MULTI-AGENT SYSTEM INSPIRED DISTRIBUTED CONTROL OF A MANIPULATOR

A Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

Soumya S.

(123029ME12F06)

MECHANICAL ENGINEERING DEPARTMENT

NATIONAL INSTITUTE OF TECHNOLOGY KARNATAKA

SURATHKAL, MANGALORE -575025

July, 2019

MULTI-AGENT SYSTEM INSPIRED DISTRIBUTED CONTROL OF A MANIPULATOR

A Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

Soumya S.

(123029ME12F06)

Under the guidance of

Dr. K.R.Guruprasad
Assistant Professor

MECHANICAL ENGINEERING DEPARTMENT

NATIONAL INSTITUTE OF TECHNOLOGY KARNATAKA

SURATHKAL, MANGALORE -575025

July, 2019

DECLARATION

I hereby declare that the Research Thesis entitled **'Multi-agent system inspired distributed control of a manipulator'** which is being submitted to the **National Institute of Technology Karnataka, Surathkal**, in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** is a **bonafide report of work carried out by me**. The material contained in this Thesis has not been submitted to any University or Institution for the award of any degree.

Soumya S.
Registration No.: ME12F06
Dept of Mechanical Engg

Place: NITK, Surathkal
Date:

CERTIFICATE

This is to certify that the Research Thesis entitled **'Multi-agent system inspired distributed control of a manipulator'** submitted by **SOUMYA S. (Register number:123029ME12F06)** as the record of the research work carried out by her, is *accepted* as the *Research Thesis submission* in partial fulfilment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. K.R. Guruprasad
Reasearch Guide
Assistant Professor
Dept of Mechanical Engg.
NITK Surathkal 575025.

Chairman - DRPC
(Signature with Date and Seal)

Place: NITK, Surathkal
Date:

Acknowledgements

Five and a half years of my Ph.D. journey was filled with lots of challenges and learning moments. The experience what I am carrying out is incredible and life-changing. It has not only given me a new dimension in life but also has changed the way I approach a problem. This journey could not have been completed without support and encouragement of few people. I would like to acknowledge individually for all what they have done to me.

I express my deep sense of gratitude and indebtedness to my research supervisor **Dr. K.R Guruprasad** for his limitless support and tireless contributions throughout this research.

I want to thank **Dr. Naveen Karanth P** (Assistant. Prof., Dept. of Mechanical engineering), and **Dr. Debashisha Jena** (Asst. Prof., Dept. EEE) for being my Research Progress Assessment Committee (RPAC) members. Very few research scholars are fortunate to get RPAC members work in the similar research area, and I was one of those. They never gave any excuses even in their busy schedule to share knowledge and attend my seminars. Their comment, observation, understanding, and discussion, encouraged me to think beyond boundaries. Thank you **Dr. Naveen Karanth** and **Dr. Debashisha Jena**, for all the support and guidance.

With deep sense of regard I wish to thank **Prof. Shrikantha S Rao,** Head of the department of Mechanical engineering for the very best support provided for the successful completion of this research work.

I humbly extend my acknowledgement to **Prof. Sumam David** (Prof., Dept. of E& C). She is an inspiration for everyone. She carries a personality that provokes us and make us to think beyond our reach. Her door was always open with a smiling face for me and ready for any discussion. Thank you madam, for being a part of my life and giving me a space in your heart.

With deep sense of regards i wish to thank **Prof. B R Jayalakshmi** (Prof., Dept. of Civil Engineering), **Prof. Prasad Krishna** (Prof., Dept. of Mechanical Engineering), **Prof. Kulkarni S M** (Prof., Dept. of Mechanical Engineering),

Prof. Vijay H Desai (Prof., Dept. of Mechanical Engineering) and **Prof. Narendranath S** (Prof., Dept. of Mechanical Engineering),for their constant encouragement and support given throughout my stay in the campus.

I would like to thank all my colleague teaching and nonteaching staff of mechanical engineering department for their constant support and cooperation.

Words fail to express my gratitude towards my friends Geedhika K, Chythanya Krishnan, Himika boliya, Anjana.P, Minu.S, Anaswara, Nanda Chowdary, Swapna Mansani, Mokkanepalli Vijay, Shruti Goyal, Rejani R, Rajalekshmi, Deepthi Antony and Jisha Prakash for their unconditional affection sincerity and encouragement.

I would like to express my gratitude to my mother **K. Suseela** and father **M Subramanya Pillai** for the support they has given this long. I am also grateful to my sister and cousins for their wishes and encouragement. I express my deep gratitude to all my teachers those who taught me so long.

Above all I would like to thank almighty God for the completion of this thesis.

Soumya S

Place:

Date:

ABSTRACT

Robotic manipulators are used in a wide variety of applications. In all the applications, the end-effector or the tool of the manipulator needs to be moved along a desired trajectory in its workspace. In this thesis we present model-based control schemes for robotic manipulators using a distributed architecture.

Inspired by multi-agent/robotic systems, first we perceive a manipulator, which is MIMO multi-body system, as a multi-agent system with the joints (or the joint-link pairs) as sub-systems or agents, which interact with each other in a distributed manner. Here, the interaction between the joint-link agents is in the form of interactive forces and moments that lead to dynamic coupling. As the adjacency graph formed by the joint-link agents as nodes and links between two joints as edges is connected, the direct interactions between the immediate neighbors result in interaction (in the form of dynamic coupling) between any two joint-link agents.

We carry out an analysis of the computational cost associated with the model-based control law for planar serial-link manipulators with degrees-of-freedom varying from 2 to 6 using Maple. Using this analysis, we establish the fact that the total computational cost associated with the model-based control law increases with the degrees-of-freedom. Toward mitigating the computational overhead associated with the conventional model-based control scheme, we propose a distributed architecture for the motion control of manipulator exploiting its multi-agent nature. Here, each joint-link agent is controlled by a dedicated controller, and the joint-level controllers communicate and cooperate among themselves. Though one of the primary motivation for the proposed distributed control scheme is to reduce the computational overhead, in this thesis we rely on the natural distributed nature of the manipulator dynamics rather than the program optimization or operation optimization techniques that are used at the algorithmic level.

We propose a simple distributed control scheme based on the conventional model-based control law and show that it can be implemented using the

distributed control architecture. Here, apart from the reduced computational lead time due to distributed computation of the control law at the joint-levels, unlike the decentralized or independent joint control schemes, the proposed control scheme fully utilizes the knowledge of the system dynamics, leading to a feedback linearized closed-loop error dynamics. Though the proposed distributed control scheme is suitable for a general serial-link manipulator, in this thesis, we focus on planar manipulators with revolute joints. We prove, that the proposed distributed control scheme makes the links of the manipulator, and hence the end-effector, follow the desired trajectory, asymptotically. We define a quantity called *distribution effectiveness* to quantify how the distributed control schemes share the computational load among the individual joint-level controllers. We also provide a discussion on implication of the discrete-time implementation of the proposed distributed control scheme in contrast to the conventional model-based control scheme. We design a distributed model-based controller for a planar 3R manipulator, to illustrate the proposed distributed control scheme and the distributed control architecture for a manipulator. For the case of planar manipulators with degrees-of-freedom $2-6$, we provide a method to reduce the computational cost associated with dynamic equations used in the control law by identifying repetitive terms, which may be generalized for any other manipulator in principle.

In an attempt to further improve the distribution effectiveness and reduce the computational lead time, we propose a cooperative control scheme for a manipulator using the distributed control architecture. While in the basic distributed control scheme proposed, joint-level controllers interact amongst themselves in terms of exchanging desired and measured states (and their derivatives), in the case of the cooperative control scheme the joint-level controller cooperate by exchanging certain computed terms between them. Even in this case, we provide a discussion on implication of the discrete-time implementation. We prove, that the proposed cooperative control law makes the links of the manipulator, and hence the end-effector, follow the desired

trajectory, asymptotically. We design a cooperative distributed model-based controller for a planar 3R manipulator, to illustrate the proposed cooperative manipulator control scheme implemented in the distributed control architecture. We also provide a discussion on computational effectiveness of the proposed cooperative distributed control scheme along with a method to further reduce the computational lead time by identifying repetitive terms in the control law.

We present a detailed analysis of computational cost associated with the dynamic equation of planar manipulators with degrees-of-freedom from 2 to 6, where we analyze the cost involved in the proposed distributed control schemes in contrast to that in the conventional centralized model-based control scheme, using Maple. We provide results which indicate that the distribution effectiveness of the proposed simple distributed control schemes improves with degrees-of-freedom of the manipulator. We also provide a detailed discussion on reducing the computational cost by identifying repetitive terms in the dynamic equations at each joint-level, for planar manipulators with degrees-of-freedom from 3 to 6.

We then present simulation results demonstrating the proposed control schemes. We present results which show how the trajectory tracking performance of the model-based control law degrades with increase in the sampling time. Then we present results which demonstrate that with the proposed distributed control schemes every joint tracks the desired trajectory satisfactorily, in comparison with the independent-joint PID control scheme. We present details of implementation of the proposed distributed manipulator control scheme using Simulink-ROS hybrid platform based on Matlab's Robotics toolbox, which provides a more realistic simulation result and it is also amenable for hardware implementation. Finally, we present a discussion to compare decentralized control schemes presented in the literature with the distributed control schemes presented in this thesis.

Keywords: Manipulator control, distributed control, nonlinear systems, feedback linearization

CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

τ	:	Joint Torque vector
θ	:	Joint Position
$\dot{\theta}$	:	Joint Velocity
$\ddot{\theta}$	:	Joint Acceleration
$M(\theta)$	:	Inertia matrix
$V(\theta, \dot{\theta})$	:	Coriolis and centripetal force
$G(\theta)$	:	Gravitational force
m	:	mass of a link
l	:	link length
E	:	Tracking Error signal
K_p	:	Position gain matrix
K_v	:	Velocity gain matrix
K_i	:	Integral gain matrix
N_T	:	Number of trigonometric operations
N_A	:	Number of additions
N_M	:	Number of multiplications
f_i	:	Force exerted on the ith link by link $i-1$
F_i	:	Net force acting on the link i
n_i	:	Moment exerted on the ith link by link $i-1$
N_i	:	Net moment acting on the link i
N_{Arith}	:	Number of arithmetic operations
C_T	:	Computational cost
C_{Tc}	:	Total computational cost
η_d	:	Distribution effectiveness

T_d : Time delay

T_d^d : Time delay in discrete time

C_A : Cost of addition

C_M : Cost of multiplication

C_{TC} : Cost of computation of a centralised controller

C_{Td} : Cost of computation of a distributed controller

CHAPTER 1

Introduction

Robotic manipulators are used in a wide variety of applications. In all the applications, the end-effector or the tool of the manipulator needs to be moved along a desired trajectory in its workspace. Thus, the motion control of the individual links, and hence of that of the end effector, is one of the fundamental problems addressed in the field of robotics. Designing a stabilizing controller for a highly nonlinear and coupled dynamic system such as a robotic manipulator is a challenging task and this problem is attracting researchers to this date, in spite of theoretical and technological advancement in the field of the control system design. Further, the complexity of the manipulator dynamics increases with the degrees-of-freedom or the number of links and joints, increasing the computational complexity of the control law in general. In this Chapter, we preview concepts from manipulator dynamics and control (Craig 2005, Spong & Vidyasagar 2008, Asada & Slotline 1986, Ghosal 2013, Saha 2008), highlighting the issues involved in the manipulator control relevant to the thesis. We introduce the concepts of multirobotic/agent systems, centralized, decentralized, and distributed control architectures.

1.1 Manipulator dynamics

A robotic manipulator is made up of a sequence of links connected by joints. Links are rigid members and the joints allow relative motion between the links. Joints can be prismatic, cylindrical, spherical, screw, and revolute. Most widely used joints in manipulators are prismatic, which allow a linear motion between the connected links, and revolute, which allows a rotary motion between them. Prismatic and revolute joints have one degree of freedom. Owing to ease of actuation by an electric motor, revolute joints are typically most preferred joints.

Based on the arrangements of the links, a manipulator may be a serial-link configuration, a parallel-link configuration or a hybrid of these two. In this thesis, we focus on serial-link manipulators which are most widely used in industries. The degrees-of-freedom of a serial link manipulator is equal to the number of joints, assuming that the joints have single degree of freedom (such as prismatic or revolute). The first link of a manipulator is typically fixed to a work table, and the last link carries a tool or end effector which performs the actual task. End effector could be a gripper, for pick and place tasks, a welding torch to carry out welding operation, or any other tool depending on the task the robot needs to perform.

For the purpose of manipulating the work (sometimes a tool), the end effector needs to be moved in a desired manner. Movement of end effector is achieved by actuation provided at the joints, which move individual links. Motion of links leads to that of the end effector (the last link). In order to move the end effector in a specified manner or along a desired trajectory, control input should be provided at joints as torques (or force in the case of a linear motion). Dynamics of the manipulator which relates torque/force provided at the joints to the motion of links is very important to design a motion controller for achieving desired tool motion. Tool motion (trajectory) is related to motion at the joint level is related by the (inverse and forward) kinematics of the manipulator.

The dynamic equation of serial-link manipulator with N degrees-of-freedom has the following standard form,

$$\tau = M(\dot{\theta})\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta) \tag{1.1}$$

where, $M(\dot{\theta})$ is $n \times n$ mass matrix,$V(\theta, \dot{\theta})$ is $n \times 1$ velocity vector, $G(\theta)$ is $n \times 1$ gravitational vector, and τ is $n \times 1$ joint torque (actuation) vector. A manipulator is a lumped parameter (rigid body assumption), time-invariant (the parameters such as mass, inertia, damping (if considered), etc, are constants), autonomous, and nonlinear system.

For example, dynamics of a two-link planar manipulator with revolute joints (2R) shown in Figure 1.1, where for simplicity, we assume that mass of each link is concentrated at the end of links, may be obtained using Newton-Euler or

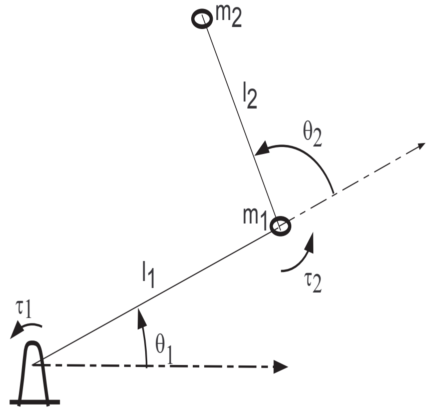


Figure 1.1: A Two-link robot with revolute joints and point masses

Lagrangian formulation methods as:

$$\begin{aligned}
\tau_1 &= m_1 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 g c_1 \\
&\quad + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1 g s_2 s_{12} + m_2 l_1 l_2 c_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\
&\quad + m_2 l_1 g c_2 c_{12} \\
\tau_2 &= m_1 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2)
\end{aligned} \tag{1.2}$$

Which can be re-written as:

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{bmatrix} m_2 l_2^2 + m_2 2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}$$

$$+ \begin{pmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{pmatrix} + \begin{pmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{pmatrix} \tag{1.3}$$

Thus,

$$M(\theta) = \begin{bmatrix} m_2 l_2^2 + m_2 2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix}$$

$$V(\theta, \dot{\theta}) = \begin{pmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{pmatrix}$$

and

$$G(\theta) = \begin{pmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{pmatrix}$$

Here, m_i and l_i are mass and lengths of the ith link, $c_i = \cos(\theta_i), s_i = \sin(\theta_i), c_{12} = \cos(\theta_1 + \theta_2)$, and $s_{12} = \sin(\theta_1 + \theta_2)$. Also note that we have not considered effect of friction here. There are three important observations to be made here:

1. The manipulator dynamics is nonlinear in nature.
2. The manipulator dynamics is coupled in the sense the motion of and force/momnet on one link will affect motion of the other link.
3. Even for a simple two-link manipulator, we may observe that the equations modeling the dynamics are complex and computationally intense.

1.1.1 Implication of nonlinearity

Nonlinearity of the dynamics affects the control system design for a manipulator. As we know for a linear system, the response depends on the location of poles. Thus, the control system design involves locating the closed-loop poles at the locations as dictated by the performance specifications. For example, desired performance may be indicated using the time domain specifications such as rise time, settling time and maximum peak overshoot based on step response of the system (Franklin, Powell & Emami-Naeini 2002). A step response of a second order linear system along with the standard time domain specifications are shown in Figure 1.2. Each of these time domain specifications provide a region in the S-plane within which the poles have to be located.

As the principle of superposition does not hold for a nonlinear system, standard test signals such as impulse, step, and sine and the system's response for these signals do not make much sense in the context of nonlinear systems. For example, the response to $sin(\omega t)$ is not twice of that for $2\sin(\omega t)$ for a nonlinear

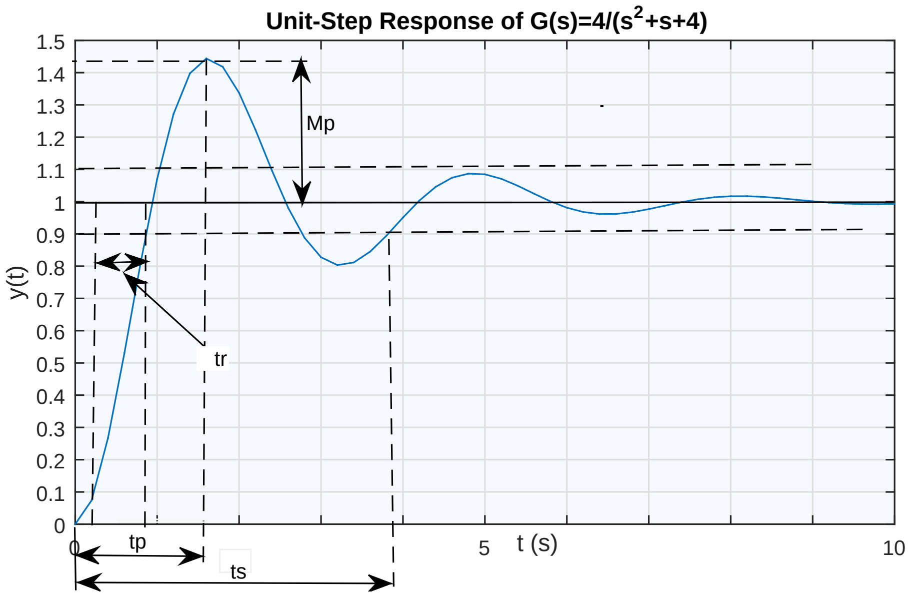


Figure 1.2: Step response of a typical under-damped second order system and the time domain specifications.

system. The standard practice for designing a controller for a nonlinear system is to linearize it about an operating point (state be precise) and then deciding on controller gains so as to locate the poles as such as those dictated by the time domain specifications. For example controlling an inverted pendulum is a classical problem which finds many practical applications such as segways, rocket launching, etc. Here, the nonlinear dynamics of the pendulum is linearized about the unstable equilibrium state $(\theta, \dot{\theta}) = (\pi, 0)$ (open-loop pole is on the RHP of the S -plane) and a controller (such as a PD control law) is designed such that the closed-loop is in the LHP of the S-plane. Further the controller gains (such as k_p, the proportional gain and k_v, the derivative gain) may be selected based on the time domain specification such as, say settling time and response (rise) time and allowable maximum overshoot. Note that a pendulum from the perspective of the mechanism and dynamics, may be seen as one degree-of-freedom manipulator with revolute joint. The limitations of linearization are primarily two. First, the closed-loop linearized equation is valid only for small errors. Thus, the control system may not restore the stability if the pendulum has deviated too much from the desired vertical position. For example, if a rocket deviates too much from the vertically upward position at the time of launch, and controller fails to restore it, eventually it will fall (as the other equilibrium state is stable in this case). Second, if the operating point is not fixed (such as in trajectory following problem), then one cannot resort to the linearization technique as the linearized equation is valid only in the vicinity of the operating point used for linearization. Thus, in the case of a manipulator which has to move the end effector anywhere in the workspace, there is no single operating point about which the dynamics can be linearized, as the control problem is a servo (trajectory tracking) problem rather than a regulator problem in the context of manipulator.

If the manipulator is actuated by stepper motors and the speed at which it needs to operate is on the slower side, the dynamic (inertial) coupling effects may not be prominent and the manipulator may be controlled in open-loop. Another practical scenario is when servo motors with high gear reduction are used to actuate the manipulator, again, the dynamics of the manipulator may not appear prominently at the actuator level, and hence a simple independent-joint linear controller such as PID controller may be used. However, linear controllers cannot be used in general situations. Two approaches used in such a scenario are: gain scheduling, robust control techniques, etc.

1.1.2 Implication of coupled dynamics

As we have seen using a simple 2R planar manipulator, the dynamics are coupled in the sense that motion of one link affects the other link and also actuation at one joint affects the other link. Thus, a control system designed has to account for the coupling affect. Any independent joint control scheme, however advanced it may, cannot truly account for the coupled dynamics. An independent joint control scheme using adaptive techniques may account for only the non-coupled component of the manipulator dynamics.

1.1.3 Implication of higher computational cost

Being a nonlinear coupled dynamic system, a nonlinear model-based control scheme, which uses the dynamic model in the control law, is ideally suited for the manipulator motion control problem. Now as the manipulator dynamics is computationally intense, it affects the sampling rate at which the control law can be implemented depending on the time required to complete the computation of the manipulator dynamics as part of the control law. We shall discuss this issueS in detail later as this is one of the main motivation for this thesis.

1.2 Manipulator Control

Designing a stabilizing controller for a highly nonlinear and coupled dynamic system such as a robotic manipulator is a challenging task. As we have discussed in the previous section, simple control schemes such as open-loop control scheme (in the case of stepper motors and slower motion requirement), simple independent-joint PID control (when the motors have high gear ratio), or more complex techniques such as robust control techniques, gain scheduling, etc. have been used for manipulator control.

