac{1}{2}+it_n, \\ t_n > 0}} b_n \cos\left(\delta_n - \frac{t_n}{2}\log(y)\right). \end{aligned} \tag{4.2.3}$$

Here we have considered $b_n e^{i\delta_n}$ as the polar representation of $\Gamma\left(\frac{\rho_n}{2}+k-1\right) L(\frac{\rho_n}{2}+k-1, \mathrm{Sym}^2(f))\,(\zeta'(\rho_n))^{-1}$. Employ (4.2.3) in (4.2.2) to the complete the proof. □

CHAPTER 5

RANKIN–COHEN BRACKETS ON HERMITIAN JACOBI FORMS AND THE ADJOINT OF SOME LINEAR MAPS

Kohnen (1991) considered certain linear maps between spaces of modular forms using the product map by a fixed cusp form and then computed the adjoint maps of these linear maps with respect to the Petersson scalar product. The Fourier coefficients of the image of a form constructed by using this method involve special values of certain Dirichlet series attached to this form. The work of Kohnen has been generalized to other automorphic forms (see Choie et al. (1995), Sakata (1998) and Wang and Pei (1995)).

Herrero (2015) generalized the work of Kohnen within the theory of modular forms. The author constructed the adjoint map of similar linear maps defined by using the Rankin–Cohen brackets by a fixed modular form instead of the usual product map. The work of Herrero has been generalized to the case of Jacobi forms, Jacobi forms of several variables and Siegel modular forms of degree 2 by Jha and Sahu (2016, 2017, 2019).

In this chapter we generalize the work of Herrero (2015) to the case of Hermitian Jacobi forms over $\mathbb{Q}(i)$.

5.1 ADJOINT CONSTRUCTION PROBLEM

Suppose $\psi \in J_{k_2,m_2}^{\delta_2,cusp}(\Gamma^J(\mathscr{O}))$ is fixed. Define the map

$$T_{\psi,\nu} : J_{k_1,m_1}^{\delta_1,cusp}(\Gamma^J(\mathscr{O})) \to J_{k_1+k_2+2\nu,m_1+m_2}^{\delta_1\delta_2(-1)^\nu,cusp}(\Gamma^J(\mathscr{O}))$$

by $T_{\psi,\nu}(\phi) = [[\phi,\psi]]_\nu$. Then $T_{\psi,\nu}$ is a $\mathbb{C}$-linear map between finite-dimensional Hilbert spaces and therefore there exists a unique adjoint map

$$T_{\psi,\nu}^* : J_{k_1+k_2+2\nu,m_1+m_2}^{\delta_1\delta_2(-1)^\nu,cusp}(\Gamma^J(\mathscr{O})) \to J_{k_1,m_1}^{\delta_1,cusp}(\Gamma^J(\mathscr{O}))$$

such that

$$\langle \varphi, T_{\psi,\nu}(\phi)\rangle = \langle T^*_{\psi,\nu}(\varphi), \phi\rangle,$$

for all $\phi \in J^{\delta_1,cusp}_{k_1,m_1}(\Gamma^J(\mathcal{O}))$ and $\varphi \in J^{\delta_1\delta_2(-1)^\nu,cusp}_{k_1+k_2+2\nu,m_1+m_2}(\Gamma^J(\mathcal{O}))$.

Theorem 5.1.1. *(S and Singh (2021) Let $k_1, k_2 > 4$. Let m_1, m_2 be positive integers. Suppose $\psi \in J^{\delta_2,cusp}_{k_2,m_2}(\Gamma^J(\mathcal{O}))$ has Fourier expansion*

$$\psi(\tau,z,w) = \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathcal{O}^\#\\ n_1m_2-|r_1|^2>0}} a(n_1,r_1)e^{2\pi i(n_1\tau+r_1z+\overline{r_1}w)}.$$