Though a manipulator has coupled nonlinear dynamics, in many practical applications, each joint is independently controlled using PID controllers. Such a strategy is called independent joint PID control. The independent joint PID control law trajectory following is:

$$\tau(t) = \ddot{\theta}_d + K_v \dot{E} + K_p E + K_i \int E dt \tag{1.4}$$

Here K_v, K_p, and K_i are controller gain matrices and they are diagonal matrices with positive entries. For example $K_p = diag(k_{p1}, k_{p2}, \ldots, k_{pN})$, with k_{pi} being the proportional gain corresponding to controller of the ith joint and N is the degrees-of-freedom of the manipulator. Here, the actual dynamics of the manipulator is not considered in the controller. Hence, nonlinearity and coupling effects are neglected. The controller performs reasonably well when the speeds and acceleration are small, as the inertial effects and hence the effect of nonlinearity and coupling small, and as the feedback control can handle reasonable model uncertainty and disturbances. However, as we discussed in the previous section, the control performance will not be uniform across the state space.

Now we shall preview a model-based nonlinear control scheme for a manipulator. Figure 1.3 shows a block diagram of the model-based manipulator control scheme.

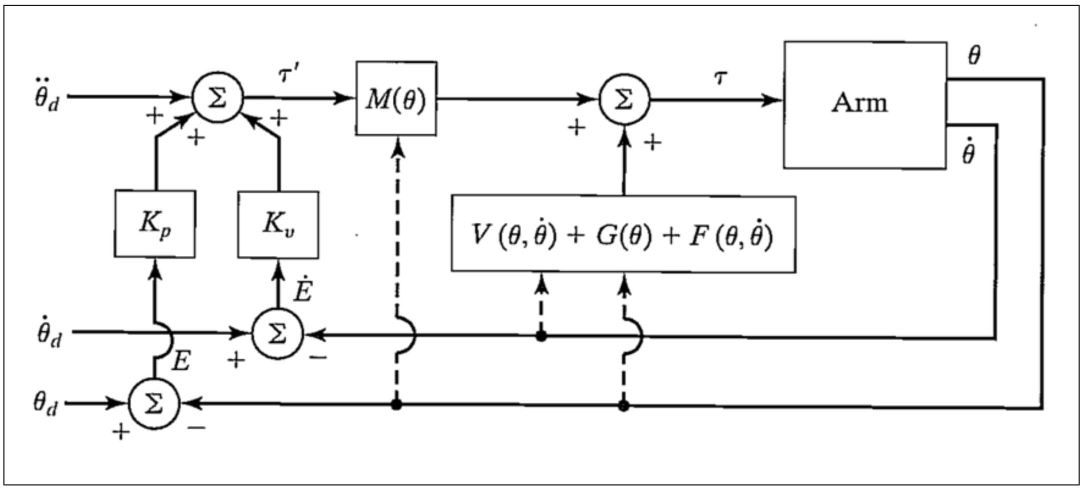


Figure 1.3: Block diagram of a model-based control scheme.

Consider the control law:

$$\tau = M(\theta)(\ddot{\theta}_d + K_v\ddot{E} + K_pE) + C(\theta, \dot{\theta}) + G(\theta) \tag{1.5}$$

Now the closed-loop dynamics of the manipulator whose dynamics is given by Eqn. (1.1) with the model-based control law given in Eqn. (1.5) may be obtained by substituting for τ from (1.5) in (1.1):s

$$\ddot{E} + K_v\dot{E} + K_pE = 0 \tag{1.6}$$

Note that the error dynamics given by Eqn. (1.6) is independent of the manipulator dynamics and more importantly de-coupled and linear. This process of lineralization is called *feedback linearization.* Unlike the standard linearization process, there is no approximation here. It can be shown that (as the corresponding poles are on LHS):

$$E \to 0, \text{ as } t \to \infty$$

for positive controller gains. The controller gains can be chosen to ensure the closed loop system performs as desired, by meeting the time domain specifications such as rise time, overshoot, and settling time. Note that the performance is uniform over the entire state-space, unlike with the independent-joint linear (PID) controller. Though we are controlling a nonlinear system (a manipulator) here, using a nonlinear controller, the tools used for a second order linear time invariant (LTI) SISO system are sufficient to design (that is, chose the controller gains) and analyze the system for stability and its performance.

1.3 Multi-agent/robot systems

Nature uses either an individual with high level of skills/capabilities or use a group of cooperating creatures with rudimentary capabilities to solve a complex problem or accomplish a complex task. In the second category, the nature solves problems by using simple behavior patterns. Several living beings such as ants, birds, fish etc., which have limited capabilities. Their simple local behavior leads to a useful collective behavior such as swarms, schools, flocks, etc. Developments in areas such as wireless communication, autonomous robots, computation, and sensors, facilitate the use of large number of agents (UAVs, mobile robots, or autonomous vehicles), which are equipped with necessary sensors, communication capabilities, and computation ability, to cooperatively achieve various tasks in a distributed manner.

Such distributed multi-agent systems (MAS) or multi-robotic systems (MRS) have been shown to achieve and maintain formations, move as flocks while avoiding obstacles. These multi-robotic systems are increasingly being used to solve many complex problems, such as autonomous lawn mowing(Cohen, Sirotin & Rave 2008), vacuum cleaning(Doty & Harrison 1993), search and rescue (Guruprasad & Ghose 2011), landmine detection (Guruprasad, Wilson & Dasgupta 2012),(Dasgupta, Baca, Guruprasad, Munoz-Melendez & Jumadinova 2015), surveillance, signal source identification, etc. The major advantages of distributed systems are adaptability to failure of individual agents, their versatility in accomplishing multiple tasks, simplicity of agents' hardware, and requirement of only minimal local information.

1.4 Centralized, decentralized and distributed control

In the context of multi-agent or networked systems, we may have primarily two possible control architectures. First a *centralized* control architecture and second, a *decentralized* control architecture. In the case of centralized architecture, a single central controller controls all the individual plants. A central controller may not be suitable in many applications due to several disadvantages, main among them being the failure of the central controller makes the entire system fail. This architecture also requires complete communication between the central controller and all the individual agents/plants, and hence the, communication overhead of the centralized system is high.

In a decentralized control architecture each plant/agent has its own dedicated controller. This architecture may not suffer in a major way when some of the agents fail. Further, one can also have a *distributed system* architecture which is somewhere between the centralized and decentralized architectures, where controllers communicate among themselves and exchange information, while the decision is taken locally. Figure 1.4 illustrates these architectures. In this thesis, though we address the problem of control of a single system (the manipulator), we will use the concepts of decentralized and distributed control architecture.

1.5 Contribution of the thesis

Inspired by multi-agent/robotic systems, first we perceive a manipulator, which is MIMO multi-body system, as a multi-agent system with the joints (or

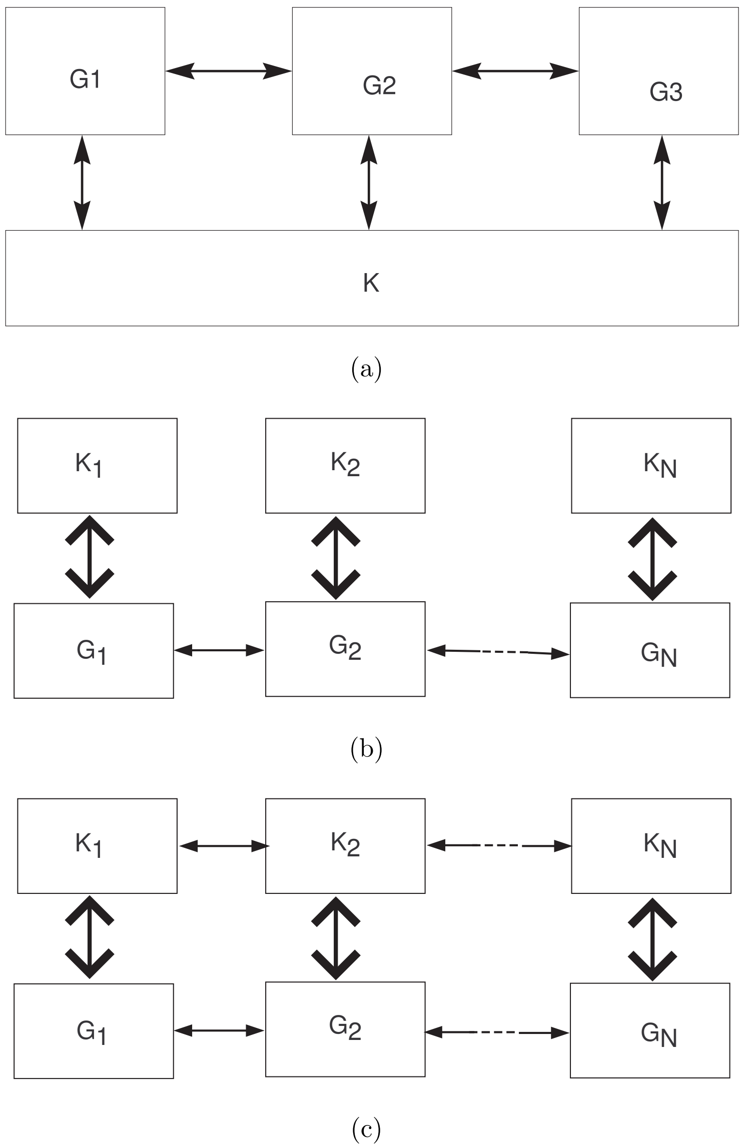


Figure 1.4: (a) Centralized (b) decentralized, and (c) distributed control architecture used in multi-agent or networked systems.

the joint-link pairs) as sub-systems or agents, which interact with each other in a distributed manner. Here, the interaction between the joint-link agents is in the form of interactive forces and moments that lead to dynamic coupling. As the adjacency graph formed by the joint-link agents as nodes and links between two joints as edges is connected, the direct interactions between the immediate neighbors result in interaction (in the form of dynamic coupling) between any two joint-link agents.

We carry out an analysis of the computational cost associated with the model-based control law for planar serial-link manipulators with this analysis, we establish the fact that the total computational cost associated with the model-based control law increases with the degrees-of-freedom. We propose a distributed architecture for the control of manipulator now considered as a multi-agent system of joint-link agents, with the primary motivation of reducing computational cost associated with the centralized control scheme. Here, each joint-link agent is controlled by a dedicated controller, and the joint-level controllers communicate and cooperate among themselves. Though one of the primary motivation for the proposed distributed control scheme is to reduce the computational overhead, in this thesis we rely on the natural distributed nature of the manipulator dynamics rather than the program optimization or operation optimization techniques that are used at the algorithmic level.

We propose a simple distributed control scheme based on the conventional model-based control law and show that it can be implemented using the distributed control architecture. Here, apart from the reduced computational lead time due to distributed computation of the control law at the joint-levels, unlike the decentralized or independent joint control schemes, the proposed control scheme fully utilizes the knowledge of the system dynamics, leading to a feedback linearized closed-loop error dynamics. Though the proposed distributed control scheme is suitable for a general serial-link manipulator, in this thesis, we focus on planar manipulators with revolute joints. We prove, that the proposed distributed control scheme makes the links of the manipulator, and hence the end-effector, follow the desired trajectory, asymptotically. We define a quantity called *distribution effectiveness* to quantify how the distributed control schemes share the computational load among the individual joint-level controllers. We also provide a discussion on implication of the discrete-time implementation of the proposed distributed control scheme in contrast to the conventional model-based control scheme. We design a distributed model-based controller for a planar 3R manipulator, to illustrate the proposed distributed control scheme and the distributed control architecture for a manipulator. For the case of planar manipulators with degrees-of-freedom $2-6$, we provide a method to reduce the computational cost associated with dynamic equations used in the control law by identifying repetitive terms, which may be generalized for any other manipulator in principle.

In an attempt to further improve the distribution effectiveness and reduce the computational lead time, we propose a cooperative control scheme for a manipulator using the distributed control architecture. While in the basic distributed control scheme proposed, joint-level controllers interact amongst themselves in terms of exchanging desired and measured states (and their derivatives), in the case of the cooperative control scheme the joint-level controller cooperate by exchanging certain computed terms between them. Even in this case, we provide a discussion on implication of the discrete-time implementation. We prove, that the proposed cooperative control law makes the links of the manipulator, and hence the end-effector, follow the desired trajectory, asymptotically. We design a cooperative distributed model-based controller for a planar 3R manipulator, to illustrate the proposed cooperative manipulator control scheme implemented in the distributed control architecture. We also provide a discussion on computational effectiveness of the proposed cooperative distributed control scheme along with a method to further reduce the computational lead time by identifying repetitive terms in the control law.

We present a detailed analysis of computational cost associated with the dynamic equation of planar manipulators with degrees-of-freedom from 2 to 6, where we analyze the cost involved in the proposed distributed control schemes in contrast to that in the conventional centralized model-based control scheme, using Maple. We provide results which indicate that the distribution effectiveness of the proposed simple distributed control scheme improves with degrees-of-freedom of the manipulator. We also provide a detailed discussion on reducing the computational cost by identifying repetitive terms in the dynamic equations at each joint-level, for planar manipulators with degrees-of-freedom from 3 to 6.

We then present simulation results demonstrating the proposed control schemes. We present results which show how the trajectory tracking performance of the model-based control law degrades with increase in the sampling time. Then we present results which demonstrate that with the proposed distributed control schemes every joint tracks the desired trajectory satisfactorily, in comparison with the independent-joint PID control scheme. We present details of implementation of the proposed distributed manipulator control scheme using Simulink-ROS hybrid platform based on Matlab's Robotics toolbox, which provides a more realistic simulation result and it is also amenable for hardware implementation. Finally, we present a discussion to compare decentralized control schemes presented in the literature with the distributed control schemes presented in this thesis.

1.6 Organization of the thesis

The rest of the Thesis is organized as follows. We discuss the relevant literature in chapter 2. Chapter 3 provides the research gap, motivation for the thesis and the objectives of this. A control scheme for a manipulator implemented in distributed architecture is presented in the Chapter 4. Chapter 5 presents a cooperative, distributed manipulator control scheme. Results and discussions are provided in Chapter 6, and the thesis is concluded in Chapter 6 with a summary and a discussion on scope for further work.

CHAPTER 2

Literature survey

In this chapter, we provide a survey of representative literature on manipulator control. A large number of work is available on the manipulator control as it is practically relevant and theoretically challenging problem.

As robotic manipulator is made up of several links coupled with joints which allow relative constrained motion between the connected links. Each joint-link pair may be considered as a subsystem. Thus, a manipulator can be considered as a networked system. In this perspective, the control scheme may have a centralized archtecture, where the manipulator is considered a single 'multi-body/MIMO system', or decentralized architecture, where manipulator is considered a network of SISO systems.

2.1 Centralized control Schemes

First we preview a few representative control schemes that can be grouped under the centralized architecture. Traditionally these control schemes are not called centralized schemes as, a manipulator here are not considered as networked or distributed system, rather as a single nonlinear coupled MIMO system. However, in contrast to the decentralized manipulator control schemes proposed in the literature, we call these control scheme as centralized control scheme, or more precisely, control schemes implemented using the centralized architecture. Figure 2.1 shows a centralized manipulator control architecture, where all the subsystems (that is, the joint-link pairs) are controlled by a single central controller.

The traditional nonlinear model based controller (Craig 2005, Asada & Slotline 1986, Spong & Vidyasagar 2008), discussed in the previous chapter, utilizes the dynamic model of the manipulator into the control law and achieves linear error dynamics through feedback. As discussed in the previous chapter, such a model-based control scheme is an ideal control scheme because of its

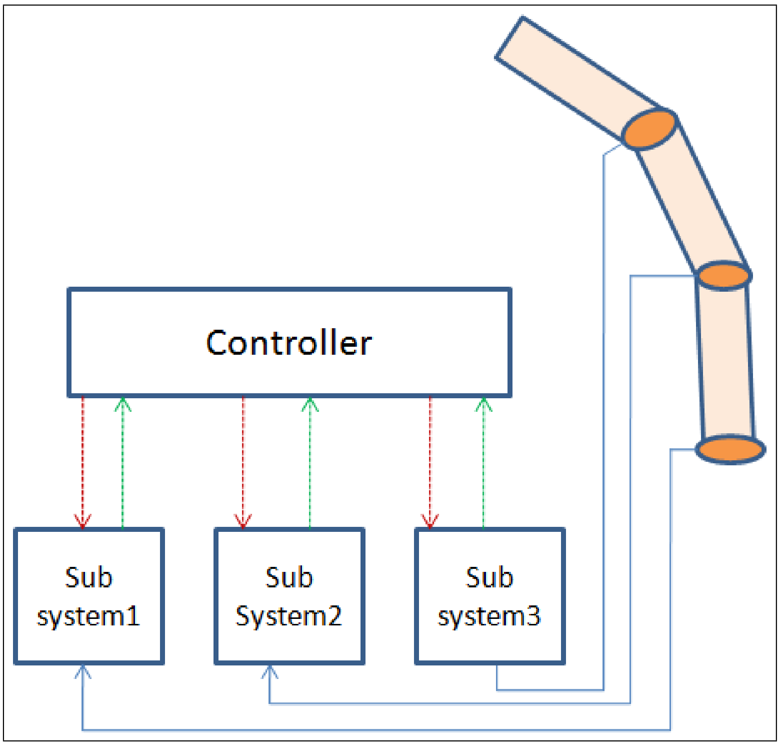


Figure 2.1: A centralized manipulator control architecture. Each of the joint-link pairs considered as a subsystem.

guaranteed performance, in terms of satisfying the time domain specifications uniformly across the state-space, or for any desired trajectory on the entire workspace. Though typically the model-based control law uses the joint-level trajectory, the control law may be redesigned to use the trajectory in task space directly. These schemes are called model-based control in Cartesian space (Ghosal 2013). The authors in (Yuan 1988) present a closed-loop manipulator control using quaternion. Feedback linearization is achieved using quaternion or Euler parameters, which are used to represent the orientation. Here, the desired trajectories are provided in the task space. The control is achieved directly in the task space by using the inverse of the Jacobean matrix. In (Yun 1988), the author considers a problem of simultaneous motion and force control robotic manipulator. Here, the authors achieve exact linearization and decoupling of the motion and force control loops using a dynamic nonlinear state feedback and a nonlinear state transformation.

In these model-based control schemes which use feedback linearization technique, the control gains can be tuned to achieve desired performance level and the performance is uniform in the entire state-space. However, main disadvantage is that the system dynamic equation need to be computed online. As we have discussed in the previous chapter, the robot dynamics in general is computationally expensive. One solution proposed to reduce the computational load is the computed torque control (Paul 1972, Markiewicz 1973). A model-based computed torque control law is:

$$\tau = M(\theta_d)(\ddot{\theta}_d + K_v\ddot{E} + K_pE) + V(\theta_d, \dot{\theta}_d) + G(\theta) \tag{2.1}$$

Here the dynamics is pre-computed based on the planned trajectory (that is, $\theta_d, \dot{\theta}_d$, and $\ddot{\theta}_d$, in place of the current states/derivatives $\theta, \dot{\theta}$, and $\ddot{\theta}$) and a feed-forward loop is added. The performance approaches that of the model-based controller when the robot is following trajectory accurately. However, when the tracking error is high (initially), nonlinear terms do not cancel and hence the performance may be poor. Further, this scheme requires large memory to store the pre-computed dynamics along the entire trajectory. Moreover, this scheme can not be used when the trajectory is generated online.

Another limitation of the model-based (and the computed torque scheme) is that the dynamic model of the robot should be known exactly. In the absence of a complete knowledge of the system dynamics, or even when known, to reduce computation, techniques such as robust control (Yim & Park 1999, Essakki, Bhat & Su 2013) adaptive control (Sastry 1984, Craig 1986, Slotine & Weiping 1988), model predictive control (Poigent & Gautier 2000), model identification based control (Paul 1972), etc., are used. Approaches based on Artificial Neural Networks (Li, Wang & Rafique 2018, Jin, Li, Yuc & inbo He n.d.), Fuzzy logic controllers, or a combination known as Adaptive Network based Fuzzy Inference System (ANFIS) (Guruprasad & Ghosal 1999), etc., are also used to account for the model uncertainty. There is a vast literature on such control schemes. However, the focus of this thesis is on control of a manipulator when the dynamic model is known. Hence, we have provided only a few representative works on control schemes that address model uncertainty.

In a recent work, (Andreev & Peregudova 2017) authors propose a control scheme using the dynamic position-feedback controller with feedforward. By using Lyapunov vector function and comparison principle, authors construct a non-linear controller with variable gain matrices and first-order linear dynamic compensator to achieve a closed-loop system that is uniformly asymptotically stable. Authors also show that controller is robust with respect to parameters uncertainties.

2.2 Decentralized Control Schemes

In contrast to the centralized control architecture, in the decentralized control schemes, each subsystem (joint-link pair) is controlled independently by a dedicated controller. This scheme is also known as independent joint control. This is illustrated in Figure 2.2. The actual system (that is, the manipulator)

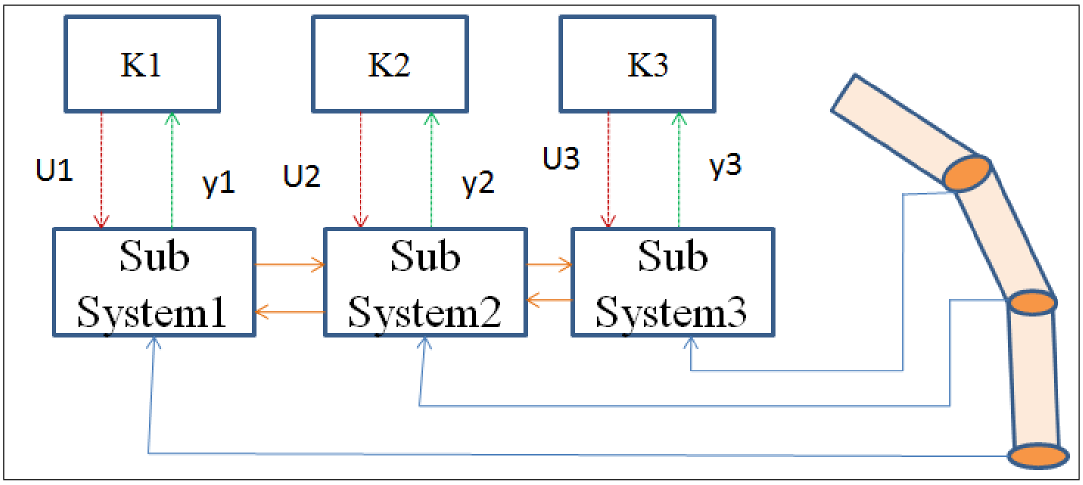


Figure 2.2: Decentralized manipulator control architecture

has physical interactions between the subsystems in terms of dynamic coupling. However, the controllers do not communicate or cooperate among themselves. A simplest decentralized control is the independent joint PD/PID control (Craig 2005), where each joint is controlled by an independent controller. As we discussed earlier, the trajectory tracking performance with such a controller will not be uniform in the entire state-space.

In (Seraji 1989) the author proposed an adaptive independent joint control scheme for a manipulator. Here, each joint is controlled using a PID control law along with a position, velocity, and acceleration feed-forward loop with adaptive gains. The authors in (Hsia & Gao 1990) proposed a decentralized linear control using the control input computed in previous time instance to estimate the coupling terms in the manipulator dynamics. Their control law approaches the model-based control law as the time delay (sampling time) approaches zero. An adaptive version of this control law (Hsia & Gao 1990) is presented in (Cho, Lee, Kim, Kuc, Chang & Jin 2016). An asymptotically stable decentralized adaptive control scheme for trajectory tracking by a manipulator has been presented in (Tarokh 1996). In (Tang & Guerrero 1998) authors provide a simple controller using linear state-feedback, with an additional signal which compensates for the coupling terms, uncertainty in parameters, and bounded disturbances. A nonlinear adaptive decentralized controller is proposed in (Liu 1997), where the author attempts to account for nonlinear coupling by using decentralized cubic feedback. In (Liu 1999) the author uses a robust nonlinear feedback term in addition to a decentralized cubic feedback, to a decentralized PD control law. In (Wang & Wend 1999) authors subdivide the dynamics of the subsystems as nominal system and uncertainties. The Riccati equation approach is used to control manipulator based on the nominal system with bounds on the uncertainties. The authors in (Narendra & Oleng 2002) provide a theoretical analysis of a strictly decentralized adaptive control systems, and show that it is possible to track the desired outputs with zero error. In (Hsu 2006) an adaptive decentralized controller using adaptive variable structure compensations has been proposed. Authors in (Yang, Fukushima & Qin 2012) proposed a decentralized robust control for a manipulator. Here, the low pass coupled uncertainties are considered as disturbances, and a disturbance observer (DOB) is introduced to compensate for the same. In (Leena & ray 2012) the authos proposed a class of stabilizing decentralized PID controllers for a general n-link robot manipulator. Authors obtain the controller gains using Kharitonov theorem (Huang & Wang 2000) and stability boundary equations. They use the concept of gain-phase margin (Franklin et al. 2002) used in control system design. In (Senda, Nishibu & Mano 2003) a decentralized control (though authors refer to it a distributed in the title) for a redundant fault tolerant manipulator using visual serving is proposed.

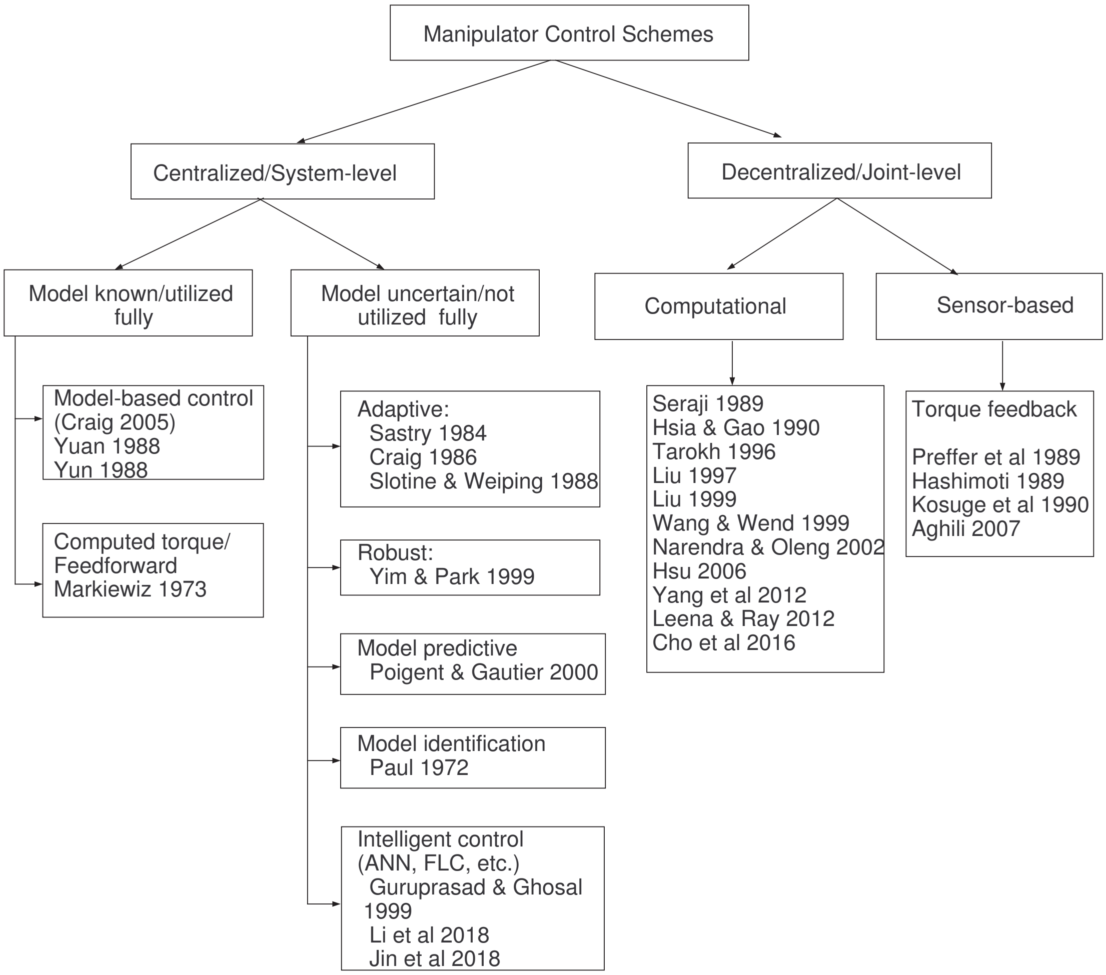


Figure 2.3: Summary of representative manipulator literature.

A class of manipulator control schemes that use torque feedback have been presented in the literature (Preffer, Khatib & Hake 1989, Hashimoti 1989, Kosuge, Takeuchi & Furuta 1990, Aghili 2007). These control scheme rely on joint torque measurement using torques sensors mounted at each joint. In principle, measured torque values replace the computation and hence the control scheme may be implemented in a purely decentralized architecture, with provable performance guarantee. However, as these schemes require torque sensors to measure the motor torques at each joint, these are not suitable for a large number of existing manipulators which lack such a sensing capability.

A few work in the literature such as (Morit, Nayat, Osatot & Kawaokatt 1996, Bohner & Luppen 1997, Jia, Zhuang, Bai, Fan & Huang 2007, Tsuji, Nakayama & Ito 1993), use concepts from MAS for manipulator control. We shall discuss them in more detail in Chapter 4, except to state that these work do not address a dynamic control of manipulators. We have summarized the manipulator control literature previewed here in Figure 2.3.

In this chapter, we previewed relevant literature in manipulator control as centralized and decentralized schemes. We identify the gap in the research that motivates the work carried out in thesis in the next chapter.

CHAPTER 3

Motivation and Objectives

In this chapter we identify the research gap based on the literature survey carried out in the previous chapter, provide motivation for the work and present the objectives of the research.

3.1 Research gap and motivation

First we make a few observations based on the literature review presented in the previous chapter.

1. Though the problem of motion control of a serial-link rigid manipulator is very old (earliest reference provided in the previous chapter dates back to the year 1972), it is relevant even today as the most recent papers we referred are from the year 2018.

2. Various decentralized control scheme have been reported in the literature indicating the importance of joint-level control schemes for a manipulator, particularly to address the computational overhead involved in traditional centralized schemes. Further, recent literature in the decentralized schemes (latest work referred in previous chapter is from the year 2016) also indicate that problem of joint-level control of a manipulator is not saturated and is relevant to date.

3.1.1 Disadvantages of the centralized control schemes

In the literature we grouped the manipulator control schemes into two major architectures, namely, centralized control architecture and decentralized control architecture. The focus of this thesis is on the scenarios where the dynamic model of a manipulator is available. This is true in most practical situations. Motivation for control schemes that consider model uncertainty are two. First, an exact model

Sl No	Operation\DOF	2	3	4	5	6
1	Trigonometric (N_T)	108	373	1015	2478	5451
2	Multiplication (N_M)	239	798	2109	6458	11250
3	Addition (N_A)	122	484	1430	3704	8353

Table 3.1: number of computations serial-link manipulator

may not be available. The model uncertainty is typically at the parametric level, such as frictional coefficients, which are difficult to model. Variation in load may also be the cause of model uncertainty. Second is the high computational cost associated with the model-based control schemes. Here, instead of incorporating the complete model into the control law, a simpler control law may be used and the controller parameters (such as gains) may be tuned on the go (adaptively) so as to achieve a balance between performance and computational cost of implementing the controller.

In this work as we consider a situation where the manipulator model is known completely, a centralized nonlinear controller such as model-based control law is most suitable because of its guaranteed performance across the state space. However, the computational overhead is an important issue to be addressed while implementing it in realtime. This disadvantage of high computational cost involved in the computer implementation of such a centralized control law is apparent from the literature as it has motivated a large quantum of research into manipulator control, mainly into the decentralized control schemes.

Major component of computational cost involved in the model-based control law comes from the manipulator dynamic equations. We made an analysis of number of arithmetic operations involved in computation of manipulator dynamic equations. Table 3.1 shows the number of arithmetic operations (N_T, trigonometric operations; N_A, addition/subtraction; and N_M multiplication) involved in computation of manipulator dynamics for serial-link planar manipulators with revolute joints with degrees-of-freedom from 2 to 6, using Maple. Figure 3.1 shows how the number of addition, multiplication, and

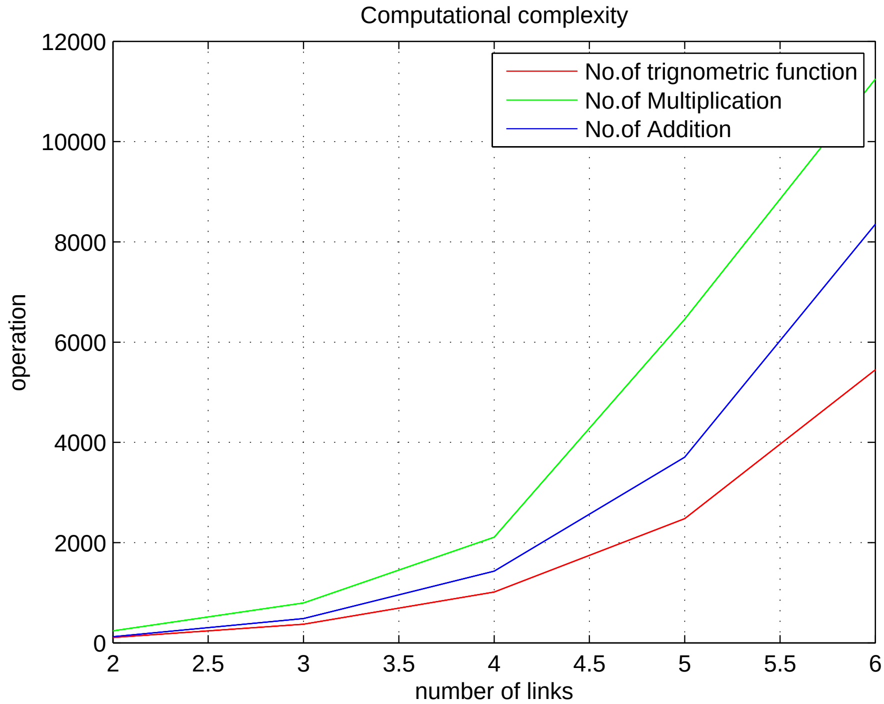


Figure 3.1: Number of arithmetic operations in the dynamic equation of planar serial-link manipulators vs the degrees-of-freedom.

trigonometric operations involved in the computation of manipulator dynamics increase with the degrees-of-freedom for planar serial manipulators.

Exact cost of computation of addition, multiplication, and trigonometric operations depend on the processor and the computational method used. However, for the purpose of obtaining a comparative cost, we considered cost of addition as 1 unit, that of multiplication as 4 units and that of trigonometric operations as 60 units, in the order of their respective computational complexity. Table 3.2 shows the computational cost associated with dynamic equation for different degrees of freedom planar manipulators. Figure 3.2 shows how the cost of addition, multiplication, and trigonometric operations involved in the computation of manipulator dynamics increase with the degrees-of-freedom for planar serial manipulators. The computational analysis carried out here is only representational and for the purpose of establishing the fact that the computational cost associated with the dynamic equations of a manipulator, and hence that of the model-based control law (in centralized architecture) is high

Sl No	cost\DOF	2	3	4	5	6
1	C_T	6480	22380	60900	148680	327060
2	C_M	956	3192	8436	25832	45000
3	C_A	122	484	1430	3704	8353

Table 3.2: computational cost associated with the dynamics equation of of serial-link manipulators.

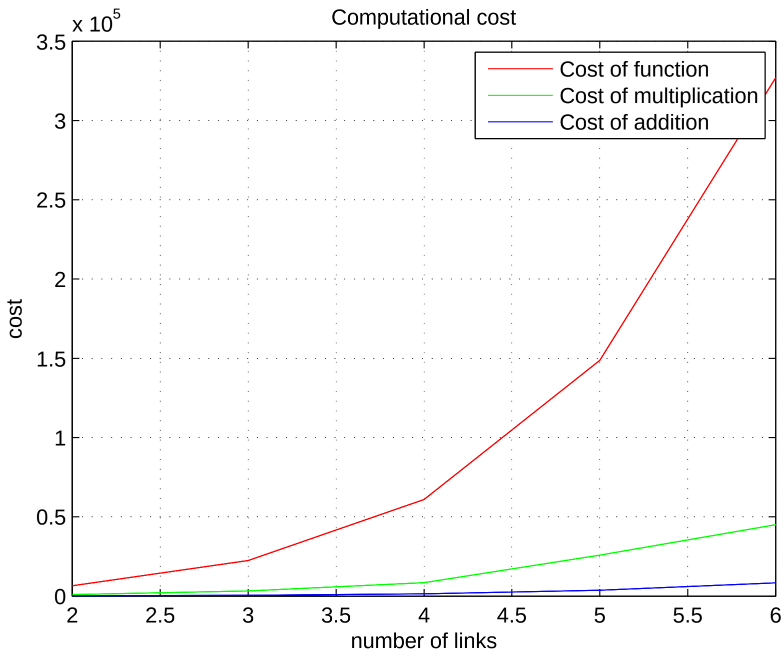


Figure 3.2: computational cost associated with dynamics of serial-link manipulators. vs degrees-of-freedom.

and also increases with the degrees-of-freedom.

Even when the model is not known (or used) completely, techniques such as adaptive control schemes can be used to estimate the model (in terms of the system parameters) or tune the control parameters to provide the desired performance level. For example, in the ANFIS based model-reference learning control scheme proposed in (Guruprasad & Ghosal 1999), the ANFIS corrector uses the function approximation characteristics of the Adaptive networks, and can learn the model by comparing the difference between the desired performance and actual performance and account for the model uncertainty including the coupling effect. In fact the results shown in (Guruprasad & Ghosal 1999) demonstrate that even when the assumed model is linear and decoupled, the ANFIS corrector was able to account for the complete model. This is achieved as the control scheme has provision to account for nonlinearity and dynamic coupling. However, even in such situations, the control schemes are expected to be computationally expensive.

Though the centralized architecture has provision for incorporating provisions to account for nonlinearity and dynamic coupling, many of the control scheme reported in the literature over simplify the control law and do not provide provisions for incorporating the nonlinearity and/or dynamic coupling effects, instead rely on robustness property of the feedback control schemes. Here by robustness we mean the feedback control schemes in general are less sensitive to the model uncertainty, particularly with higher gains.

3.1.2 Concerns with the decentralized control schemes

Though decentralized control laws typically result in lower computational lead time, being one of the primary motivation, they do not consider the dynamic coupling between links, instead, use either robust control techniques or adaptive control techniques. Theoretically, a purely decentralized architecture cannot account for the dynamic coupling effects.

Consider the manipulator dynamics provided in Chapter 1, neglecting gravity.

$$\tau = M(\dot{\theta})\ddot{\theta} + V(\theta, \dot{\theta}) \tag{3.1}$$

We may rewrite Eqn. (3.1) for a two-link manipulator as:

$$\begin{aligned} \tau_1 &= M_{11}(\theta)\ddot{\theta}_1 + M_{12}\ddot{\theta}_2 + V_1(\theta, \dot{\theta}) \\ \tau_2 &= M_{21}(\theta)\ddot{\theta}_1 + M_{22}\ddot{\theta}_2 + V_2(\theta, \dot{\theta}) \end{aligned} \tag{3.2}$$

Here, $\theta = (\theta_1, \theta_2)'$, $V = (V_1, V_2)'$, and

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Here,

$$\begin{aligned} M_{11} &= m_2 l_2^2 + m_2 2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 \\ M_{12} &= m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ M_{21} &= m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ M_{22} &= m_2 l_2^2 \\ V_1 &= -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ V_2 &= m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{aligned} \tag{3.3}$$

Note that the τ_1 equation contains $\cos(\theta_2), \sin(\theta_2), \dot{\theta}_2$, and $\ddot{\theta}_2$, the states corresponding the second link. Similarly the τ_2 equation contains $\dot{\theta}_1$ and $\ddot{\theta}_1$. Let $X_1 = (\theta_1, \dot{\theta}_1)'$ and $X_2 = (\theta_2, \dot{\theta}_2)'$, the states of the two link manipulator. We may write Eqn. (3.2) as:

$$\begin{aligned} \tau_1 &= F_1(X_1, X_2, \dot{X}_1, \dot{X}_2) \\ \tau_1 &= F_1(X_1, X_2, \dot{X}_1, \dot{X}_2) \end{aligned} \tag{3.4}$$

Consider a control law of the form to control the two-link manipulator:

$$\tau = K(G, X_1, X_2, \dot{X}_1, \dot{X}_2) \tag{3.5}$$

Here, τ is the joint torque vector. A suitable control law $K(\cdot)$ and controller parameters/gain G may be used in principle to approximate the manipulator dynamics, including the coupling terms. Centralized adaptive controllers use such a model, where the controller parameters are tuned to improve a suitably defined performance metric.

Now consider a pure decentralized control of the form:

$$\begin{aligned} \tau_1 &= k_1(G_1, X_1, \dot{X}_1) \\ \tau_2 &= k_2(G_2, X_2, \dot{X}_2) \end{aligned} \tag{3.6}$$

In this case the control laws cannot account for coupling terms in the dynamics given in Eqn. (3.4), with any form of k_i and any value of control parameters G_i. Here τ_1 depends on only the states of the first joint, X_1 and its derivative, and hence it cannot approximate the effect of X_2 (and $\dot{X}_2$), the states of the second joint. The success of the such a controller depends purely on robustness properties of the corresponding control laws, such as gain/phase margins they provide.

Thus, we observe here that though the decentralized control scheme may successfully reduce the computational overhead and guarantee satisfactory performance, they cannot replicate the ideal performance a model-based control achieves. Their performance level depends on the states, and hence the control system designer has to ensure that the worst possible performance should be better than that desired. In fact purposefully neglecting the available nonlinear dynamics, particularly the dynamic coupling effects, and using additional techniques to account for them, may also lead to additional computational overhead.

3.1.3 Summary of research gap

To summarize, when the dynamic model of the manipulator is fully available, as is assumed in this work, the centralized control schemes though guarantee best possible trajectory tracking performance, are computationally expensive. Also, as discussed earlier using the control scheme presented in (Guruprasad & Ghosal 1999), even in the situation where the model is not fully known (or utilized), it is possible in the case of the centralized architecture to have provisions to account for the nonlinearity and dynamic coupling. The decentralized control architecture suffers from its inability to truly account for the dynamic coupling between the links of a manipulator, though it can have provision for incorporating nonlinearity. Both centralized control schemes that do not utilize the manipulator dynamic model into control law and the decentralized control schemes, rely on robustness properties of feedback control and concepts such as gain margins, and focus on worst case performance level.

3.2 Objective

The objective of the research work is to envisage a computationally efficient joint-level control architecture for the manipulator control, fully utilizing its dynamic model, and without compromising on the performance as compared to that with the traditional feedback linearizeation based model-based control scheme.

In the following chapters we provide control architecture and control schemes that attempt to achieve the objective of this work.

CHAPTER 4

Distributed control of manipulator

In this chapter, we present a distributed control architecture for manipulator control and provide a simple model-based control scheme that is implemented in the proposed distributed control architecture.

4.1 Manipulator as a multi-agent system

A serial-link manipulator consists of several links with joints which allow motion between them. Links physically interact with each other in terms of interactive forces and moments between them through these joints. As illustrated in Figure 4.1, the link $(i-1)$ exerts a force f_i and a moment n_i on the link i. Similarly, the link $(i+1)$ exerts a force $-f_{i+1}$ and a moment $-n_{i+1}$ on the link i. With these interactive forces and moments from connected links, the ith link experiences a net force of F_i and a net moment of N_i. The actuator applies a moment (or force in the case of prismatic joints) about (along) the joint axes Z_i. We may consider a 'joint-link pair' as a subsystem or an agent, interacting with other subsystems/agents. In this sense, a serial-link robot is a multi-agent system. However, unlike in a typical multi-agent system, where the coupling between any two agents is at the behavioral level, interactions between the agents (joint-link pairs) in a manipulator at the physical level.

Note that the ith link directly interacts only with the neighboring links $i-1$ (through the joint i) and $i+1$ (through the joint $i+1$). However, interaction of link $i+2$ with $i+1$ is experienced on the link i through the link $i+1$. In this way, motion of (and the force/torque on) every link affects every other link. Such an indirect interaction is also seen in distributed multi-agent systems. Here, direct local interactions lead to interaction between every (connected) agents.