Then the image of $\varphi \in J^{\delta_1\delta_2(-1)^\nu,cusp}_{k_1+k_2+2\nu,m_1+m_2}(\Gamma^J(\mathcal{O}))$ with Fourier expansion

$$\varphi(\tau,z,w) = \sum_{\substack{n_2\in\mathbb{Z}, r_2\in\mathcal{O}^\#\\ n_2(m_1+m_2)-|r_2|^2>0}} b(n_2,r_2)e^{2\pi i(n_2\tau+r_2z+\overline{r_2}w)}$$

under $T^*_{\psi,\nu}$ is given by

$$T^*_{\psi,\nu}(\varphi)(\tau,z,w) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^\#\\ nm_1-|r|^2>0}} c(n,r)e^{2\pi i(n\tau+rz+\bar{r}w)}$$

where

$$c(n,r) = \frac{(m_1+m_2)^{k_1+k_2+2\nu-3}}{m_1^{k_1-3}}\frac{\Gamma(k_1+k_2+2\nu-2)}{\Gamma(k_1-2)}\frac{(nm_1-|r|^2)^{k_1-2}}{(4\pi)^{k_2+2\nu}}$$

$$\times\sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)(4nm_1-4|r|^2)^l$$

$$\times \sum_{\substack{n_1>0\\ r_1\in\mathcal{O}^\#\\ n_1m_2-|r_1|^2>0\\ (n+n_1)(m_1+m_2)-|r+r_1|^2>0}} \frac{(4n_1m_2-4|r_1|^2)^{\nu-l}\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{((n+n_1)(m_1+m_2)-|r_1+r_2|^2)^{k_1+k_2+2\nu-2}}$$

and $A_l(k_1,m_1,k_2,m_2;\nu) = (-1)^l\binom{k_1+\nu-2}{\nu-l}\binom{k_2+\nu-2}{l}m_1^{\nu-l}m_2^l$.

We state the following two lemmas which will be required to prove Theorem 5.1.1. The proof of Lemma 5.1.2 follows from a direct computation. The proof of Lemma 5.1.3 follows from the usual Rankin unfolding argument and Lemma 3.9 of Martin (2016).

Lemma 5.1.2. *Let $\phi, \psi : \mathbb{H} \times \mathbb{C}^2 \to \mathbb{C}$ be holomorphic functions. For $\delta_1, \delta_2 \in \{\pm 1\}$, we have*

$$[[\phi|_{k_1,m_1,\delta_1}\gamma, \psi|_{k_2,m_2,\delta_2}\gamma]]_\nu = [[\phi, \psi]]_\nu \ |_{k_1+k_2+2\nu, m_1+m_2, \delta_1\delta_2(-1)^\nu}\gamma.$$

Lemma 5.1.3. *Let ψ, φ be as in Theorem 5.1.1. The sum*

$$\sum_{\gamma\in\Gamma^J_\infty(\mathscr{O})\backslash\Gamma^J(\mathscr{O})} \int_{\Gamma^J(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2} \Bigg(\Big| \varphi(\tau,z,w)\overline{[[e^{2\pi i(n\tau+rz+\bar{r}w)}|_{k_1,m_1,\delta}\gamma, \psi(\tau,z,w)]]_\nu} \times e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2} v^{k_1+k_2+2\nu} \Big| \Bigg) \mathrm{d}V$$

converges.

5.2 PROOF OF 5.1.1 AND SOME REMARKS

Proof. Let

$$T^*_{\psi,\nu}(\varphi)(\tau,z,w) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathscr{O}^\#\\ m_1n-|r|^2>0}} c(n,r)e^{2\pi i(n\tau+rz+\bar{r}w)}.$$

By the definition of adjoint map, we have

$$\langle T^*_{\psi,\nu}(\varphi), P^{k_1,m_1,\delta_1}_{n,r}\rangle = \langle \varphi, T_{\psi,\nu}(P^{k_1,m_1,\delta_1}_{n,r})\rangle = \langle \varphi, [[P^{k_1,m_1,\delta_1}_{n,r}, \psi]]_\nu\rangle.$$

By Lemma 1.3.3, we have

$$\langle T^*_{\psi,\nu}(\varphi), P^{k_1,m_1,\delta_1}_{n,r}\rangle = c(n,r)\frac{m^{k_1-3}\Gamma(k_1-2)}{\pi^{k_1-2}(4m_1n-4|r_1|^2)^{k_1-2}}.$$

This implies that

$$c(n,r) = \frac{\pi^{k_1-2}(4m_1n-4|r_1|^2)^{k_1-2}}{m^{k_1-3}\Gamma(k_1-2)}\langle \varphi, [[P^{k_1,m_1,\delta_1}_{n,r}, \psi]]_\nu\rangle. \tag{5.2.1}$$

Now we shall compute $\langle \varphi, [[P^{k_1,m_1,\delta_1}_{n,r}, \psi]]_\nu\rangle$. From the definition of the Petersson scalar product we have

$$\langle \varphi, [[P^{k_1,m_1,\delta_1}_{n,r}, \psi]]_\nu\rangle = \int_{\Gamma^J(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2} \varphi(\tau,z,w)\overline{[[P^{k_1,m_1,\delta_1}_{n,r}, \psi]]_\nu} e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2} v^{k_1+k_2+2\nu}\mathrm{d}V$$

$$= \int_{\Gamma^J(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2} \Bigg(\varphi(\tau,z,w)\overline{[[\sum_{\gamma\in\Gamma^J_\infty(\mathscr{O})\backslash\Gamma^J(\mathscr{O})} e^{2\pi i(n\tau+rz+\bar{r}w)}|_{k_1,m_1,\delta_1}\gamma, \psi]]_\nu}$$

$$\times e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2} v^{k_1+k_2+2\nu}\Big) \mathrm{d}V$$

$$= \int\limits_{\Gamma^J(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2} \sum_{\gamma\in\Gamma^J_\infty(\mathscr{O})\backslash\Gamma^J(\mathscr{O})} \Big(\varphi(\tau,z,w)\overline{[[e^{2\pi i(n\tau+rz+\bar{r}w)}|_{k_1,m_1,\delta_1}\gamma,\psi]]_\nu}$$

$$\times e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2} v^{k_1+k_2+2\nu}\Big) \mathrm{d}V.$$

By Lemma 5.1.3, the last expression converges absolutely and hence we can interchange the integral and the sum. Therefore we have

$$\langle\varphi,[[P^{k_1,m_1,\delta_1}_{n,r},\psi]]_\nu\rangle = \sum_{\gamma\in\Gamma^J_\infty(\mathscr{O})\backslash\Gamma^J(\mathscr{O})} \int\limits_{\Gamma^J(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2} \Big(\varphi(\tau,z,w)\overline{[[e^{2\pi i(n\tau+rz+\bar{r}w)}|_{k_1,m_1,\delta_1}\gamma,\psi]]_\nu}$$

$$\times e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2} v^{k_1+k_2+2\nu}\Big) \mathrm{d}V.$$

Changing variable (τ,z,w) to $\gamma^{-1}\cdot(\tau,z,w)$ in the above identity and using the definition 1.3.1 and Lemma 5.1.2, we have

$$\langle\varphi,[[P^{k_1,m_1,\delta_1}_{n,r},\psi]]_\nu\rangle = \sum_{\gamma\in\Gamma^J_\infty(\mathscr{O})\backslash\Gamma^J(\mathscr{O})} \int\limits_{\gamma\cdot\left(\Gamma^J(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2\right)} \Big(\varphi(\tau,z,w)\overline{[[e^{2\pi i(n\tau+rz+\bar{r}w)},\psi]]_\nu}$$

$$\times e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2} v^{k_1+k_2+2\nu}\Big) \mathrm{d}V.$$

Applying the Rankin unfolding argument, the above identity is equivalent to

$$\langle\varphi,[[P^{k_1,m_1,\delta_1}_{n,r},\psi]]_\nu\rangle = \int\limits_{\Gamma^J_\infty(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2} \Big(\varphi(\tau,z,w)\overline{[[e^{2\pi i(n\tau+rz+\bar{r}w)},\psi]]_\nu}$$

$$\times e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2} v^{k_1+k_2+2\nu}\Big) \mathrm{d}V.$$

Now using the definition of the Rankin–Cohen bracket (1.3.3), we have

$$\langle\varphi,[[P^{k_1,m_1,\delta_1}_{n,r},\psi]]_\nu\rangle = \int\limits_{\Gamma^J_\infty(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2} \Big(\varphi(\tau,z,w)\sum_{l=0}^{\nu}(-1)^l\binom{k_1+\nu-2}{\nu-l}\binom{k_2+\nu-2}{l}m_1^{\nu-l}m_2^l$$

$$\times \overline{L^l_{m_1}(e^{2\pi i(n\tau+rz+\bar{r}w)})L^{\nu-l}_{m_2}(\psi)}e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2} v^{k_1+k_2+2\nu}\Big) \mathrm{d}V. \tag{5.2.2}$$