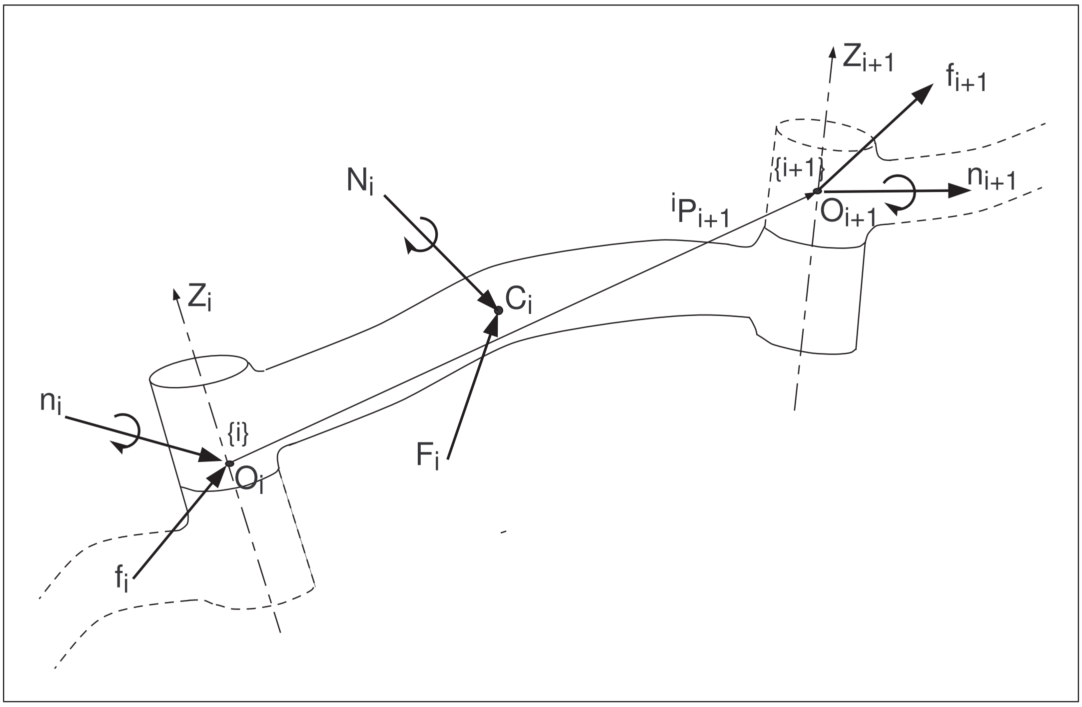


Figure 4.1: Connected links exert forces and moments through the joints. The axis of motion between links $(i-1)$ and i is Z_i. A force f_i, and a moment n_i, are exerted by $(i-1)$th link on the ith link. F_i and N_i are the net force and moment acting on the link i.

4.2 Computational cost of manipulator dynamics

The equation modeling the dynamics of a serial-link manipulator has the form

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta) \tag{4.1}$$

Here, τ is the vector of joint torques with size $N \times 1$; $M(\cdot)$ is the mass matrix with size $N \times N$; $\theta, \dot{\theta}$, and $\ddot{\theta}$ are joint angle, joint velocity, and joint acceleration vectors, respectively, all of size $N \times 1$; $V(\theta, \dot{\theta})$ is the vector involving centrepetal/coriolis accelerations, of size $N \times 1$; and $G(\theta)$ is the vector of gravity terms of size $N \times 1$. Here, N is the degrees-of-freedom of the manipulator. The standard model-based control law is (as given in Chapter 1):

$$\tau = M(\theta)(\ddot{\theta}_d + K_v \ddot{E} + K_p E) + V(\theta, \dot{\theta}) + G(\theta) \tag{4.2}$$

Here, θ_d is the vector of desired joint angles, $E = \theta_d - \theta$ is the tracking error, and K_p and K_v are diagonal matrices of controller gains.

The model-based nonlinear control law uses the dynamic model of the manipulator for computing the control input. Thus, computational cost of the dynamic equations dictates the frequency at which the control input can be updated. Higher the computational cost, higher is the computational lead time. The computational cost associated with the dynamic equations of a manipulator increases with degrees-of-freedom.

We carried out a simple analysis to find out how the number of computations and hence the computational cost depends on the degrees-of-freedom, using Maple. We considered planar manipulators with degrees-of-freedom from 2 to 6. We used iterative Newton-Euler formulation method to obtain the manipulator dynamics using Maple. The details of the process of obtaining manipulator dynamics is given in Appendix A1 using the Maple output for the case of a 3R planar manipulator. Similar process was used for planar manipulators with degrees-of-freedom $2 - 6$. Manipulator dynamics as obtained by Maple for degrees-of-freedom 4 and 5 are provided in Appendix A2[1]. In computing computational cost, we have considered cost of addition/subtraction as 1 unit. Multiplication operation is definitely computationally more expensive than addition, though the actual relative cost depends on the algorithm used and the processor itself. Here, for the purpose of comparative cost analysis, we have assumed the computation of multiplication operation is 4 times more expensive than the addition. As trigonometric terms appear at most twice per degree of freedom, in form of cosine and sine, we have not considered them, though they are computationally more expensive. The number of arithmetic operations and the corresponding cost involved in computation of the dynamic equation of a planar manipulator with revolute joints for different degrees-of-freedom are plotted in Figure 4.2. The number of arithmetic operations too were obtained using Maple. Based on these results, we may obtain an empirical relationship for number of arithmetic operations (N_{Arith}) in the dynamic equation

[1]Dynamics of 2R and 3R manipulators are provided in the body of thesis. Dynamics of 6R manipulator is not given as it runs into several pages.

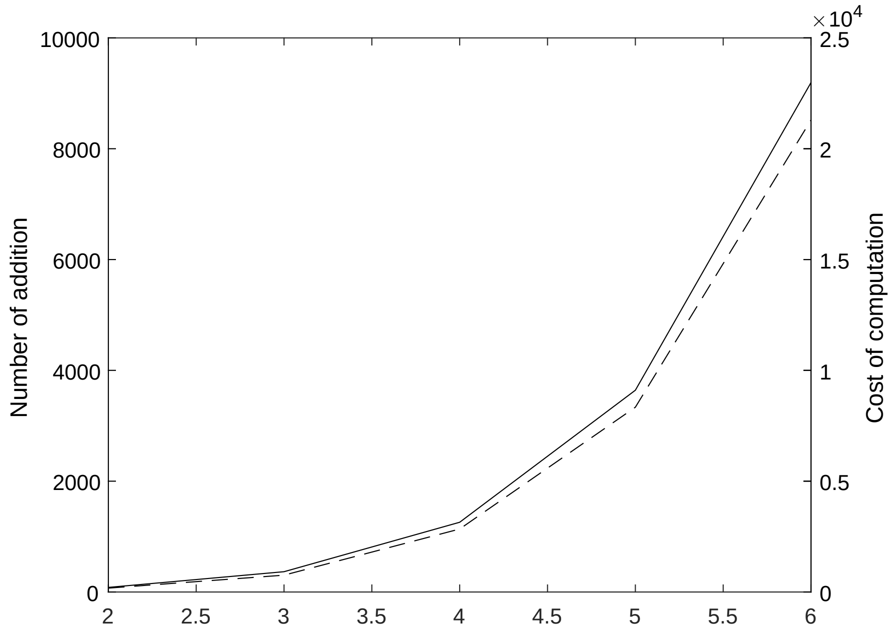


Figure 4.2: Variation of number of arithmetic operations (shown with dashed line) and total cost of computation (shown with solid line) of the dynamic equations of a planar manipulator with the degrees-of-freedom.

of a planar manipulator as a function of the degrees-of-freedom N as,

$$N_{Arith} = 35.583N^4 - 370.5N^3 + 1676.9N^2 - 3424N + 2605 \tag{4.3}$$

which is polynomial in N. A similar trend is expected in a general serial manipulator, or even parallel and hybrid manipulators, though the dynamic coupling (between neighboring links) effect is maximum in the case of a planar serial manipulator. The computational issues may not be very crucial for control of manipulators with small degrees-of-freedom or those which can use high performance processors for the implementation of control law. However, as the degrees-of-freedom increases, particularly in redundant or hyper redundant manipulators, higher computational effort may start affecting trajectory tracking performance.

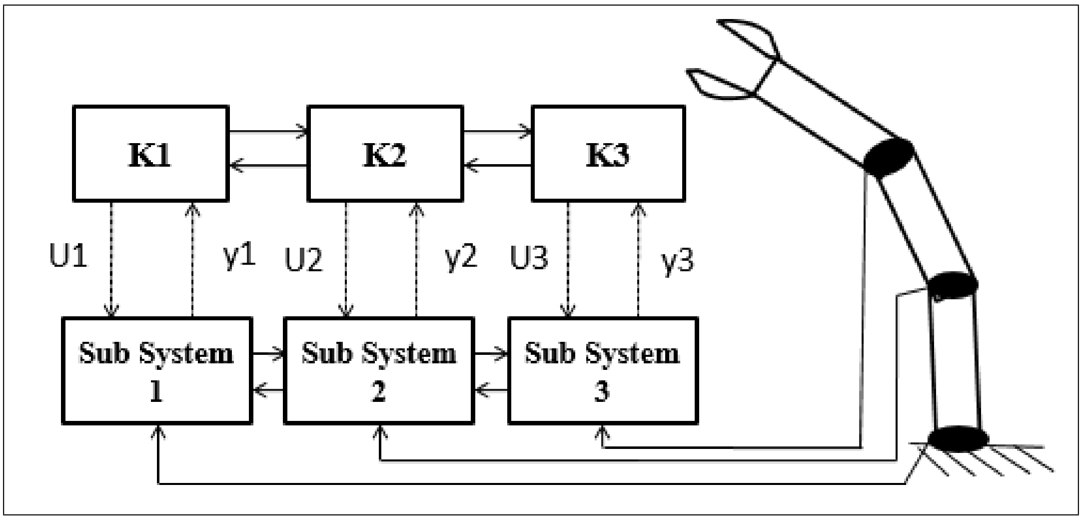


Figure 4.3: Distributed manipulator control architecture

4.3 Distributed manipulator control architecture

Now we propose a distributed control architecture as illustrated in Figure 4.3 for a manipulator, exploiting the distributed nature of manipulator dynamics as discussed earlier. Here, each joint-link agent is controlled by a dedicated joint-level controller. While the joint-link agents interact (directly) with neighboring agents (and indirectly with other agents), the joint-level controllers interact in form of communication with the neighboring controllers (and indirectly with all other controllers).

There has been some attempt in the literature to perceive manipulator as a multi agent system. In (Morit et al. 1996) the authors consider a joint-link pair as an agent and use the multi-agent system concept for manipulator control. These agents are software agents rather than physical agents. Further, this paper addresses kinematics rather than dynamic control of the manipulator. The inverse kinematic problem is solved using a distributed architecture which provided input to a high level control. The authors in (Bohner & Luppen 1997) present a reactive planning and control system for redundant manipulators. Here a 'joint-agent' is responsible for planning and controlling the motion of one joint, by integrating sensor data, such as tactile sensors. In (Jia et al. 2007), the authors proposed distributed architecture for a light space manipulator. However, they do not consider manipulator dynamics. The authors in (Tsuji et al. 1993) presented a distributed control for redundant manipulators based on a concept of virtual arms. Though the authors present control at dynamic level, the subsystems here are virtual arms rather than the join-link pairs.

4.3.1 A simple distributed control scheme

Now we present a simple model-based distributed control scheme based on the proposed distributed architecture, without use of any additional sensors.

The model-based control law given in Eqn. (4.2) represents a control law using the centralized control architecture, where a single central controller computes control inputs, $\tau_i, i = 1, \ldots, N$, for all the joints. Note that here all the variables $\tau, \theta, \dot{\theta}, \ddot{\theta}, E, \ddot{E}$, etc. are obtained in time t. Consider a simple distributed control law for the ith joint-level controller as:

$$\tau_i \quad = \quad \Sigma_{j=1}^{N} M_{ij}(\theta)(\ddot{\theta}_{jd} + K_{vj}\dot{E}_j + K_{pj}E_j) + V_i(\theta, \dot{\theta}) + G_i(\theta) \qquad (4.4)$$

Here, the index j (or i) is used to indicate the corresponding component of a vector, and M_{ij} is the jth element in ith row of M. Note that the Eqn. (4.4) is the ith component of the model-based control law as given in Eqn. (4.2). Let K_i be a controller using the control law given in the Eqn. (4.4). For the model-based control law given in Eqn. (4.4), $i = 1, \ldots, N$, we make following observations:

1. Output of each controller K_i is τ_i, the control input to the ith joint of the manipulator or G_i, the ith joint-link agent.

2. Controller K_i requires inputs from joint-link agents $j \neq i$, which it may receive through the corresponding joint-level controllers $K_j, j \neq i$.

3. The controller $K_i, i = 2, \ldots N-1$ is connected to neighboring controllers K_{i-1} and K_{i+1}, in the sense that it can send and receive signals.

4. Now the adjacency graph of $K_i, i = 1, \ldots N$ is connected.

5. Thus, the controller K_i can receive (send) signals from (to) any controller $K_j, j \neq i$ through a distributed (multi-hop, if required) communication.

6. The adjacency graph formed by the joint-level controllers is identical to that formed by the joint-link agents.

Thus, the control architecture, where each joint-level controller K_i controls the ith joint while obtaining necessary information (such as feedback values of joint states and the desired states), has a natural distributed architecture, which in fact is the result of the distributed nature of the manipulator dynamics.

Thus, though the control law corresponding to K_i may contain terms corresponding to every joint/link, not only those corresponding to the immediate neighboring agents, the control scheme based on the Eqn. (4.4) is naturally amenable for a distributed implementation. As observed, the joint-level controllers are allowed to communicate directly with the immediate neighbors, and as every joint-level controller is indirectly connected to every other controller, the required information may be obtained through (multi-hop) distributed communication between the joint-level controllers.

Theorem 1 *The control law given by the Eqn. (4.4), with positive gains, makes the links of the manipulator whose dynamics is given by the Eqn. (4.1), follow the desired trajectory* $\theta_d(t)$*, asymptotically.*

Proof. The closed-loop error dynamics may be obtained from Eqn (4.4) and Eqn. (4.1) as,

$$\ddot{E}_j + K_{vj}\dot{E}_j + K_{pj}E_j = 0 \qquad \forall j \in 1, 2....N \tag{4.5}$$

Thus, we have $E_j \to 0$, as $t \to \infty, \forall j \in \{1, 2, \ldots, N\}$, for positive gains. □

Remark 1 *Here, the closed-loop error dynamics is identical to that obtained using the conventional model-based control scheme given by Eqn. (4.2). This is not surprising for two reasons: First, the joint-level controllers* K_i *and the distributed control scheme presented here is based on the control laws given in Eqn (4.4), which itself is based the control law given in Eqn. (4.2); Second, any control* *law that cancels the nonlinear and coupled dynamics using feedback, or in other words, feedback linearization, should lead to linear decoupled closed-loop error dynamics as given in Eqn. (4.5). The contributions here are: identifying a natural distributed nature of the model-based control law given in Eqn. (4.2), presenting a control scheme that is amenable for implementation in the distributed control architecture, and obtaining feedback linearization leading to guaranteed state independent trajectory tracking performance, unlike the decentralized (or independent joint) control schemes presented in the literature.*

Remark 2 *The distributed control scheme given by the Eqn. (4.4), is only a simple example of a control scheme/law that can be implemented in the proposed distributed manipulator control architecture. In principle, several model-based control schemes may be designed within the proposed distributed architecture.*

4.3.2 Distribution effectiveness

The main objective of a distributed control law for a manipulator being reduction in the cost of computation involved in the control law, Here we introduce a quantity called *distribution effectiveness* to quantify how the distributed control schemes share the computational load among the individual joint-level controllers. Let C_{Ti} be the computational cost associated with the i_{th} joint-level controller, and C_{Tc} be that associated with the corresponding centralized controller. We define the distribution effectiveness for N degree of freedom robot as:

$$\eta_d = \frac{C_{Tc}/N}{max_i(C_{Ti})} \tag{4.6}$$

In an ideal situation, when the computation is distributed uniformly among the N joint-level controllers, we get $\eta_d = 1$.

4.4 Discrete-time implementation: Effect of time delay

The model-based (centralized) control law as given in Eqn. (4.2) is in continuous time domain. However, in realty, this control law is realized in discrete time. The model-based control law in discrete time is given by,

$$\begin{aligned}\tau(t) &= M(\theta(t-T_d)\ddot{\theta}_d(t-T_d) + K_v(t-T_d)\dot{E}(t-T_d) + K_pE(t-T_d)) \\ &\quad +V(\theta(t-T_d), \dot{\theta}(t-T_d)) + G(\theta(t-T_d))\end{aligned} \tag{4.7}$$

Where, T_d is the time delay introduced due to the sampling time T_d. The sampling time depends on the time required to compute the control law Eqn. (4.7), along with any other processing required. Note that with the discrete control law given in Eqn. (4.7), feedback linearization is not achieved. We can achieve the feedback linearization only when $T_d = 0$. However, due to continuity of the dynamics of the manipulator and the model-based control law, tracking performance is expected to degrade gracefully with increasing T_d.

Now consider the discrete-time distributed control law based on that given in Eqn. (4.4),

$$\begin{aligned}\tau_i(t) &= \Sigma_{j=1}^{N} M_{ij}(\theta(t-T_d^d))(\ddot{\theta}_{jd}(t-T_d^d) + K_{vj}(t-T_d^d)\dot{E}_j(t-T_d^d) \\ &\quad +K_{pj}E_j(t-T_d^d)) + C_i(\theta(t-T_d^d), \dot{\theta}(t-T_d^d)) + G_i(\theta(t-T_d))\end{aligned} \tag{4.8}$$

Here, T_d^d is the time delay (due to sampling time) in the discrete-time distributed model-based control law. Note that it is expected that $T_d^d < T_d$ as the computational effort associated with the control law is now shared among the individual controllers. Hence, it is expected that the trajectory tracking performance of manipulator with the discrete time distributed model-based control law (4.8) is superior to that with the centralized, discrete-time model-based control law given in Eqn. (4.7).

4.5 Distributed control for a 3R planar manipulator

Now we shall illustrate the control scheme given by Eqn. (4.4) (or (4.8)) implemented in the proposed distributed control architecture using a simple 3R (three link manipulator with revolute joints) planar manipulator. Figure 4.4 shows a schematic of a 3R planar manipulator with joint axes fixed using the standard convention (Craig 2005). The D-H parameters are listed in Table 4.1.

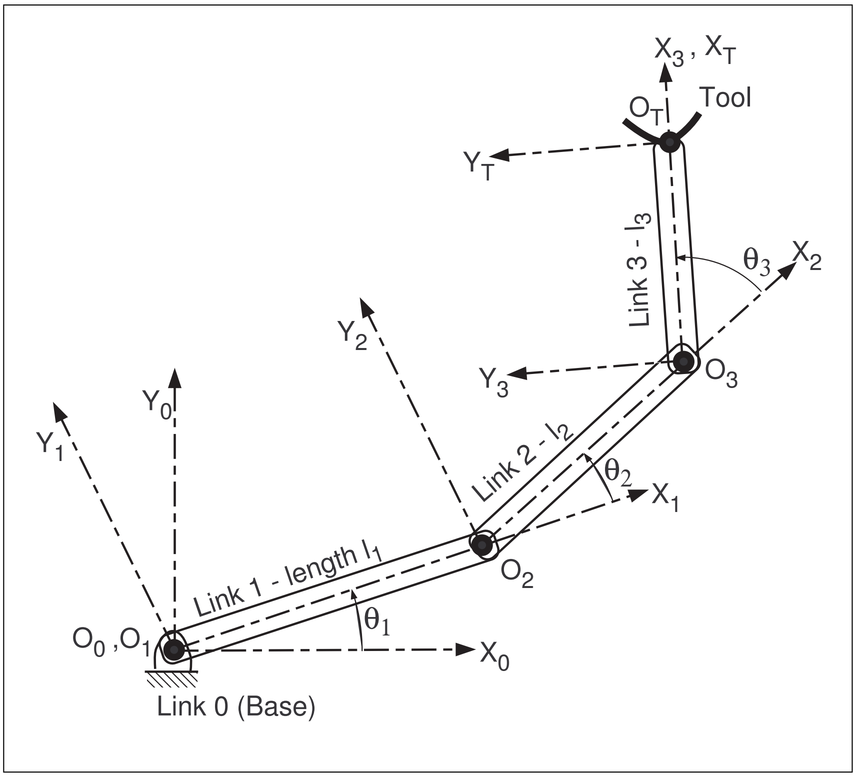


Figure 4.4: Schematic of a 3R planar manipulator with joint axes and D-H parameters.

i	α_{i-1}	d_i	θ_i	a_{i-1}
1	0	0	θ_1	0
2	0	0	θ_2	l_1
3	0	0	θ_3	l_2
4/T	0	0	0	l_3

Table 4.1: D-H parameters for a 3R planar manipulator shown in Figure 4.4.