We have

$$L_{m_1}(e^{2\pi i(n\tau+rz+\bar{r}w)}) = (4nm_1-4|r|^2)e^{2\pi i(n\tau+rz+\bar{r}w)}.$$

By induction we have

$$L_{m_1}^{l}(e^{2\pi i(n\tau+rz+\bar{r}w)}) = (4nm_1-4|r|^2)^l e^{2\pi i(n\tau+rz+\bar{r}w)} \tag{5.2.3}$$

and

$$L_{m_2}^{\nu-l}(\psi) = \sum_{\substack{n_1\in\mathbb{Z},r_1\in\mathscr{O}^{\#}\\ m_2n_1-|r_1|^2>0}} (4n_1m_2-4|r_1|^2)^{\nu-l}a(n_1,r_1)e^{2\pi i(n_1\tau+r_1z+\bar{r}_1w)}. \tag{5.2.4}$$

Substituting the Fourier expansion of φ and (5.2.3), (5.2.4) in (5.2.2), we have

$$\langle\varphi,[[P_{n,r}^{k_1,m_1,\delta_1},\psi]]_\nu\rangle = \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)\int\limits_{\Gamma_\infty^J(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2}\Bigg(\sum_{\substack{n_2\in\mathbb{Z},r_2\in\mathscr{O}^{\#}\\ (m_1+m_2)n_2-|r_2|^2>0}} b(n_2,r_2)$$

$$\times\, e^{2\pi i(n_2\tau+r_2z+\bar{r}_2w)}\sum_{\substack{n_1\in\mathbb{Z},r_1\in\mathscr{O}^{\#}\\ m_2n_1-|r_1|^2>0}}(4m_2n_1-4|r_1|^2)^{\nu-l}(4nm_1-4|r|^2)^l$$

$$\times\overline{a(n_1,r_1)e^{2\pi i((n+n_1)\tau+(r+r_1)z+\overline{(r+r_1)}w)}}e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2}v^{k_1+k_2+2\nu}\Bigg)\mathrm{d}V.$$

Taking out the summation outside the integral we have

$$\langle\varphi,[[P_{n,r}^{k_1,m_1,\delta_1},\psi]]_\nu\rangle = \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)\sum_{\substack{n_2\in\mathbb{Z},r_2\in\mathscr{O}^{\#}\\ (m_1+m_2)n_2-|r_2|^2>0}}\ \sum_{\substack{n_1\in\mathbb{Z},r_1\in\mathscr{O}^{\#}\\ m_2n_1-|r_1|^2>0}}\overline{a(n_1,r_1)}$$

$$\times b(n_2,r_2)(4m_2n_1-4|r_1|^2)^{\nu-l}(4nm_1-4|r|^2)^l$$

$$\times\int\limits_{\Gamma_\infty^J(\mathscr{O})\backslash\mathbb{H}\times\mathbb{C}^2}\Bigg(e^{2\pi i(n_2\tau+r_2z+\bar{r}_2w)}\overline{e^{2\pi i((n+n_1)\tau+(r+r_1)z+\overline{(r+r_1)}w)}}e^{\frac{-\pi(m_1+m_2)}{v}|w-\bar{z}|^2}v^{k_1+k_2+2\nu}\Bigg)\mathrm{d}V,$$

where $\tau=u+iv,\ z=x_1+iy_1,\ w=x_2+iy_2$. Choose the fundamental domain for the action of $\Gamma_\infty^J(\mathscr{O})$ on $\mathbb{H}\times\mathbb{C}^2$ to be $\mathscr{F}=([0,1]\times(0,\infty))\times([0,1]\times[0,1])\times(\mathbb{R}\times\mathbb{R})$.