We consider a 3R planar manipulator for several reasons: First, most manipulators use revolute joints, which result in nonlinearities and also dynamic coupling; second, serial planar manipulator has maximum dynamic coupling between its links; third, it is a simplest (in terms of degrees-of-freedom) manipulator with at least one intermediate link, and fourth, it is a simplest (in terms of degrees-of-freedom) redundant manipulator (considering only tool position in a plane without considering its orientation). Figure 4.5 shows the

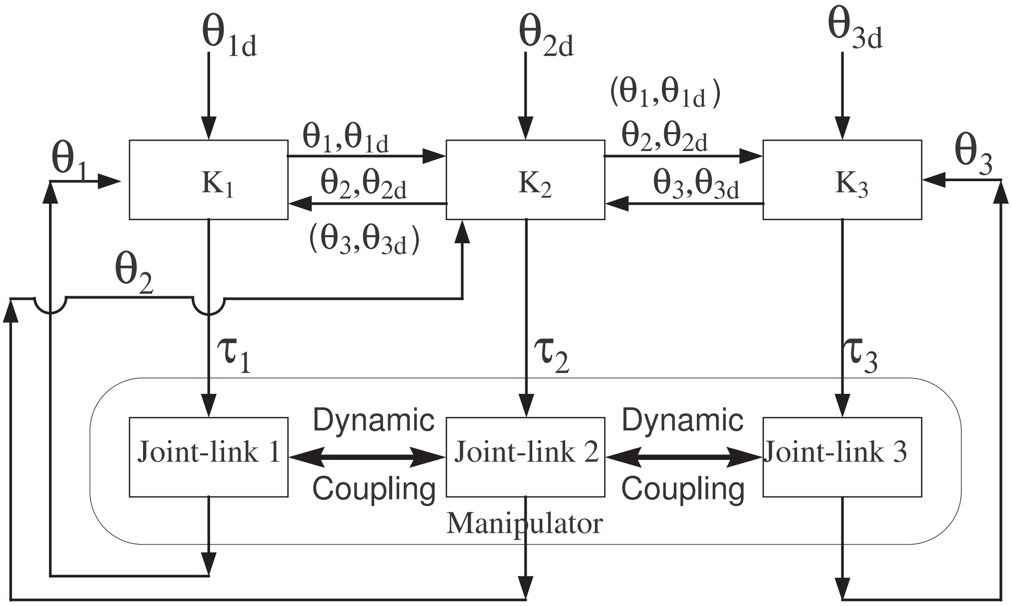


Figure 4.5: Block diagram of proposed model-based control of a 3R planar manipulator in the proposed distributed architecture. Joint-level controllers K_1, K_2, and K_3 communicate among themselves while the dynamically coupled three joint-link agents form the 3R manipulator. Each joint level controller K_i provides the control input τ_i to corresponding joint-link agent/motor.

block diagram of the control law (Eqn. (4.4)) implemented in the proposed distributed architecture. The communication links, along with the information exchange between the neighboring controllers is also shown in the diagram. Controller K_i receives θ_i (and its first derivative) as feedback from the joint-link agent i, θ_{id} (and its first and second derivative) as the desired value. Controllers communicate the values of corresponding joint variable (feedback) and desired joint variable (along with necessary derivatives not shown in the figure), that they received, to the immediate neighbors ($K_1 \leftrightarrow K_2, K_2 \leftrightarrow K_3$). The intermediate controller K_2 communicates the values of θ_1 (feedback it received via K_1, along with derivatives) and θ_{1d} (desired value it received via K_1) to K_3 (shown as (θ_1, θ_{1d}) in bracket) and the values of θ_3 (feedback it received via K_3, along with derivatives) and θ_{3d} (desired value it received via K_3) to K_1 (shown as (θ_3, θ_{3d}) in bracket), through the multi-hop communication. Now with this distributed communication between the individual joint-level controllers, each of them has all the necessary data to compute the corresponding control law. Finally, the controller K_i provides the control input τ_i to the ith joint (or the joint-link agent) using Eqn. (4.4) or (4.8).

Remark 3 *The control scheme provided in Figure 4.5 may be implemented in hardware. Each joint-level controller K_i may be implemented on an embedded hardware with provision for necessary communication between them. In the case of distributed control of a manipulator, unlike in a typical multi-agent/robotic system, it is possible to use wired communication between the joint-level controllers. However a detailed discussion on the hardware implementation is beyond the scope of this thesis.*

4.5.1 Computational cost and distribution effectiveness

As one of the motivation for the distributed control scheme is high computational cost associated with the conventional model-based control scheme, in this section we discuss the computational effectiveness of the proposed distributed control scheme. Table 4.2 shows the number of additions (N_A), multiplications (N_M), corresponding costs (C_A, C_M), and the total cost (C_T), involved in computation of the dynamics at each joint. We obtain a distribution effectiveness of 0.66 in this case, as against an ideal value of 1. The (maximum) computational cost with the distributed implementation now reduces from 944 units to 480 units (corresponding to the first joint controller), that is about 50% of the cost of centralized implementation. This implies that the sampling time of a discretized implementation of the control law in the

Link #	N_A	N_M	C_A	C_M	C_T
1	64	104	64	416	480
2	50	78	50	312	362
3	14	22	14	88	102
Total	128	204	128	816	944

Table 4.2: Number of additions (N_A), multiplications (N_M), corresponding costs (C_A, C_M), and the total cost (C_T), involved in computation of the dynamics at each joint.

DOF	η_d
2	0.64
3	0.66
4	0.69
5	0.72
6	0.75

Table 4.3: Distribution effectiveness with the degrees-of-freedom of planar manipulators.

distributed architecture is about half that in the centralized architecture. If the computational load were distributed equally among the joint-level computations, then the computational cost, and hence the sampling time, with the distributed implementation would have been 33% of that with the centralized implementation. Thus, though the model-based control law implemented in both centralized architecture (Eqn. (4.2)) and the proposed distributed architecture (Eqn. (4.4) are identical theoretically, in reality when the control law is implemented in discrete time, the performance of the trajectory performance with the control law in distributed architecture is expected to be superior compared to that in the centralized architecture as $T_d^d = 0.5T_d$. If we carefully design the distributed control law such that $\eta_d = 1$, then we get $T_d^d = 0.33T_d$, the least possible sampling time. As shown in Table 4.3, the distribution effectiveness η_d improves with degrees-of-freedom of the manipulator. It may be observed that manipulator control in distributed architecture is more useful for higher degrees-of-freedom manipulator due to higher computational cost of the centralized control law and better distribution effectiveness.

Sl:no	Repeating terms	Joint 1	Joint 2	Joint 3
1	$-l_1\dot{\theta_1}^2 + S_1 g$	9	7	2
2	$l_1\ddot{\theta} + C_1 g$	10	7	2
3	$-l_2(\dot{\theta}_1 + \dot{\theta}_2)^2$	4	3	1
4	$l_2(\ddot{\theta}_1 + \ddot{\theta}_2)$	5	4	1
5	$l_3(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3)$	2	2	1
6	$-l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2$	1	1	0

Table 4.4: Terms appearing multiple times in 3R manipulator dynamic equation

Remark 4 *We may observe that the distributed control scheme based on Eqn. (4.4) presented here is based on identification of a natural distributed nature of the manipulator dynamics itself and that of the model-based control law (4.2). The reduction in computational lead time with the distributed control scheme is achieved purely because of the distribution of the computational effort among the joint-level controllers, based on the natural distributed nature of the dynamics, rather than the program optimization or operation optimization techniques that is used at the algorithmic level.*

4.5.2 Reducing computational cost

With a careful observation, we can identify several repetitive terms in the dynamics of a 3R planar manipulator. We have used Maple to aid in this exercise. Such repetitive terms are shown in Table 4.4 along with the number of repetitions. For example the term $l_1\ddot{\theta} + C_1 g$ repeats 10 times in the equation corresponding to the first joint, 7 times in that corresponding to the second joint, and twice in that corresponding to the third joint.

Now if we compute each of the terms that are listed in the Table 4.4 only once, we may further reduce the computation cost associated with dynamics (and hence the control law) at each joint. Note that this reduction is achieved without neglecting any of the terms. Table 4.5 shows number of addition, multiplication, along with the corresponding computational cost and total computational cost associated with the dynamics at each joint level after this refinement. It may be

Link ID	N_A	N_M	C_A	C_M	C_T
1	30	40	30	160	190
2	24	32	24	12	152
3	6	8	6	32	38
Total	60	80	60	320	30

Table 4.5: Number of addition and multiplication, corresponding computational cost after avoiding repetitive computation of terms shown in Table 5.4.

observed that the computational cost at each joint level is now reduced by about 60% compared to that shown in Table 4.2. However, the distribution effectiveness $\eta_d = 0.66$ even in this case, indicating this exercise of reducing computations by avoiding repeated computation of certain repetitive terms does not affect how the computation load is shared among the individual controllers. We provide a detailed analysis of computational cost in Chapter 6.

Remark 5 *Apart from the reduction in computational overhead due to the natural distribution of the computational effort among the joint-level controllers, we have achieved further reduction in the computational load here by identifying repetitive terms in the manipulator dynamics/control law. As demonstrated by the fact that the distribution effectiveness is unaffected by this exercise, this process of reduction in computational load is independent of the distributed property of the manipulator dynamics or the proposed distributed control scheme.*

CHAPTER 5

Cooperative control of manipulators

In this Chapter, we present a cooperative control scheme for a planar manipulator within the proposed distributed manipulator control architecture. We consider a 3R planar manipulator to illustrate the proposed scheme.

5.1 Cooperative nature of planar manipulator dynamics

First we revisit the manipulator dynamic equation and observe a special distributed and cooperative structure in planar manipulators with revolute joints, As we know the dynamics of a N degrees of freedom serial link manipulator has the form,

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta) \tag{5.1}$$

Here, τ is a $N \times 1$ joint torque vector; $M(\cdot)$ is the $N \times N$ mass matrix; $\theta, \dot{\theta}$, and $\ddot{\theta}$ are is $N \times 1$ joint vector, joint velocity, and joint acceleration vectors, respectively; $V(\cdot, \cdot)$ is a $N \times 1$ vector of centrepetal/coriolis accelerations, gravity and other terms such as frictional forces[1].

Consider the dynamics of a two-link planar manipulator with revolute joint having concentrated mass at the end of the links:

$$\begin{aligned} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} &= \begin{bmatrix} m_2 l_2^2 + 2 m_2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} \\ &+ \begin{pmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{pmatrix} + \begin{pmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{pmatrix} \end{aligned} \tag{5.2}$$

[1]Though we have provided these equation in earlier chapters, we provide them here for sake of completeness and ease of presentation.

Let us define:

$$\begin{aligned} \tau_1 &= \tilde{\tau}_2 + \tilde{\tau}_1 \qquad (5.3) \\ \tau_2 &= \tilde{\tau}_2 \qquad (5.4) \end{aligned}$$

Now from the Eqn. (5.2) we obtain:

$$\begin{pmatrix} \tilde{\tau}_1 \\ \tilde{\tau}_2 \end{pmatrix} = \begin{bmatrix} m_2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 & m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}$$

$$+ \begin{pmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_1^2 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{pmatrix} + \begin{pmatrix} (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{pmatrix} \qquad (5.5)$$

or

$$\tilde{\tau} = \tilde{M}(\theta))\ddot{\theta} + \tilde{V}(\theta, \dot{\theta}) + \tilde{G}(\theta) \qquad (5.6)$$

We have analyzed the dynamic equations of planar manipulators with revolute joints for degrees-of-freedom from 2 to 6 using Maple. A sample output is provided the results obtained from Maple in Appendix A4. We have obtained similar form of dynamic equations as observed for the 2R manipulator given in Eqn. (5.3). Thus, for $2 \leq N \geq 6$ we have:

$$\begin{aligned} \tau_N &= \tilde{\tau}_N \\ \tau_{N-1} &= \tilde{\tau}_N + \tilde{\tau}_{N-1} \\ &\vdots \\ \tau_j &= \tilde{\tau}_j + \tilde{\tau}_{j-1} + \cdots + \tilde{\tau}_1 \\ &\vdots \\ \tau_1 &= \tilde{\tau}_N + \tilde{\tau}_{N-1} \ldots \tilde{\tau}_2 + \tilde{\tau}_1 \end{aligned} \qquad (5.7)$$

We expect that the observation provided in Eqn. (5.7) may be generalized for a planar manipulator with revolute joints, with any degrees-of-freedom.

5.2 Distributed cooperative control scheme

Using the special structure of dynamics of planar manipulators discussed in the previous section, we now develop a distributed cooperative control scheme for a 3R planar manipulator.

For a 3R planar manipulator, the dynamics (Eqn. (5.1)) may be now re-written as,

$$\begin{aligned} \tau_3 &= \tilde{\tau}_3 \\ \tau_2 &= \tilde{\tau}_3 + \tilde{\tau}_2 \\ \tau_1 &= \tilde{\tau}_3 + \tilde{\tau}_2 + \tilde{\tau}_1 \end{aligned} \tag{5.8}$$

Or

$$\begin{aligned} \tilde{\tau}_3 &= \tilde{M}_3(\theta)\ddot{\theta} + \tilde{V}_3(\theta, \dot{\theta}) + \tilde{G}_3(\theta) \\ \tilde{\tau}_2 &= \tilde{M}_2(\theta)\ddot{\theta} + \tilde{V}_2(\theta, \dot{\theta}) + \tilde{G}_2(\theta) \\ \tilde{\tau}_1 &= \tilde{M}_1(\theta)\ddot{\theta} + \tilde{V}_1(\theta, \dot{\theta}) + \tilde{G}_1(\theta) \end{aligned} \tag{5.9}$$

Here $\tilde{M}_j(\theta)$ is the jth row of $\tilde{M}(\theta)$ and $\tilde{V}_j(\theta, \dot{\theta})$ is the jth element of the vector $\tilde{V}(\theta, \dot{\theta})$, and $\tilde{G}_j$ is the jth element of the vector $\tilde{G}(\theta)$

Where,

$$\begin{aligned} \tilde{\tau}_3 &= m_3 l_3 \ddot{\theta}_1 + m_3 l_3 \ddot{\theta}_2 + m_3 l_3 \ddot{\theta}_3 + m_3 s_3 l_2 \dot{\theta}_1^2 + 2 m_3 s_3 l_2 \dot{\theta}_1 \dot{\theta}_2 + m_3 s_3 l_2 \dot{\theta}_2^2 \\ &+ m_3 s_3 c_2 l_1 \dot{\theta}_1^2 - m_3 s_3 c_2 s_1 g - m_3 s_3 s_2 l_1 \ddot{\theta}_1 - m_3 s_3 s_2 c_1 g + m_3 c_3 l_2 \ddot{\theta}_1 \\ &+ m_3 c_3 l_2 \ddot{\theta}_2 + m_3 c_3 s_2 l_1 \dot{\theta}_1^2 - m_3 c_3 s_2 s_1 g + m_3 c_3 c_2 l_1 \ddot{\theta}_1 + m_3 c_3 c_2 c_1 g \end{aligned} \tag{5.10}$$

$$\tag{5.11}$$

$$
\begin{aligned}
\tilde{\tau_2} \;=\;& -m_3l_3s_3l_2\dot{\theta}_1^2 - m_3l_3s_3l_2\dot{\theta}_2^2 - l_2s_3m_3l_3\dot{\theta}_3^2 + {s_3}^2m_3{l_2}^2\ddot{\theta}_1 + {s_3}^2m_3{l_2}^2\ddot{\theta}_2 \\
& +m_3l_3c_3l_2\ddot{\theta}_1 + m_3l_3c_3l_2\ddot{\theta}_2 + l_2c_3m_3l_3\ddot{\theta}_3 + {c_3}^2m_3{l_2}^2\ddot{\theta}_1 + {c_3}^2m_3{l_2}^2\ddot{\theta}_2 \\
& -2m_3l_3s_3l_2\dot{\theta}_1\dot{\theta}_2 - 2l_2s_3m_3l_3\dot{\theta}_1\dot{\theta}_3 - 2l_2s_3m_3l_3\dot{\theta}_2\dot{\theta}_3 + l_2{s_3}^2m_3s_2l_1\dot{\theta}_1^2 \\
& -l_2{s_3}^2m_3s_2s_1g + l_2{s_3}^2m_3c_2l_1\ddot{\theta}_1 + l_2s3^2m_3c_2c_1g + l_2{c_3}^2m_3s_2l_1\dot{\theta}_1^2 \\
& -l_2{c_3}^2m_3s_2s_1g + l_2{c_3}^2m_3c_2l_1\ddot{\theta}_1 + l_2{c_3}^2m_3c_2c_1g
\end{aligned} \tag{5.12}
$$

$$
\begin{aligned}
\tilde{\tau_1} \;=\;& {c_2}^2m_2{l_1}^2\ddot{\theta}_1 + s2^2m_2{l_1}^2\ddot{\theta}_1 + l_1{c_2}^2m_2c_1g + l_1{s_2}^2m_2c_1g - m_2l_2s_2l_1\dot{\theta}_1^2 \\
& -2l_1s_2m_2l_2\dot{\theta}_1\dot{\theta}_2 - l_1s_2m_2l_2\dot{\theta}_2^2 + m_2l_2c_2l_1\ddot{\theta}_1 + l_1c_2m_2l_2\ddot{\theta}_2 \\
& +m_1{l_1}^2\ddot{\theta}_1 + m_1l_1c1g
\end{aligned} \tag{5.13}
$$

Now consider the conventional model-based control law

$$
\tau = M(\theta)(\ddot{\theta}_d + K_v\dot{E} + K_pE) + V(\theta, \dot{\theta}) \tag{5.14}
$$

Here, θ_d is the desired joint angle value and $E = \theta_d - \theta$ is the tracking error, and K_p and K_v are diagonal matrices of controller gains.

As we have seen earlier, in the conventional model-based control scheme, a single (central) controller will compute control torques $\tau_i(t)$ to be applied at all the joints. In the simple distributed manipulator control scheme we had proposed in the previous chapter, each joint level controller computes only the corresponding torque τ_i.

Now we propose a distributed cooperative control law based on the model-based control law (5.14) as,

$$
\begin{aligned}
\tau_3 &= \tilde{\tau_3} \\
\tau_2 &= \tilde{\tau_3} + \tilde{\tau_2} \\
\tau_1 &= \tilde{\tau_3} + \tilde{\tau_2} + \tilde{\tau_1}
\end{aligned} \tag{5.15}
$$

Where,

$$\begin{aligned}
\tilde{\tau}_3 &= \tilde{M}_3(\theta)(\ddot{\theta}_d + K_v\dot{E} + K_pE) + \tilde{V}_3(\theta,\dot{\theta}) + \tilde{G}_3(\theta) \\
\tilde{\tau}_2 &= \tilde{M}_2(\theta)(\ddot{\theta}_d + K_v\dot{E} + K_pE) + \tilde{V}_2(\theta,\dot{\theta}) + \tilde{G}_2(\theta) \\
\tilde{\tau}_1 &= \tilde{M}_1(\theta)(\ddot{\theta}_d + K_v\dot{E} + K_pE) + \tilde{V}_1(\theta,\dot{\theta}) + \tilde{G}_1(\theta)
\end{aligned} \tag{5.16}$$

Note that the cooperative distributed control law (5.15) is also a distributed implementation of the conventional model-based control law (5.14). The proposed cooperative distributed control law has two major advantages. Firstly, the computation load associated with the control law is now distributed amongst the joint-level controllers leading to faster computation and hence decreased computation lead time. Second, as observed in Eqns (5.16), certain terms appear in computation of more than one joint torque. $\tilde{\tau}_3$ appears thrice (in all three joint torques) and $\tilde{\tau}_2$ appears twice (in first and second joint torque equations). However, in the proposed cooperative control law, each terms, that is, $\tilde{\tau}_i$ is computed exactly once by the ith joint controller, and communicated to all other joint-level controllers (Controllers $i-1,\ldots,1$). Each joint level controller computes the total control input (that is the joint torque) at the corresponding joint by simply adding all relevant $\tilde{\tau}_j$s. The latter advantage is due to the cooperative nature of the control law in contrast to the distributed control scheme proposed in previous chapter. Here we have exploited the structure of dynamic equation (as shown in Eqn. (5.8)) so that the joint-level controllers cooperate amongst themselves in computing the corresponding joint control torques more efficiently, unlike inn the basic distributed control scheme proposed in the previous chapter where each joint-level controller only communicates the state variables with the neighboring joint-level controllers. The block diagram illustrating the architecture of the proposed cooperative distributed control strategy for a 3R manipulator is shown in Figure 5.1.

Theorem 2: The proposed cooperative distributed control law given by the Equations (5.15) and (5.16) makes the manipulator with dynamics given by the Eqn. (5.1) follow the desired trajectory $\theta_d(t)$ asymptotically, for positive controller gains.

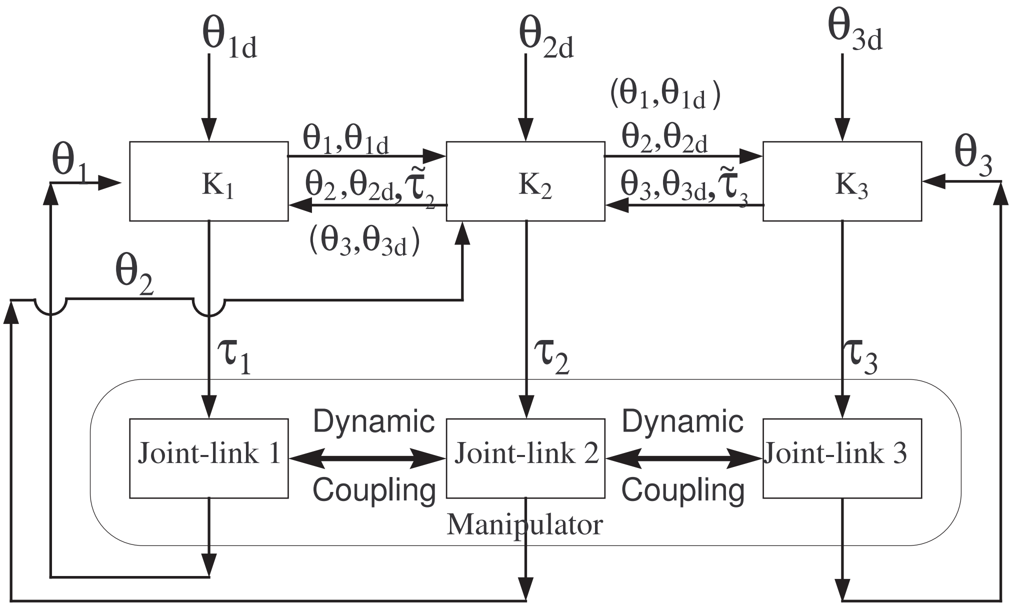


Figure 5.1: Architecture of proposed cooperative control scheme for a planar 3R manipulator.

Proof. From the control law as given in Equations (5.15) and (5.16), and the distributed dynamics given by Eqn. (5.1) we obtain the closed loop equation,

$$\ddot{E}_j + K_{vj}\dot{E}_j + K_{pj}E_j = 0 \tag{5.17}$$

For all $j \in \{1, 2, \ldots, N\}$. Thus we have

$$E_j \to 0, \text{as} \quad t \to \infty, \forall j \in \{1, 2, \ldots, N\}$$

for positive gains. □

This result is not surprising as the proposed control law given in Equations (5.15) and (5.16) is a cooperative and distributed implementation of the nonlinear model-based control law, which achieves linearization through feedback and leads to asymptotically stable trajectories for positive control gains, irrespective of the system dynamics/parameters. Remark 1 in Chapter 4 is also valid here.