Integrating on this region and substituting $z' = \overline{w} - z = \alpha + i\beta$ we have

$$\langle \varphi, [[P_{n,r}^{k_1,m_1,\delta_1}, \psi]]_\nu \rangle = \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_2\in\mathbb{Z}, r_2\in\mathcal{O}^\# \\ (m_1+m_2)n_2-|r_2|^2>0}} \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathcal{O}^\# \\ m_2n_1-|r_1|^2>0}} \overline{a(n_1,r_1)} b(n_2,r_2)$$

$$\times (4m_2n_1 - 4|r_1|^2)^{\nu-l}(4nm_1 - 4|r|^2)^l \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1\int_0^1\int_0^{\infty}\int_0^1 \big(e^{-2\pi v(n_2+n_1+n)} e^{2\pi i u(n_2-(n_1+n))}$$

$$\times e^{4\pi i Re((r_2-(r+r_1))z)+2\pi i(\overline{r_2 z'}-(r+r_1)z')} e^{\frac{-\pi(m_1+m_2)}{v}|z'|^2} v^{k_1+k_2+2\nu} v^{-4}\big) \mathrm{d}u\mathrm{d}v\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}\alpha\mathrm{d}\beta.$$

Integrating with respect to u, x_1 and y_1 first, we have

$$\langle \varphi, [[P_{n,r}^{k_1,m_1,\delta_1}, \psi]]_\nu \rangle = \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathcal{O}^\# \\ m_2n_1-|r_1|^2>0 \\ (m_1+m_2)(n+n_1)-|r+r_1|^2>0}} \overline{a(n_1,r_1)} b(n+n_1, r+r_1)$$

$$\times (4m_2n_1 - 4|r_1|^2)^{\nu-l}(4nm_1 - 4|r|^2)^l \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(v^{k_1+k_2+2\nu-4} e^{-4\pi v(n+n_1)}$$

$$\times e^{2\pi i(\overline{(r+r_1)z'}-(r+r_1)z')} e^{\frac{-\pi(m_1+m_2)}{v}|z'|^2}\Big) \mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}v.$$

Suppose $r + r_1 = \frac{t_1}{2} + i\frac{t_2}{2}$. We have

$$\langle \varphi, [[P_{n,r}^{k_1,m_1,\delta_1}, \psi]]_\nu \rangle = \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathcal{O}^\# \\ m_2n_1-|r_1|^2>0 \\ (m_1+m_2)(n+n_1)-|r+r_1|^2>0}} \overline{a(n_1,r_1)} b(n+n_1, r+r_1)$$

$$\times (4m_2n_1 - 4|r_1|^2)^{\nu-l}(4nm_1 - 4|r|^2)^l$$

$$\times \int_0^{\infty} v^{k_1+k_2+2\nu-4} e^{-4\pi v(n+n_1)} \left(\int_{-\infty}^{\infty} e^{-4\pi\left(\frac{(m_1+m_2)}{4v}\alpha^2 - t_2\alpha/2\right)} \mathrm{d}\alpha\right) \left(\int_{-\infty}^{\infty} e^{-4\pi\left(\frac{(m_1+m_2)}{4v}\beta^2 - t_1\beta/2\right)} \mathrm{d}\beta\right) \mathrm{d}v. \tag{5.2.5}$$

For $a \geq 0$ and $b \in \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} e^{-(at^2+bt)} dt = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}.$$

Using the above identity to solve integrations with respect to α and β in (5.2.5), we have

$$\langle \varphi, [[P_{n,r}^{k_1,m_1,\delta_1}, \psi]]_\nu\rangle = \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu) \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathscr{O}^\# \\ m_2n_1-|r_1|^2>0 \\ (m_1+m_2)(n+n_1)-|r+r_1|^2>0}} \frac{\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{m_1+m_2}$$

$$\times (4m_2n_1-4|r_1|^2)^{\nu-l}(4nm_1-4|r|^2)^l \int_0^\infty v^{k_1+k_2+2\nu-3} e^{\frac{-4\pi v}{m_1+m_2}((n+n_1)(m_1+m_2)-|r+r_1|^2)} \mathrm{d}v.$$

This implies that

$$\langle \varphi, [[P_{n,r}^{k_1,m_1,\delta_1}, \psi]]_\nu\rangle = \frac{(m_1+m_2)^{k_1+k_2+2\nu-3}\Gamma(k_1+k_2+2\nu-2)}{(4\pi)^{k_1+k_2+2\nu-2}} \sum_{l=0}^{\nu} A_l(k_1,m_1,k_2,m_2;\nu)$$

$$\times (4nm_1-4|r|^2)^l \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathscr{O}^\# \\ m_2n_1-|r_1|^2>0 \\ (m_1+m_2)(n+n_1)-|r+r_1|^2>0}} \frac{\overline{a(n_1,r_1)}b(n+n_1,r+r_1)(4m_2n_1-4|r_1|^2)^{\nu-l}}{((m_1+m_2)(n+n_1)-|r+r_1|^2)^{k_1+k_2+2\nu-2}}.$$