Link #	N_A	N_M	C_{TC}	C_{TD}
1	14	26	118	480
2	50	56	260	260
3	14	22	102	102
Total			480	944

Table 5.1: Number of additions (N_A), multiplications (N_M), the total cost (C_{TC}), involved in computation $\tilde{\tau}_i$, and total cost C_{TD} involved in computation of τ_i is distributed control scheme presented in Chapter 4.

5.3 Computational effectiveness of the cooperative control scheme

Now we shall examine the computational effectiveness of the proposed distributed cooperative control scheme. Table 5.1 shows the number of addition (N_A), multiplication (N_M), the corresponding computational cost C_{TC} involved in computation of $\tilde{\tau}_i$ used in the cooperative control scheme along with the joint-level total cost (C_{TD}) in computing τ_i using the distributed control scheme presented in Chapter 4. We may observe that the maximum computational cost with the cooperative control scheme presented here is reduced to 260 units from 480 units with the distributed control scheme presented in Chapter 4 and from 944 units with the conventional (centralized) model-based control scheme. That is, the maximum computational cost (which decides the sampling time) with the proposed cooperative control scheme is 54% of the distributed control scheme presented in Chapter 4 and 27.5% of the conventional model-based control scheme. Observe that this reduction in computation is achieved purely by nature of the manipulator dynamics as given in Eqn. (5.8). The distribution effectiveness η_d with proposed cooperative control scheme is 0.615, which is marginally less than that with the distributed control scheme presented in Chapter 4.

Now consider a six degrees of freedom serial link planar (6R) manipulator with revolute joints. The computation associated with the dynamic equation, corresponding to each joint torque is shown in Table 5.2. For the purpose of comparison of computational cost, we use an unit cost of 1 for computation of addition and 4 for multiplication operation. Note that a centralized model-based controller has to perform 3846 additions and 4950 multiplications, expending 23646 units of computational cost. For the simple distributed control scheme presented in Chapter 4, we obtain a distribution effectiveness $\eta_d = 0.75$, and a reduction in maximum computation to 5241 (corresponds to the first joint) from 23646, as compared to the centralized implementation of the model-based control law, that is a reduced to 22%.

Link ID	N_A	N_M	C_{TD}
1	841	1100	5241
2	827	1074	5123
3	791	1018	4863
4	708	902	4316
5	528	666	3192
6	151	190	911
Total	3846	4950	23646

Table 5.2: Number of computations involved in each degree of freedom. Here N_A number of additon/substraction, N_M is number of multiplications, $C_{TC} = N_A + 4 \times N_M$, the total computational cost, and C_{TD} is the cost of computing corresponding joint torque (τ_i) using the distributed control law presented in chapter 4.

Now we consider computation effort associated with the proposed distributed cooperative control scheme. Table 5.3 shows the number of arithmetic operations at each joint level controller using the proposed control scheme. We may observe, that in contrast to the simple distributed scheme (as shown in Table 5.2), with the proposed distributed cooperative control scheme (as shown in Table 5.3), there is a substantial reduction in computation at each joint level. The maximum computation cost ($max_i(C_{Ti})$) is 2281 units (corresponds to 5th joint) against 5241 units with the distributed control scheme (43.5%) and against 23646 units with the centralized model-based control scheme (9.6%). The distribution effectiveness $\eta_d = 0.39$. Thus, though the distribution effectiveness of the proposed cooperative control scheme is considerably lower than that of the simple distributed control scheme, we obtain considerable reduction in maximum computation cost and hence in the lead time with the proposed cooperative control law in contrast to the simple distributed and the centralized control schemes.

If we carefully look at the equations corresponding to $\tilde{\tau}_i$, we can identify a

Link ID	N_A	N_M	C_{TC}
1	14	26	118
2	36	56	260
3	83	116	547
4	180	236	1124
5	377	476	2281
6	151	190	911
Total	841	1100	5304

Table 5.3: Number of computations involved in each degree of freedom ($\tilde{\tau}_i$). Here N_A number of additon/substraction, N_M is number of multiplications, $C_{TC} = N_A + 4 \times N_M$, the total cost, associated with computing $\tilde{\tau}$.

Sl:no	Repeating terms	$\tilde{\tau}_1$	$\tilde{\tau}_2$	$\tilde{\tau}_3$
1	$-l_1\dot{\theta}_1^2 + gs_1$	2	5	2
2	$l_1\ddot{\theta} + gc_1$	3	5	2
3	$-l_2(\dot{\theta}_1 + \dot{\theta}_2)^2$	1	2	1
4	$l_2(\ddot{\theta}_1 + \ddot{\theta}_2)$	1	3	1
5	$l_3(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3)$	0	1	1
6	$-l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2$	0	1	0

Table 5.4: Repetitive terms in a 3R cooperative manipulator dynamic equation

few computational terms repeating several times. For example, the term $l_1\ddot{\theta}_1 + \cos(\theta_1)g$ repeats thrice in $\tilde{\tau}_1$, 5 times in $\tilde{\tau}_2$, and twice in $\tilde{\tau}_3$ equation in a 3R planar manipulator. If we identify these terms and avoid repetitive computation, we can reduce the computation cost associated with each of the joints. Table 5.4 lists a few terms which appear multiple times in the dynamic equation.

Link	C_{Td}	C_{TC}	C'_{TC}
1	480	118	38
2	362	260	114
3	102	102	38
Total	944	480	190

Table 5.5: Number of computation involved in each degree of freedom of a 3R planar manipulator with distributed control, cooperative control, and cooperative control with reduced computation by identifying repetitive terms.

Table 5.5 shows computational cost associated with each joint level controllers for a 3R manipulator, with the simple distributed control scheme

D.O.F	C_C	C_{TC}	C'_{TC}
2	208	118 (57%)	51 (24.5%)
3	916	260 (28%)	106 (11.5%)
4	3147	547(17.4%)	194 (6%)
5	9104	1124 (12.3%)	401(4.4%)
6	23646	2281 (9.7%)	801(3.4%)

Table 5.6: Comparative computational cost: Centralized vs cooperative model based control schemes

(C_{TD}), cooperative control scheme proposed in this paper (C_{TC}), and with the cooperative control scheme by avoiding duplicate computation of identified repetitive terms (C'_{TC}). Here, the maximum computational cost (which decides the sampling time) with the basic cooperative control scheme is 260 units, (that is, 54.1%) as against 480 units with the basic distributed control scheme) and further reduces to 114 units (that is 23.75%) after the exercise of avoiding repeated computation. Thus, we can successfully reduce the computational cost of control, first by exploiting the nature of manipulator dynamics, and then by a careful computational optimization technique of identifying repetitive terms and avoiding duplicate computations.

Table 5.6 provides a comparison of computational cost with the centralized model-based control scheme and the proposed cooperative control scheme, for planar manipulators with different degrees of freedom. Here C_C is the computational cost involved with computation of system dynamics for centralized (conventional) model-based control scheme, $C_{TC} = max_i(C_{Ti})$, with the proposed cooperative control scheme, and $C'_{TC} = max_i(C'_{TC_i})$, that with cooperative control scheme after avoiding repetitive computation. It may be observed that cooperative control scheme leads to substantial reduction in the computation cost, which is further reduced by the exercise of identifying repetitive terms and avoiding duplicate computation.

5.4 Discrete implementation of cooperative control scheme

As discussed in Chapter 4, in reality, a control law is implemented using a computer digitally. Thus the control law now takes the discrete form while the robot dynamics remains a continuous time system. The model-based control law in discrete time is given by,

$$
\begin{aligned}
\tau(t) &= M(\theta(t-T_d)\ddot{\theta}_d(t-T_d) + K_v(t-T_d)\dot{E}(t-T_d) + \\
& \quad K_pE(t-T_d)) + C(\theta(t-T_d), \dot{\theta}(t-T_d))
\end{aligned} \tag{5.18}
$$

Where, T_d is the time delay introduced due to the sampling time. The sampling time T_d depends on the time required to compute the control law Eqn. (4.7) along with any other processing required. Note that with the discrete control law given in Eqn. (4.7), feedback linearization is not achieved unless $T_d = 0$. However, due to continuity of the dynamics of the manipulator and the model-based control law, tracking performance is expected to degrade gracefully with increasing T_d.

Now consider the discrete version of the proposed cooperative control law (5.15) and (5.16)

$$
\begin{aligned}
\tau_3(t) &= \tilde{\tau}_3(t) \\
\tau_2(t) &= \tilde{\tau}_3(t) + \tilde{\tau}_2(t) \\
\tau_1(t) &= \tilde{\tau}_3(t) + \tilde{\tau}_2(t) + \tilde{\tau}_1(t)
\end{aligned} \tag{5.19}
$$

Where,

$$
\begin{aligned}
\tilde{\tau}_3(t) &= \tilde{M}_3(\theta((t-T_d^c))(\ddot{\theta}_d(t-T_d^c) + K_v\dot{E}(t-T_d^c) \\
& \quad +K_pE(t-T_d^c)) + \tilde{V}_3(\theta(t-T_d^c), \dot{\theta}(t-T_d^c)) + \tilde{G}_3(\theta((t-T_d^c)) \\
\tilde{\tau}_2(t) &= \tilde{M}_2(\theta(t-T_d^c))(\ddot{\theta}_d(t-T_d^c) + K_v\dot{E}(t-T_d^c) \\
& \quad +K_pE(t-T_d^c)) + \tilde{V}_2(\theta(t-T_d^c), \dot{\theta}(t-T_d^c)) + \tilde{G}_2(\theta((t-T_d^c)) \\
\tilde{\tau}_1(t) &= \tilde{M}_1(\theta(t-T_d^c))(\ddot{\theta}_d(t-T_d^c) + K_v\dot{E}(t-T_d^c) \\
& \quad +K_pE(t-T_d^c)) + \tilde{V}_1(\theta(t-T_d^c), \dot{\theta}(t-T_d^c)) + \tilde{G}_1(\theta((t-T_d^c))
\end{aligned} \tag{5.20}
$$

Here, T_d^c is the time delay due to sampling time in the discrete time implementation of the distributed cooperative control law given by Equations (5.15) and (5.16). As in the case of the discrete time implementation of the (centralized) model-based control law (Eqn. (5.18), the discrete time distributed control law (Equations (5.19) and (5.20)) too fail to achieve feedback linearization, unless the sampling time (T_d^c) is zero. Further as noted in the case of the model-based control scheme, the performance with the discrete time distributed cooperative control scheme degrades gracefully with the increase in T_d^c. With the distribution of computational load in the proposed distributed control scheme, it is expected that $T_d^c < T_d$. Thus, we may expect the trajectory tracking performance with the proposed control scheme to be at least marginally better than the conventional (centralized) model-based control scheme, when implemented digitally.

CHAPTER 6

Results and discussions

In this chapter we present the results of analysis and simulation experiments carried out and discuss their implications. First we provide results of analysis of the computational cost associated with model-based control law in centralized and distributed architectures, carried out using Maple, and establish the advantage of the proposed distributed control schemes. Then we present the results of simulation experiments carrier out to demonstrate the trajectory tracking performance of the distributed model-based control schemes.

6.1 Computational cost: Centralized vs Distributed control

Now we provide results of analysis of the computational cost associated with the model-based control law for planar manipulators with revolute joints, comparing the cost with the centralized and the distributed control architectures. The analysis was carried out using Maple, and sample results obtained from the Maple is provided in the Appendix. In this chapter, we provide the results comprehensively, though some of the results have already been presented in earlier chapters for the purpose of illustrations.

First we consider a two-link (2R) planar manipulator. We illustrate the details of the analysis for this 2R manipulator. The dynamics of a 2R manipulator is:

$$\begin{aligned}
\tau_1 &= m_2 l_2(l_2^2(\ddot{\theta}_1+\ddot{\theta}_2) - s_2(-l_1\dot{\theta}_1^2 + s_1 g)^2 + c_2(l_1\ddot{\theta}_1 + c_1 g)) \\
&\quad + m_1 l_1(l_1\ddot{\theta}_1 + c_1 g + m_2 l_1 s_2((-l_2\dot{\theta}_1 + \dot{\theta}_2)^2 \\
&\quad + c_2(-l_1\dot{\theta}^2 + s_1 g) + s_2(l_1\ddot{\theta}_1 + c_1 g)) + c_2 m_2(l_2(\ddot{\theta}_1\ddot{\theta}_2) \\
&\quad - s_2(-l_1\dot{\theta}_1^2 + s_1 g) + c_2(l_1\ddot{\theta}_1 + c_1 g)) \\
\tau_2 &= m_2 l_2(l_2^2(\ddot{\theta}_1+\ddot{\theta}_2) - s_2(-l_1\dot{\theta}_1^2 + s_1 g)^2 + c_2(l_1\ddot{\theta}_1 + c_1 g)
\end{aligned} \tag{6.1}$$

Here, m_i and l_i are the mass and length of the ith link, $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$.

Now let us find N_A, the number of additions, and N_M, the number of multiplications, involved in computing τ_1 and τ_2 in Eqn. (6.1). Here, we do

Torque	N_A	N_M	C_T
τ_1	19	36	163
τ_2	5	10	45
Total			208

Table 6.1: Number of computations in a two-link planar Manipulator

not consider trigonometric operations as they appear exactly once or at most twice in form of cosine and sine of the joint variables. Further, for the purpose of comparison, as discussed earlier, we assume that the computational cost of addition is 1 unit and that of multiplication is 4 units. Table 6.1 shows the number N_A, N_M, and C_T, the computational cost ($C_T = N_A + 4N_M$) for links 1 and 2. From the Table 6.1, we observe that the computational cost associated with the first joint, $C_{T1} = 163$ units and that of the second joint, $C_{T2} = 45$ units, and total cost associated with computation of the dynamics $C_{Tc} = 208$ units. Now, if the control law is implemented by a classical central controller the cost involved in computation is 208 units. If the controller is implemented in the distributed architecture (Chapter 4), cost associated with the first joint-level controller K_1 is 163 units and that with second joint-level controller K_2 is 45 units. Now the distribution effectiveness $\eta_d = \frac{208/2}{163} = 0.64$. With the use of the distributed architecture, ideally we expect the total computational cost to be 104, half that with the centralized architecture, thus reducing the computational lead time by half. However, actual lead time is now that corresponding computational cost of 163 units (in the place of 104 units), which is 78% of that of the centralized control scheme.

Now we carry out similar analysis for planar manipulators with degrees-of-freedom $3-6$. Table 6.2 shows the number of addition (N_A), multiplication (N_M), and the corresponding computational cost (C_T) associated with a three-link planar manipulator (3R). The distribution effectiveness (η_d) for the 3R manipulator is obtained as 0.66.

Table 6.3 shows the number of addition (N_A), multiplication (N_M), and the corresponding computational cost (C_T) associated with a four-link planar manipulator (4R). In the case of a 4R manipulator we obtain $\eta_d = 0.69$

Torque	N_A	N_M	C_T
τ_1	64	104	480
τ_2	50	78	362
τ_3	14	22	102
Total			944

Table 6.2: Number of computations in a three-link planar Manipulator

Torque	N_A	N_M	C_T
τ_1	166	244	1142
τ_2	152	218	1024
τ_3	116	162	764
τ_4	33	46	217
Total			3147

Table 6.3: Number of computations in a four-link planar Manipulator

The number of addition (N_A), multiplication (N_M), and the corresponding computational cost (C_T) associated with a five-link planar manipulator (5R) are shown in the Table 6.4. We obtain $\eta_d = 0.72$ for the 5R manipulator.

Finally, Table 6.5 shows the number of addition (N_A), multiplication (N_M), and the corresponding computational cost (C_T) associated with a six-link planar manipulator (6R). For the 6R manipulator we obtain $\eta_d = 0.75$. As we considered planar manipulators which operates in $\mathbb{R}^2$, the 2-dimensional Euclidean space, the tool requires two spacial degrees of freedom and one orientational degree-of-freedom. Thus, a 6R planar manipulator, in fact any planar manipulator with degrees-of-freedom more than 2, is a redundant manipulator.

Table 6.6 summarizes the results for computational cost associated with

Torque	N_A	N_M	C_T
τ_1	385	528	2497
τ_2	371	502	2379
τ_3	335	446	2119
τ_4	252	330	1572
τ_5	72	94	448
Total			9015

Table 6.4: Number of computations in a five-link planar Manipulator

Torque	N_A	N_M	C_T
τ_1	841	1100	5241
τ_2	827	1074	5123
τ_3	791	1018	4863
τ_4	708	902	4316
τ_5	528	666	3192
τ_6	151	190	911
Total			23646

Table 6.5: Number of computations in a six-link planar Manipulator

N	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	C_T	η_d
2	163	45					208	0.63
3	480	362	102				944	0.66
4	1142	1024	764	217			3147	0.69
5	2497	2379	2119	1572	448		9015	0.72
6	5241	5123	4863	4316	3192	911	23646	0.75

Table 6.6: Computational cost in planar manipulators with different degrees-of-freedom.

the dynamics of manipulators with degrees-of-freedom from $2 - 6$. One important observation from the above analysis, as shown in Table 6.7, is that the distribution effectiveness η_d monotonically improves with N, the degrees-of-freedom of the manipulator. We may expect that in the limit $\eta_d \rightarrow 1$, the ideal value. Further, note that the distributed control architecture is useful for higher degrees-of-freedom manipulators. The analysis carried out here justifies the distributed control architecture for a redundant or hyper-redundant manipulator (such as a snake robot, for example), as the computational lead time is gets drastically reduced due to higher distribution effectiveness, where the computational load is shared more uniformly among multiple joint-level controllers.

Now we present consolidate result for computational cost with the cooperative control scheme presented in Chapter 5. Table 6.8 shows the cost at each joint for planar manipulators with degrees-of-freedom varying from 2 to 6. We may observe that relative maximum computational cost with the cooperative control scheme presented in Chapter 5 as compared with that with the

DOF	η_d
2	0.64
3	0.66
4	0.69
5	0.72
6	0.75

Table 6.7: Distribution effectiveness with the degrees-of-freedom of planar manipulators.

N	$\tilde{\tau}_1$	$\tilde{\tau}_2$	$\tilde{\tau}_3$	$\tilde{\tau}_4$	$\tilde{\tau}_5$	$\tilde{\tau}_6$	$\max C_{TD}$	$\max C_{TC}$
2	118	45					163	118 (72%)
3	118	260	102				480	260 (54%)
4	118	260	547	217			1142	547 (47%)
5	118	260	547	1124	448		2497	1124 (45%)
6	118	260	547	1124	2281	911	5241	2281 (43%)

Table 6.8: Computational cost in planar manipulators with different degrees-of-freedom.

distributed control scheme presented in Chapter 4, reduces monotonically with increase in the degrees-of-freedom of the manipulator (from (72%) for 2R manipulator to (43%) for the 6R manipulator). This indicates that the cooperative control scheme proposed in Chapter 5 is more effective for the manipulators with higher degrees-of-freedom.

6.1.1 Reducing the computational cost

In this section we discuss reduction in computation cost associated with the manipulator dynamics in the model-based control law (4.4) without any approximation.

If we carefully observe the dynamic equations of a manipulator, we can identify several terms that appear repetitively. Here, we try to identify such terms and avoid repetitive computation of these terms thereby reducing the computational cost involved with dynamic equations in the control laws Eqn.s (4.2) and (4.4). We perform this exercise for planar manipulators with degrees-of-freedom ranging from 3 to 6 with aid of Maple.

Repeating terms	τ_1	τ_2	τ_3
$l_1\dot{\theta}^2 + s_1 g$	9	7	2
$l_1\ddot{\theta} + c_1 g$	10	7	2
$l_2(\ddot{\theta}_1 + \ddot{\theta}_2)^2$	4	3	1
$l_2(\dot{\theta}_1 + +\dot{\theta}_2)^2$	5	4	1
$l_3(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3)$	2	2	1
$l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2$	1	1	0

Table 6.9: Repetitive terms in 3R Manipulator.

Equation	N_A	N_M	C_T	**Rel** C_T
τ_1	30	40	190	0.4
τ_2	24	32	152	0.42
τ_3	6	8	38	0.37
Total			380	0.4

Table 6.10: Number of computations and total cost after avoiding repetitive computation in 3R Manipulator

The terms that repeat in the dynamic equation of a $3R$ along with the number of times they appear in equation corresponding to each joint level equation are shown in Table 6.9. Here for example, the term $l_1\dot{\theta}^2 + s_1 g$ appears 9 times in the τ_1 equation, 7 times in the τ_2 equation and twice in the τ_3 equation. The term $l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2$ appears twice, once in equation corresponding to τ_1 and once in that corresponding to τ_2. If instead of computing these terms multiple times, we compute them once and reuse whenever they appear, we can reduce both total (centralized architecture) computational cost an that at the joint level (distributed architecture). Table 6.10 shows the computational cost at each joint level with this exercise of avoiding repetitive computation. The computational cost at each joint level and also the total computational cost is about 40% ($37 - 42\%$, to be precise) of the corresponding cost without this exercise of avoiding repetitive computation as shown in Table 6.2. However, in spite of reduction in computational cost the distribution effectiveness (0.67) does not change significantly.

Result of a similar exercise carried out for a 4R manipulator is shown in Tables 6.11 and 6.12. The repetitive terms and number of times they appear in each of the four joint torque equations are listed in Table 6.11. Corresponding cost is shown in Table 6.12. Even in this case the cost in Table 6.12 is about 40%

Repeating terms	τ_1	τ_2	τ_3	τ_4
$l_1\dot{\theta}^2 + s_1 g$	21	19	14	4
$l_1\ddot{\theta} + c_1 g$	22	19	14	4
$l_2(\ddot{\theta}_1 + \ddot{\theta}_2)$	11	10	7	2
$l_2(\dot{\theta}_1 + +\dot{\theta}_2)^2$	10	9	7	2
$l_3(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3)$	5	5	4	1
$l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2$	5	5	4	1
$l_4(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3 + \ddot{\theta}_4)$	2	2	2	1
$l_4(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4)^2$	1	1	1	0

Table 6.11: Repetitive terms in 4R Manipulator

Equation	N_A	N_M	C_T	**Rel** C_T
τ_1	75	89	431	0.38
τ_2	68	78	380	0.37
τ_3	51	57	279	0.37
τ_4	17	17	85	0.39
Total			1175	0.37

Table 6.12: Number of computations and total cost after avoiding repetitive computation in 4R Manipulator

($37 - 39\%$, to be precise) of that in Table 6.3, and $\eta_d = 0.68$.