Substituting the value of $\langle \varphi, [[P_{n,r}^{k_1,m_1,\delta_1}, \psi]]_\nu\rangle$ in (5.2.1) we get the required result. □

Remark 5.2.1. *As an illustration of the above theorem, we consider the following example:*
Let $S = \left\{0, \frac{1}{2}, \frac{i}{2}, \frac{(1+i)}{2}\right\}$ *be the set of coset representatives of* $\mathscr{O}^\#/\mathscr{O}$. *For* $s \in S$, *we define*

$$\theta_{1,s}(\tau,z,w) = \sum_{r\in\mathscr{O}^\#,\ r\equiv s \pmod{\mathscr{O}}} e^{|r|^2\tau + rz + \bar{r}w}.$$

We define the Hermitian Jacobi forms $\phi_{k,1}^+ \in J_{k,1}^+(\Gamma^J(\mathscr{O}))$ *for* $k = 4, 8$ *by*

$$\phi_{4,1}^+ = \frac{1}{2}(x^6+y^6)\theta_{1,0} + \frac{1}{2}u^6(\theta_{1,1/2}+\theta_{1,i/2}) + \frac{1}{2}(x^6-y^6)\theta_{1,(1+i)/2},$$

$$\phi_{8,1}^+ = \frac{1}{2}(x^{14}+y^{14})\theta_{1,0} + \frac{1}{2}u^{14}(\theta_{1,1/2}+\theta_{1,i/2}) + \frac{1}{2}(x^{14}-y^{14})\theta_{1,(1+i)/2}$$

and

$$\phi_{10,1}^{+,cusp} = \frac{1}{64}x^6y^6u^6(\theta_{1,1/2}-\theta_{1,i/2}) \in J_{10,1}^{+,cusp}(\Gamma^J(\mathscr{O})),$$

where

$$x = 1+2\sum_{n=1}^\infty e^{\frac{n^2}{2}\tau}, \quad y = 1+2\sum_{n=1}^\infty (-1)^n e^{\frac{n^2}{2}\tau}, \quad u = 2e^{\frac{1}{8}\tau}\sum_{n=0}^\infty e^{\frac{n(n+1)}{2}\tau}.$$

Let $\psi \in J_{k_2,m_2}^{\delta_2,cusp}(\Gamma^J(\mathscr{O}))$ *be fixed. Suppose that* $J_{k_1,m_1}^{\delta_1,cusp}(\Gamma^J(\mathscr{O}))$ *is a one dimensional* *space generated by ϕ. Then by 5.1.1, for each $\varphi \in J^{\delta_1\delta_2(-1)^\nu,cusp}_{k_1+k_2+2\nu,m_1+m_2}(\Gamma^J(\mathscr{O}))$ we have*

$$T^*_{\psi,\nu}(\varphi)(\tau,z,w) = \beta_\varphi \phi(\tau,z,w),$$

where β_φ is a constant depending on φ. Now equating the (n,r)-th Fourier coefficient on each side, we get a relationship among special values of Rankin–Selberg type convolutions of the Hermitian Jacobi forms φ and ψ with the Fourier coefficients of ϕ. For example, we take $\psi = \phi^+_{8,1} - E_4\phi^+_{4,1} \in J^{+,cusp}_{8,1}(\Gamma^J(\mathscr{O}))$, where E_4 is the normalized Eisenstein series of weight 4 for $SL_2(\mathbb{Z})$. We take $k_1 = 10$, $m_1 = 1$ and $\delta = +$ such that $J^{+,cusp}_{10,1}(\Gamma^J(\mathscr{O}))$ is a one-dimensional Hermitian Jacobi cusp space generated by $\phi^{+,cusp}_{10,1}$. Then we get the following relation

$$\beta_\varphi c(n,r) = 2^{15+2\nu}\frac{\Gamma(16+2\nu)(n-|r|^2)^8}{\Gamma(8)(4\pi)^{8+2\nu}}\sum_{l=0}^{\nu} A_l(10,1,8,1;\nu)(4n-4|r|^2)^l$$