Table 6.13 shows the identified repetitive terms in a 5R manipulator and the corresponding reduced computational cost is listed for each joint in the Table 6.14. The cost shown in Table 6.14 is about $36 - 37\%$ of that shown in Table 6.4, with $\eta_d = 0.71$.

Finally we show results for the 6 degrees-of-freedom planar manipulator in Tables 6.15 and 6.16. We found 12 repetitive terms in this case appearing $418, 419, 210, 208, 105, 102, 52, 50, 24, 21, 11$, and 5 times. The exercise of avoiding repetitive computation of these terms leads to a cost reduction as shown in Table 6.16. The cost as shown in Table 6.16 was found to be $35 - 36\%$ of that shown in Table 6.5. The value of distribution effectiveness was found to be 0.75. We have provided the detailed results obtained with the aid of Maple program to identify repetitive terms in computing τ_1 for equation corresponding to the first joint (τ_1) for degrees-of-freedom $2 - 6$, in Appendix A3.

The result after the exercise of reduction in computational cost by avoiding

Repeating terms	τ_1	τ_2	τ_3	τ_4	τ_5
$l_1\dot{\theta}^2 + s_1 g$	45	43	38	28	8
$l_1\ddot{\theta} + c_1 g$	46	43	38	28	8
$l_2(\ddot{\theta}_1 + \ddot{\theta}_2)$	23	22	18	13	4
$l_2(\dot{\theta}_1 + +\dot{\theta}_2)^2$	22	21	19	14	4
$l_3(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3)$	11	11	10	7	2
$l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2$	10	10	9	7	2
$l_4(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3 + \ddot{\theta}_4)$	5	5	5	4	1
$l_4(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4)^2$	4	4	4	3	1
$l_5(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3 + \ddot{\theta}_4 + \ddot{\theta}_5)$	2	2	2	2	1
$l_5(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 + \ddot{\theta}_5)^2$	1	1	1	1	0

Table 6.13: Repetitive terms in 5R Manipulator

Equation	N_A	N_M	C_T	**Rel** C_T
τ_1	168	186	912	0.37
τ_2	161	175	861	0.36
τ_3	144	154	760	0.36
τ_4	107	113	559	0.36
τ_5	30	32	158	0.35
Total			3250	0.36

Table 6.14: Number of computations and total cost after avoiding repetitive computation in 5R Manipulator

Repeating terms	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
$l_1\dot{\theta}^2 + s_1 g$	93	91	86	76	56	16
$l_1\ddot{\theta} + c_1 g$	94	91	86	76	56	16
$l_2(\ddot{\theta}_1 + \ddot{\theta}_2)$	47	46	43	38	28	8
$l_2(\dot{\theta}_1 + +\dot{\theta}_2)^2$	46	45	43	38	28	8
$l_3(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3)$	23	23	22	19	14	4
$l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2$	22	22	21	19	14	4
$l_4(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3 + \ddot{\theta}_4)$	11	11	11	10	7	2
$l_4(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4)^2$	11	11	10	9	7	2
$l_5(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3 + \ddot{\theta}_4 + \ddot{\theta}_5)$	5	5	5	5	3	1
$l_5(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 + \ddot{\theta}_5)^2$	4	4	4	4	4	1
$l_6(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3 + \ddot{\theta}_4 + \ddot{\theta}_5 + \ddot{\theta}_6)$	2	2	2	2	2	1
$l_6(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 + \dot{\theta}_5 + \dot{\theta}_6)^2$	1	1	1	1	1	0

Table 6.15: Repetitive terms in 6R Manipulator

Equation	N_A	N_M	C_T	Rel C_T
τ_1	357	379	1873	0.36
τ_2	350	368	1822	0.36
τ_3	333	347	1721	0.35
τ_4	296	306	1520	0.35
τ_5	219	225	1119	0.35
τ_6	62	64	318	0.35
Total			8373	0.35

Table 6.16: Number of computations and total cost after avoiding repetitive computation in 6R Manipulator

repetitive computation is summarized in Table 6.17 for planar manipulators with degrees-of-freedom $3 - 6$. The observation based on this table is that the computational cost after the exercise of avoiding repetitive computation is about 40% that without it, in general. Also, the distribution effectiveness does not change significantly. Thus, though this exercise leads to a substantial reduction in computational cost, does not affect how the computational load is distributed among the joint-level controllers.

DOF	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	C_T	Rel C_T	η_d
3	190	152	38				380	0.4	0.67
4	431	380	279	85			1175	0.37	0.68
5	912	861	760	559	158		3250	0.36	0.71
6	1873	1822	1721	1520	1119	318	8373	0.35	0.75

Table 6.17: Computational cost in planar manipulators with different degrees-of-freedom after avoiding repetitive computation.

Similar results as in Table 6.17 is shown in Table 6.18 for the cooperative control scheme presented in Chapter 5. Here too, as observed in Table 6.8, we may observe that relative maximum computational cost with the cooperative control scheme presented in Chapter 5 as compared with that with the distributed control scheme presented in Chapter 4, reduces monotonically with increase in the degrees-of-freedom of the manipulator (from (60%) for 3R manipulator to (43%) for the 6R manipulator).

N	$\tilde{\tau}_1$	$\tilde{\tau}_2$	$\tilde{\tau}_3$	$\tilde{\tau}_4$	$\tilde{\tau}_5$	$\tilde{\tau}_6$	$\max C_{TD}$	$\max C_{TC}$
3	38	114	38				190	114 (60%)
4	51	101	194	85			431	194 (45%)
5	51	101	201	401	158		912	401 (44%)
6	51	101	201	401	801	318	1873	801 ((43%)

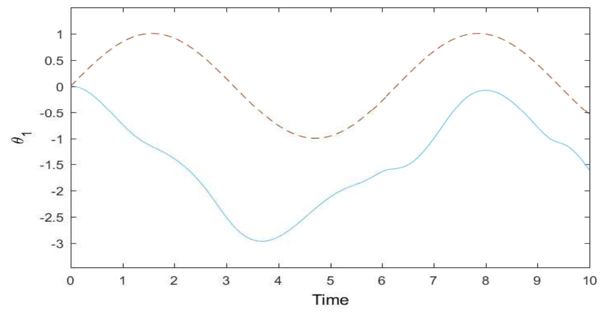
Table 6.18: Computational cost in planar manipulators using the cooperative control scheme with different degrees-of-freedom after avoiding repetitive computation.

6.2 Matlab simulations

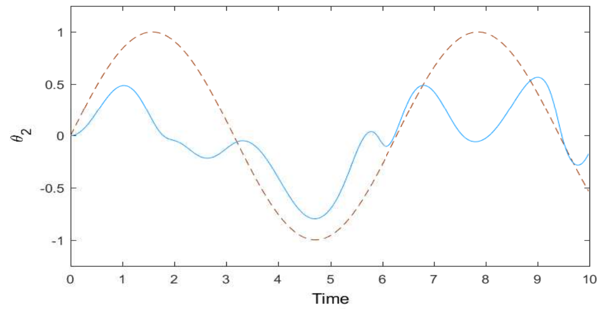
Now we present the results of simulations experiments carried out to demonstrate the trajectory tracking performance of a 3R planar manipulator using the proposed distributed control schemes carried out using Matlab Simulink.

Recollect that the closed-loop error dynamics with the proposed distributed control schemes, namely, the simple distributed control and the cooperative control, is identical to that with the conventional model-based control scheme. Thus, simulation results for all these schemes are expected to be identical. For a 3R planar manipulator, we have implemented the the proposed simple distributed control scheme as shown in Figure 4.5, the cooperative control scheme as shown in Figure 5.1, and the conventional (centralized) model-based control scheme as shown in Figure 1.3 in Simulink. As expected, the results of all three control schemes were identical. We considered a manipulator with $m_1 = 10kg$, $m_2 = 10kg$, and $m_3 = 10kg$, and $l_1 = 5m, l_2 = 6m$, and $l_3 = 5m$. We considered a step and sinusoidal signals as desired trajectories at the joint levels.

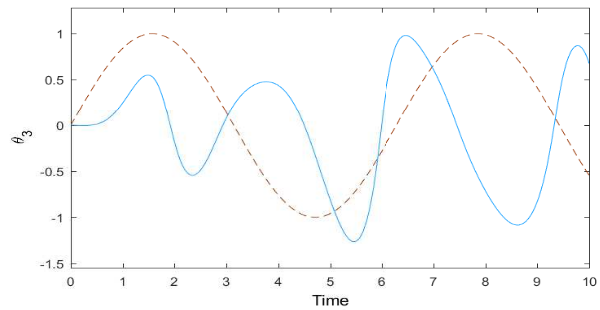
First we show the results with the independent joint PID control scheme. Figures 6.1(a)-(c) show the trajectory tracking performance of the first, second, and third joints of the manipulator with the decentralized (independent joint) PID controller. Dashed lines show the desired (sinusoidal) trajectories and the solid lines show the actual trajectories. It may be noted that the exact tracking performance varies with the desired trajectory to be tracked due to nonlinearity of manipulator and the closed-loop dynamics, apart from the values of controller gains chosen.



(a)



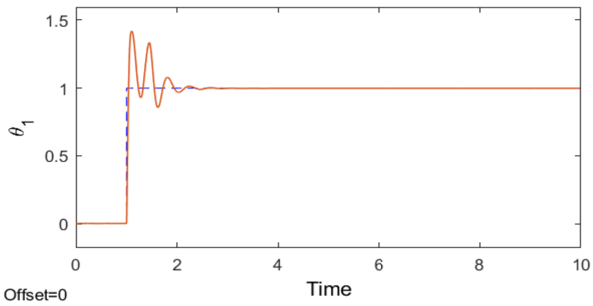
(b)



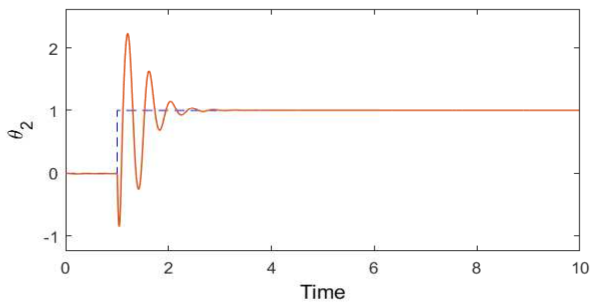
(c)

Figure 6.1: Trajectory tracking performance of a) first, b) second joint, and c) third joint, of a $3R$ planar manipulator with independent joint PID controller. Dashed lines show the desired trajectories and the solid lines show the actual trajectories. Angles $(\theta_1, \theta_2, \theta_3)$ are in radians and time is in seconds.

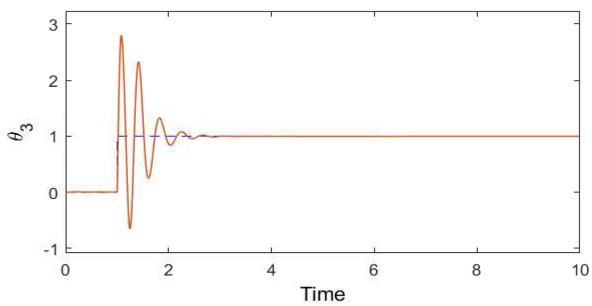
Now we present a set of simulation results carried out using the model-based control schemes, which are applicable for both the centralized and distributed scheme. Step response of (a) joint 1, (b) joint 2, and (c) joint 3 of a 3R manipulator using the model-based control is shown in Figures 6.2(1)-(c).



(a)



(b)

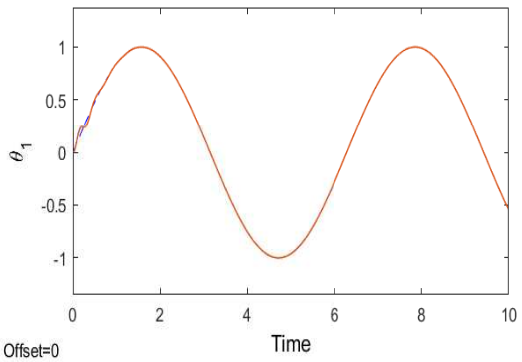


(c)

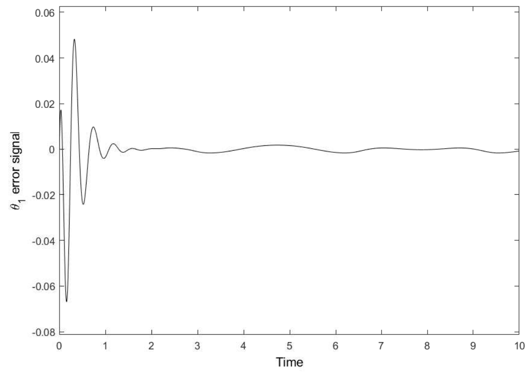
Figure 6.2: Step response of (a) joint 1, (b) joint 2, and (c) joint 3 of a 3R manipulator using the model-based control. Angles $(\theta_1, \theta_2, \theta_3)$ are in radians and time is in seconds.

Figure 6.3 shows trajectory tracking performance of a 3R manipulator with model-based control in continuous time with slow varying sinusoidal trajectory (frequency of 1rad/s) at each joint. Desired (solid lines) and actual (dashed lines) trajectories are shown in Figures 6.3(a) for joint 1, 6.3(c) for joint 2, and 6.3(e) joint 3. Corresponding error dynamics are shown in Figures 6.3(b), (d), and (f), respectively.

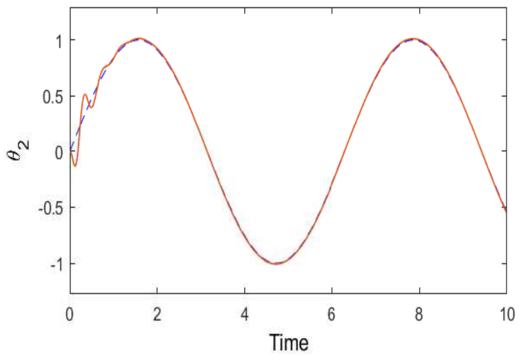
Figure 6.4 shows trajectory tracking performance of a 3R manipulator with



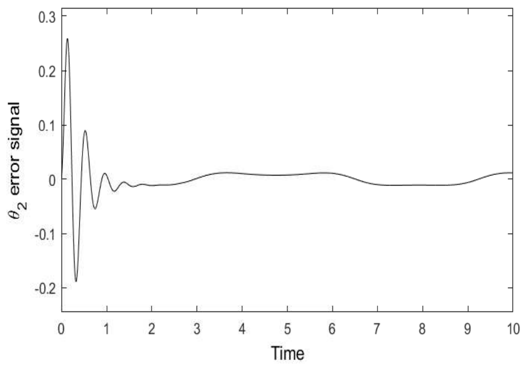
(a)



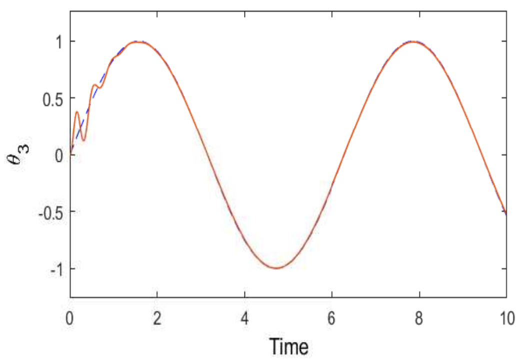
(b)



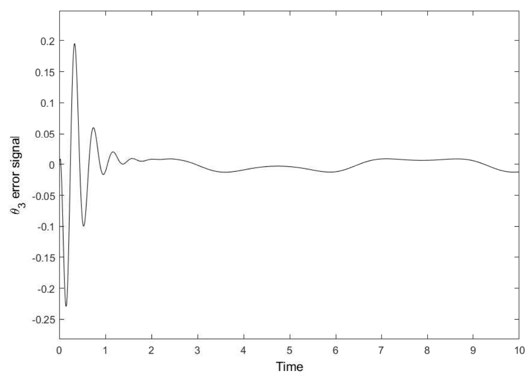
(c)



(d)



(e)



(f)

Figure 6.3: Trajectory tracking performance of a 3R manipulator with model-based control in continuous time with slow varying sinusoidal trajectory at each joint. Desired and actual trajectories are shown for (a) joint 1, (c) joint 2, and (e) joint 3, and corresponding error dynamics are shown in (b), (d), and (f), respectively. Solid lines show the desired trajectories and the dashed lines show the actual trajectories. Angles $(\theta_1, \theta_2, \theta_3)$, errors (e_1, e_2, e_3) are in radians and time is in seconds.

model-based control with low sampling time (0.01 units) and with slow varying sinusoidal trajectory (frequency of 1rad/s) at each joint. Desired (solid lines) and actual (dashed lines) trajectories are shown in Figures 6.4(a) for joint 1, 6.4(c) for joint 2, and 6.4(e) joint 3. Corresponding error dynamics are shown in Figures 6.4(b), (d), and (f), respectively.

Figure 6.5 shows trajectory tracking performance of a 3R manipulator with model-based control with a higher sampling time (0.05 units) and with slow varying sinusoidal trajectory (frequency of 1rad/s) at each joint. Desired (solid lines) and actual (dashed) trajectories are shown in Figures 6.5(a) for joint 1, 6.8(c) for joint 2, and 6.5(e) joint 3. Corresponding error dynamics are shown in Figures 6.5(b), (d), and (f), respectively.

Figure 6.6 shows trajectory tracking performance of a 3R manipulator with model-based control in continuous time with a faster varying sinusoidal trajectory (frequency of 2rad/s) at each joint. Desired (solid lines) and actual (dashed lines) trajectories are shown in Figures 6.3(a) for joint 1, 6.6(c) for joint 2, and 6.6(e) joint 3. Corresponding error dynamics are shown in Figures 6.6(b), (d), and (f), respectively.

Figure 6.7 shows trajectory tracking performance of a 3R manipulator with model-based control with lower sampling time (0.01 units) and with a faster varying sinusoidal trajectory (frequency of 2rad/s) at each joint. Desired (solid lines) and actual (dashed lines) trajectories are shown in Figures 6.7(a) for joint 1, 6.7(c) for joint 2, and 6.7(e) joint 3. Corresponding error dynamics are shown in Figures 6.7(b), (d), and (f), respectively.

Figure 6.8 shows trajectory tracking performance of a 3R manipulator with model-based control with a higher sampling time (0.05 units) and with faster varying sinusoidal trajectory (frequency of 2rad/s) at each joint. Desired (solid lines) and actual (dashed lines) trajectories are shown in Figures 6.8(a) for joint 1, 6.8(c) for joint 2, and 6.8(e) joint 3. Corresponding error dynamics are shown in Figures 6.8(b), (d), and (f), respectively.

If we compare Figures 6.3 and 6.6, Figures 6.4 and 6.7, and Figures 6.5 and

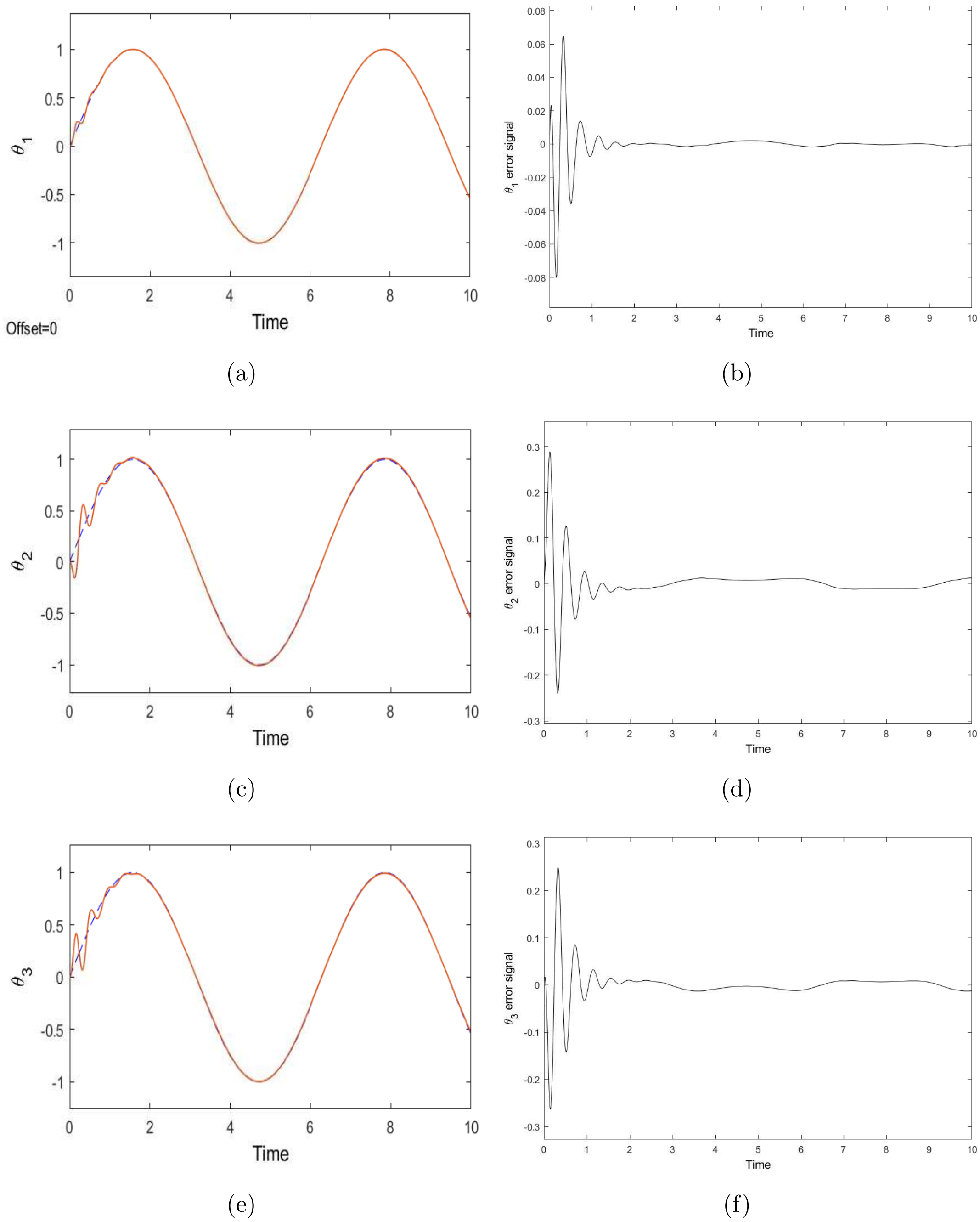


Figure 6.4: Trajectory tracking performance of a 3R manipulator with model-based control with low sampling time (0.01 units) and with slow varying sinusoidal trajectory at each joint. Desired and actual trajectories are shown for (a) joint 1, (c) joint 2, and (e) joint 3, and corresponding error dynamics are shown in (b), (d), and (f), respectively. Solid lines show the desired trajectories and the dashed lines show the actual trajectories. Angles $(\theta_1, \theta_2, \theta_3)$, errors (e_1, e_2, e_3) are in radians and time is in seconds.