$$\times \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathscr{O}^\# \\ n_1-|r_1|^2>0 \\ 2(n+n_1)-|r+r_1|^2>0}} \frac{(4n_1-4|r_1|^2)^{\nu-l}\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(2(n+n_1-|r+r_1|^2))^{16+2\nu}},$$

for all $n \in \mathbb{Z}$, $r \in \mathscr{O}^\#$ such that $n-|r|^2 > 0$, where $a(p,q)$, $b(p,q)$ and $c(p,q)$ are the (p,q)-th Fourier coefficients of $\phi^+_{8,1} - E_4\phi^+_{4,1}$, φ and $\phi^{+,cusp}_{10,1}$ respectively. Also if $\nu = 0$ in the above example, we get the special values of Rankin–Selberg type convolutions of φ and $\phi^+_{8,1} - E_4\phi^+_{4,1}$ in terms of the Fourier coefficients of $\phi^{+,cusp}_{10,1}$

$$\beta_\varphi c(n,r) = \frac{\Gamma(16)(n-|r|^2)^8}{\Gamma(8)2\pi^8} \sum_{\substack{n_1\in\mathbb{Z}, r_1\in\mathscr{O}^\# \\ n_1-|r_1|^2>0 \\ 2(n+n_1)-|r+r_1|^2>0}} \frac{\overline{a(n_1,r_1)}b(n+n_1,r+r_1)}{(2(n+n_1-|r+r_1|^2))^{16}}.$$

Remark 5.2.2. *Martin and Senadheera (2017) have studied Rankin–Cohen type differential operators for Hermitian Jacobi forms. The method used in the proof of Theorem 5.1.1 can also be used in the computation of the adjoint linear functions constructed using these operators.*

CHAPTER 6

CONCLUSION AND FUTURE SCOPE

Inspired by the conjecture of Zagier (1981), works of Hafner and Stopple (2000) and Chakraborty et al. (2017), we have considered a few interesting Lambert series in Chapter 2, Chapter 3 and Chapter 4. In the second chapter, we have studied a Lambert series associated to a cusp form and the Möbius function. Using the functional equation of the L-function associated to the cusp form and the functional equation of the Riemann zeta function, we have established an exact formula for the Lambert series $\sum_{n=1}^{\infty}[a_f(n)*\mu(n)]e^{-ny}$ in terms of the non-trivial zeros of the Riemann zeta function and as a consequence, under the assumption of the Riemann Hypothesis and simplicity of the non-trivial zeros, we have also observed that $y^{1/2}\sum_{n=1}^{\infty}[a_f(n)*\mu(n)]e^{-ny}$ has an oscillatory behaviour when $y \to 0^+$.

In the third chapter, we have established an exact formula for the Lambert series $\sum_{n=1}^{\infty}[a_f(n)\psi(n)*\mu(n)\psi'(n)]e^{-ny}$ in terms of the non-trivial zeros of $L(s,\psi')$, where $a_f(n)$ is the nth Fourier coefficient of a cusp form f over a congruence subgroup, and ψ and ψ' are primitive Dirichlet characters, thereby generalizing our earlier result to congruence subgroups.

In the fourth chapter, we have established an exact formula for the Lambert series $y^k\sum_{n=1}^{\infty}a_f(n^2)e^{-ny}$, and we found that the main term can be expressed in terms of the non-trivial zeros of $\zeta(s)$, and the error term is expressed in terms of the hypergeometric function ${}_3F_2(a,b,c;d;z)$.

In the identity (1.0.1) of Hardy and Littlewood, and also in Theorem 2.1.1 and Theorem 4.1.1, we have assumed that all the non-trivial zeros are simple, whereas Corollary 2.1.2 and Corollary 4.2.1 is true under the additional assumption of the Riemann Hypothesis. Similarly, while Theorem 3.1.1 requires only the assumption of simplicity of zeros of the Dirichlet L-function $L(s,\psi')$, the Corollary 3.3.1 requires an additional assumption of generalized Riemann Hypothesis. In 2013, Bui and Heath-Brown (2013) proved that at least 70% of the non-trivial zeros are simple, under the assumption of the

Riemann Hypothesis. This was previously established by Conrey et al. (1998) under the assumption of the Generalized Riemann Hypothesis. Simplicity of the non-trivial zeros implies $|\zeta'(\rho)| > 0$, but till today we do not have much information about the lower bound for $|\zeta'(\rho)|$ without the assumption of any hypothesis. Due to this difficulty, even after assuming Riemann Hypothesis, Hardy and Littlewood mentioned that the convergence of the infinite series over ρ in (1.0.1) is not immediate. We know $|\zeta'(\rho)| \gg |\rho|^{-1}$ under the assumption of a weak Mertens Hypothesis (Titchmarsh, 1986, p. 377, Equation (14.29.3)). In a private communication, Prof. Steven Gonek had informed us that he had previously conjectured that $|\zeta'(\rho)| \gg |\rho|^{-\frac{1}{3}+\varepsilon}$ for any $\varepsilon > 0$. If we assume one of these two results, then using Stirling's formula (2.2.1) for the Gamma function, one can straight away prove the convergence of the series over ρ present in (1.0.1), (2.1.2), (4.1.2). Not only that, these series converge very rapidly. Over the years there has been a lot of research going on the distribution of the moments of the derivative of the Riemann zeta function at the non-trivial zeros. Interested readers can see Fujii (2012), Gonek (1984), Hejhal (1989), Hiary and Odlyzko (2011) and the references therein.

In 2018, Banerjee and Chakraborty (2019) established an asymptotic expansion for the Lambert series $\sum_{n=1}^{\infty} a_f(n)\overline{a_g(n)}e^{-ny}$, where $a_f(n)$ and $a_g(n)$ are nth Fourier coefficients of Hecke-Maass cusp forms f and g respectively. Recently, the same Lambert series corresponding to the Fourier coefficents of Hilbert modular forms has been studied by Agnihotri (2021). It would be an interesting problem to study a more general Lambert series $y^k \sum_{n=1}^{\infty} |a_f(n)|^N e^{-ny}$ for $N \geq 3$. It would also be a challenging problem to classify automorphic forms for which constant terms will have an asymptotic expansion in terms of the non-trivial zeros of $\zeta(s)$ or the Dirichlet L-function.

In the fifth and the penultimate chapter, inspired by the works of Kohnen (1991), Herrero (2015) and Jha and Sahu (2016), we have defined a family of linear operators between spaces of Hermitian Jacobi cusp forms using Rankin–Cohen brackets for a fixed Hermitian Jacobi cusp form. We have computed the adjoint maps of such a family with respect to the Petersson scalar product. The Fourier coefficients of the Hermitian Jacobi cusp forms constructed using this method involve special values of certain Dirichlet series associated to Hermitian Jacobi cusp forms. Krieg (1985) has developed the theory of Modular forms on half-spaces of quaternions. It would be interesting to define and study Rankin–Cohen brackets on this space.

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LIST OF SYMBOLS

$\mathbb{N}$	:	Set of natural numbers
$\mathbb{Z}$	:	Set of integers
$\mathbb{Q}$	:	Set of rational numbers
$\mathbb{R}$	:	Set of real numbers
$\mathbb{C}$	:	Set of complex numbers
$\mathbb{H}$	:	Complex upper half plane
$\mathbb{H}^*$	:	Extended complex half plane
$\mathbb{Z}[i]$	:	$\{a+ib \mid a,b \in \mathbb{Z}\}$
$\mathbb{Q}[i]$	:	$\{a+ib \mid a,b \in \mathbb{Q}\}$
$\Re(z)$	:	Real part of a complex number z
$\Im(z)$	:	Imaginary part of a complex number z
$M_n(R)$	:	Set of all $n \times n$ matrices with elements in a ring R
$SL_n(R)$	:	Set of all $n \times n$ matrices with elements in a ring R with determinant 1
$\mu(n)$	:	Möbius function
$\zeta(s)$	:	Riemann zeta function
Δ	:	Ramanujan cusp form
$\tau(n)$	:	Ramanujan tau function

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1. Abhishek Juyal, Bibekananda Maji, and Sumukha Sathyanarayana, *An exact formula for a Lambert series associated to a cusp form and the Möbius function*, Ramanujan J., 57, 769–784, 2022. doi: 10.1007/s11139-020-00375-7.

2. Bibekananda Maji, Sumukha Sathyanarayana, and B. R. Shankar, *An asymptotic expansion for a twisted Lambert series associated to a cusp form and the Möbius function: level aspect*, Results Math., 77, 123, 2022. doi:10.1007/s00025-022-01655-y.

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