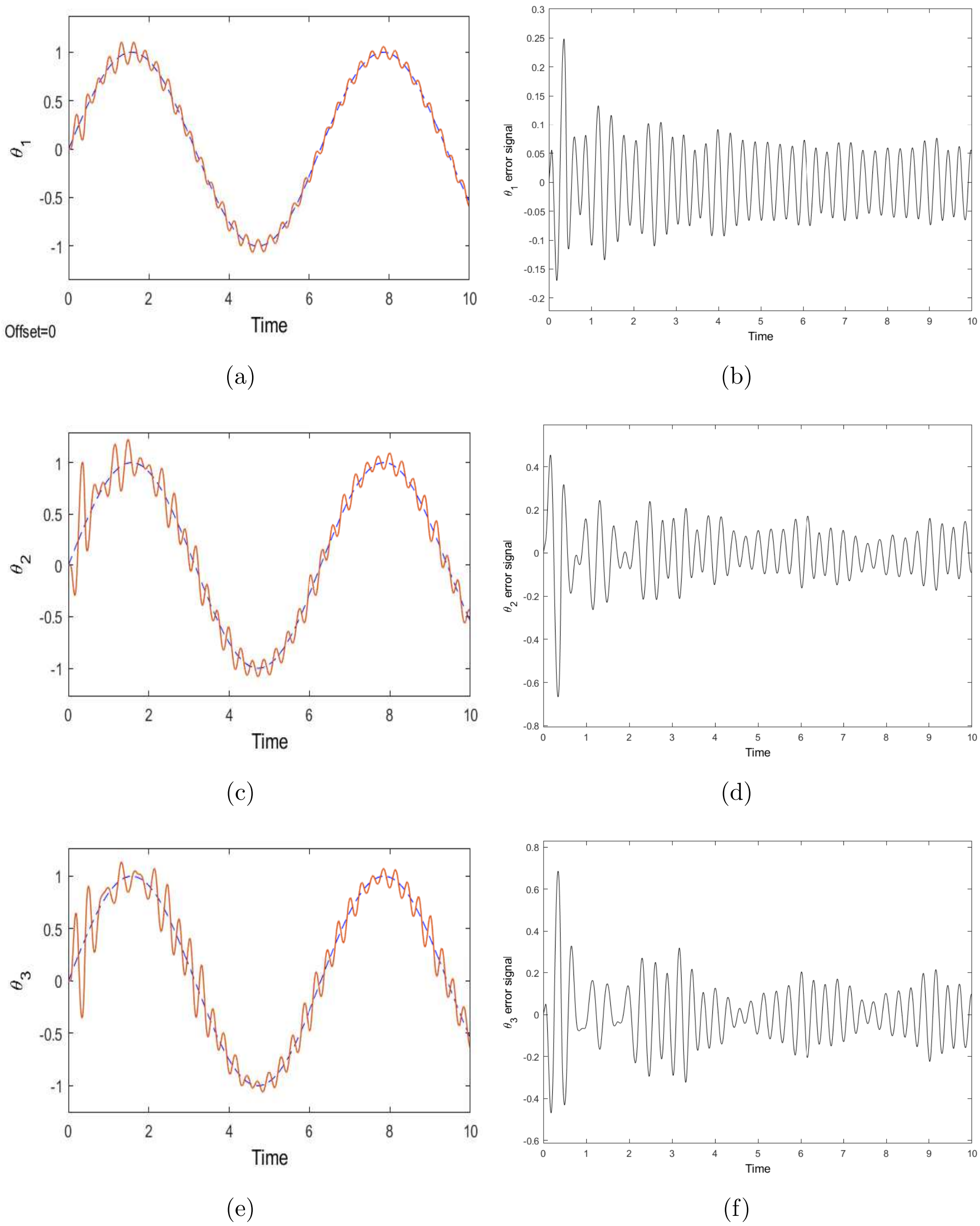
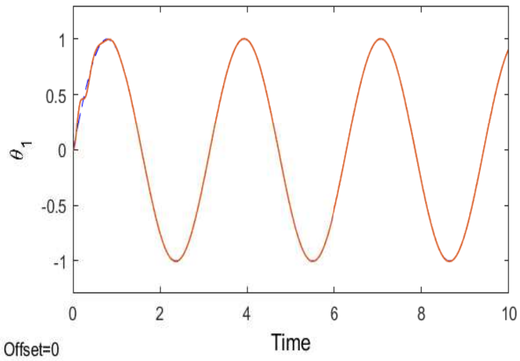
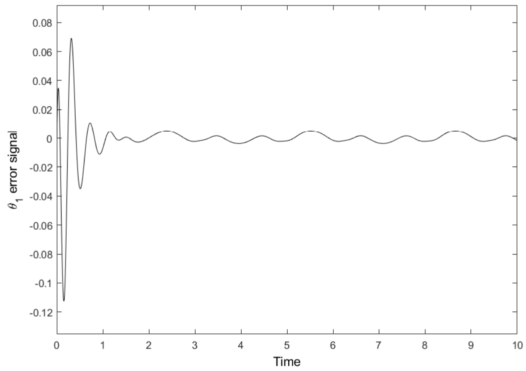


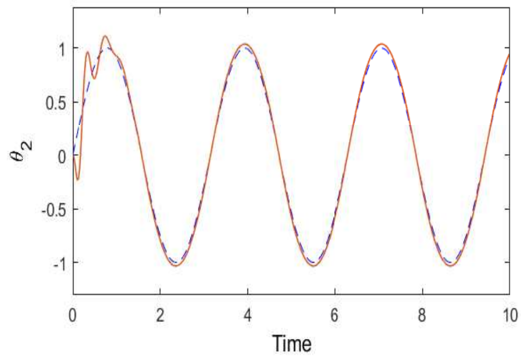
Figure 6.5: Trajectory tracking performance of a 3R manipulator with model-based control with higher sampling time (0.05 units) and with slow varying sinusoidal trajectory at each joint. Desired and actual trajectories are shown for (a) joint 1, (c) joint 2, and (e) joint 3, and corresponding error dynamics are shown in (b), (d), and (f), respectively. Solid lines show the desired trajectories and the dashed lines show the actual trajectories. Angles $(\theta_1, \theta_2, \theta_3)$, errors (e_1, e_2, e_3) are in radians and time is in seconds.



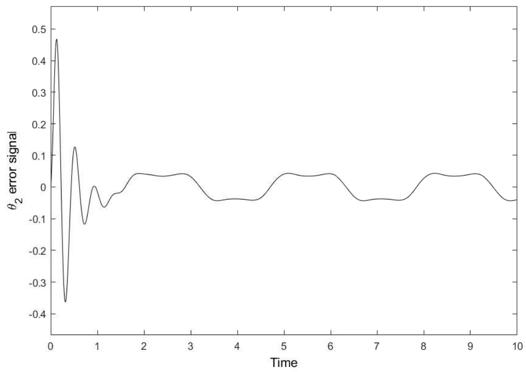
(a)



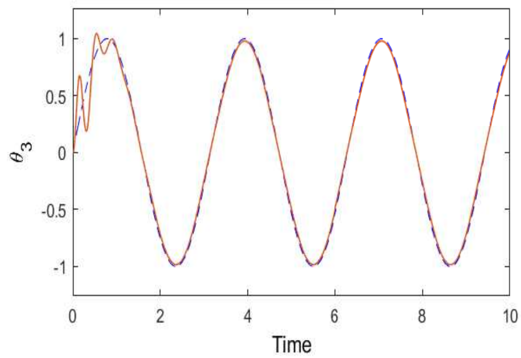
(b)



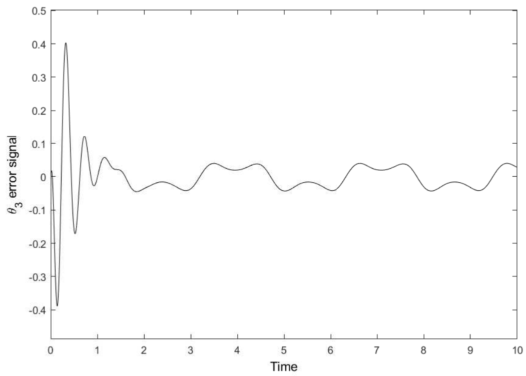
(c)



(d)



(e)



(f)

Figure 6.6: Trajectory tracking performance of a 3R manipulator with model-based control in continuous time and with faster varying sinusoidal trajectory at each joint. Desired and actual trajectories are shown for (a) joint 1, (c) joint 2, and (e) joint 3, and corresponding error dynamics are shown in (b), (d), and (f), respectively. Solid lines show the desired trajectories and the dashed lines show the actual trajectories. Angles $(\theta_1, \theta_2, \theta_3)$, errors (e_1, e_2, e_3) are in radians and time is in seconds.

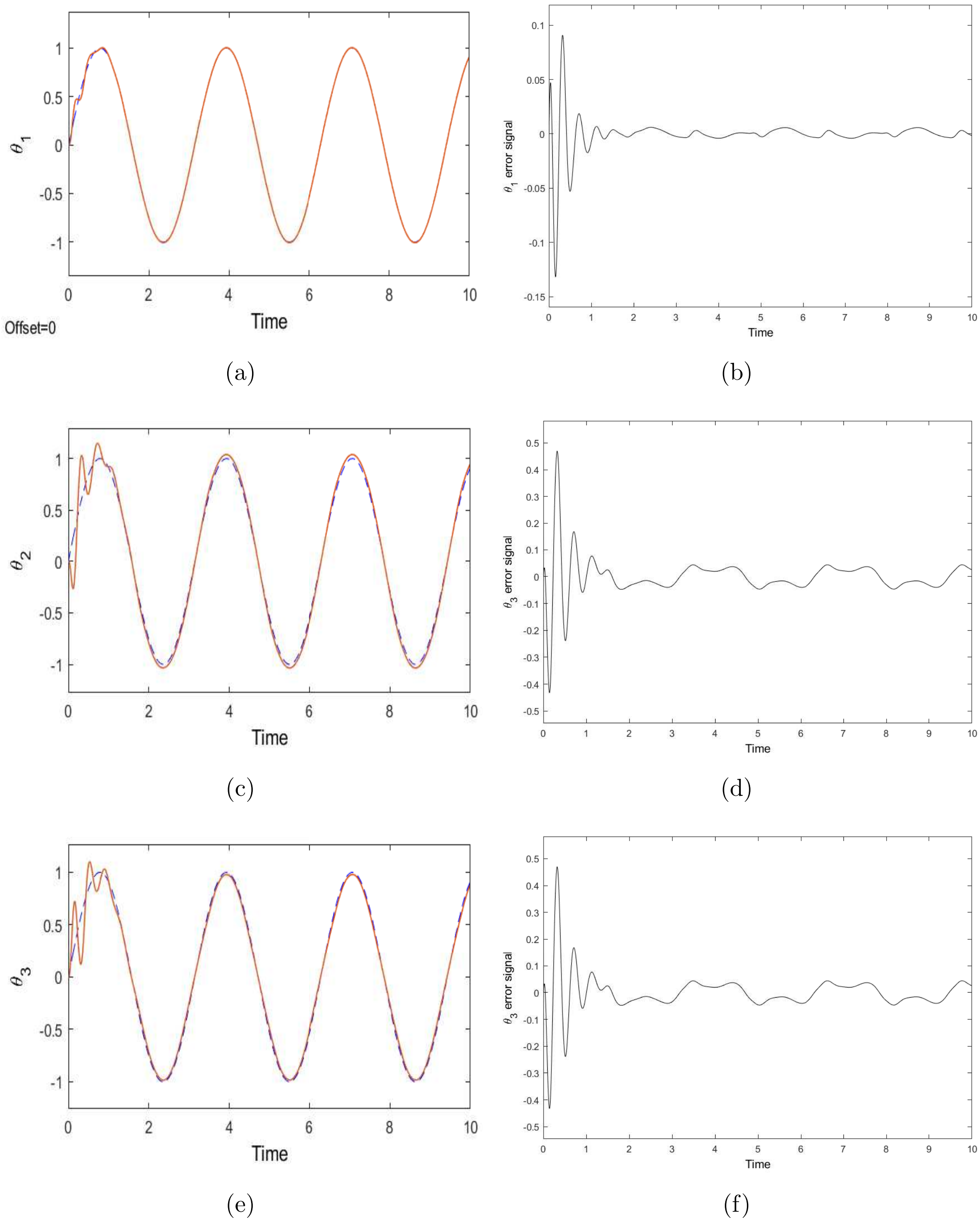
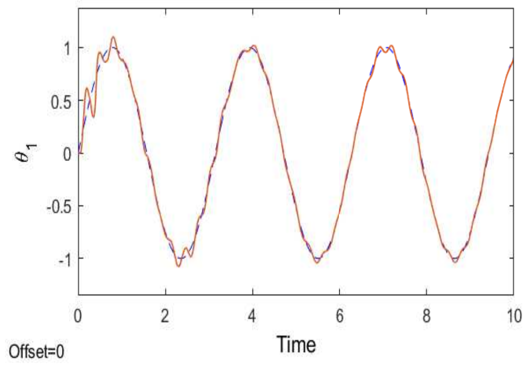
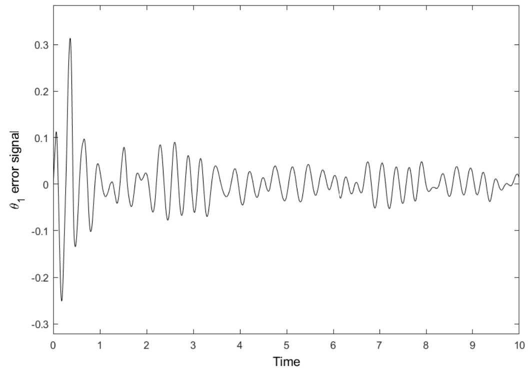


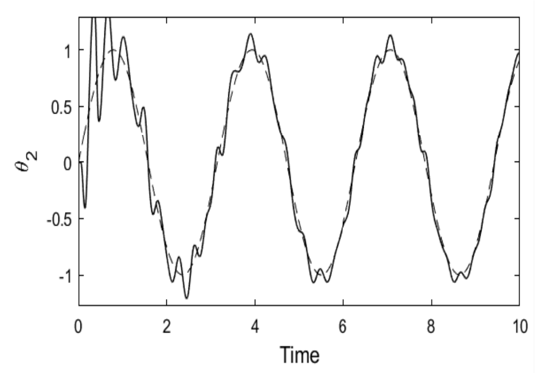
Figure 6.7: Trajectory tracking performance of a 3R manipulator with model-based control with low sampling time (0.01 units) and with faster varying sinusoidal trajectory at each joint. Desired and actual trajectories are shown for (a) joint 1, (c) joint 2, and (e) joint 3, and corresponding error dynamics are shown in (b), (d), and (f), respectively. Solid lines show the desired trajectories and the dashed lines show the actual trajectories. Angles $(\theta_1, \theta_2, \theta_3)$, errors (e_1, e_2, e_3) are in radians and time is in seconds.



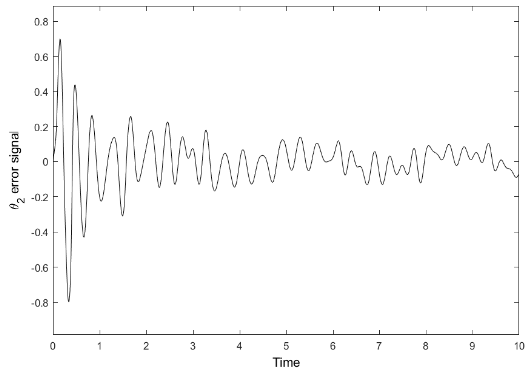
(a)



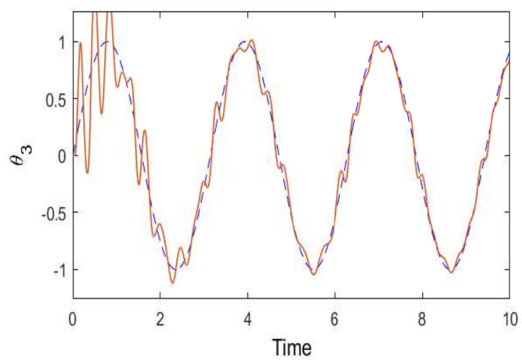
(b)



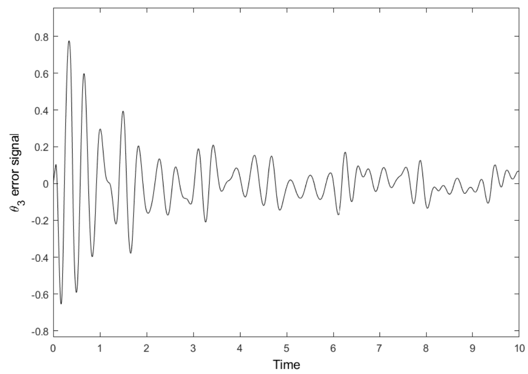
(c)



(d)



(e)



(f)

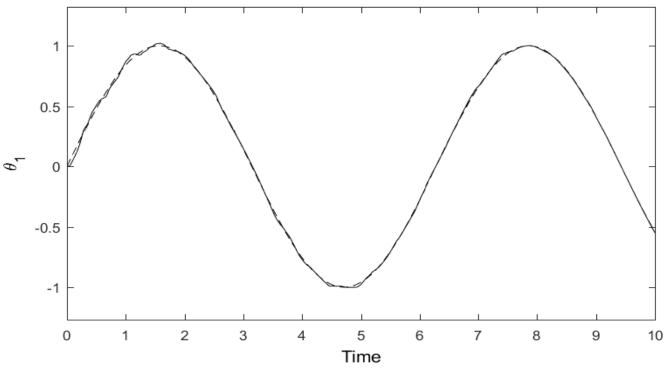
Figure 6.8: Trajectory tracking performance of a 3R manipulator with model-based control with higher sampling time (0.05 units) and with faster varying sinusoidal trajectory at each joint. Desired and actual trajectories are shown for (a) joint 1, (c) joint 2, and (e) joint 3, and corresponding error dynamics are shown in (b), (d), and (f), respectively. Solid lines show the desired trajectories and the dashed lines show the actual trajectories. Angles $(\theta_1, \theta_2, \theta_3)$ and errors (e_1, e_2, e_3) are in radians and time is in seconds.

Figure 6.8, we observe that for a given sampling time, the model-based control schemes lead to similar trajectory tracking performance, with both slow and fast sinusoidal signals (or desired trajectories at the joint level). Further we may observe that in both the cases, the trajectory tracking performance degrades with increase in the sampling time, as discussed in Chapter 4 and 5. However, compared to the trajectory tracking performance with the decentralized PID control scheme as shown in Figure 6.1, as expected, the trajectory tracking performance with the model-based schemes is superior.

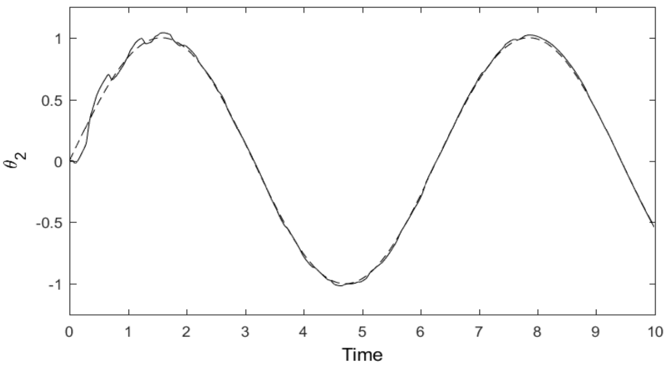
Figures 6.9(a), (b), and (c) show the desired and actual trajectories of a manipulator with the proposed distributed control scheme. Figures 6.10(a), (b), and (c) show the desired and actual trajectories of a manipulator with the proposed distributed cooperative control scheme. These results are shown to demonstrate the trajectory tracking performance with the proposed distributed model-based control schemes are identical to that with the conventional (centralized) model-based control scheme as discussed earlier in this section and Chapters 4 and 5.

6.3 Implementation of the distributed control in ROS environment

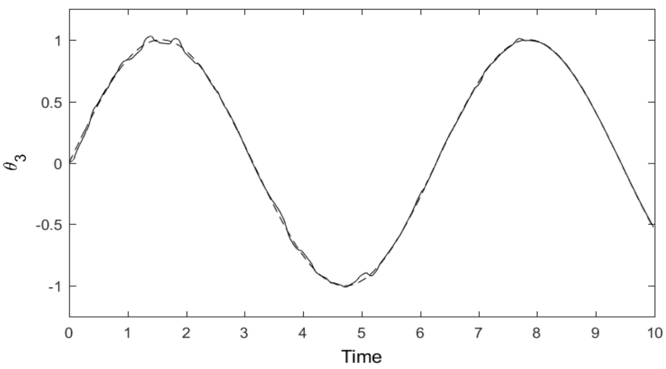
In this section we provide the details of implementation of the proposed distributed control schemes in Simulink-ROS environment. At the simulation level, in contrast to the Matlab-Simulink environment, ROS provides a truly distributed simulation environment. Major advantage of implementing the proposed distributed control scheme within ROS environment is that, the same programs may be implemented on hardware using a physical robot, at least in principle. In this sense with ROS, the simulation is more realistic and a step closer to hardware implementation. We have implemented the proposed distributed control schemes using Simulink-ROS hybrid platform (using Robotics toolbox in Matlab). Here, Simulink handles simulation of manipulator dynamics with control, and ROS handles communication between modules/nodes in form of 'topics'. The block diagrams of the Simulink-ROS implementation are shown



(a)

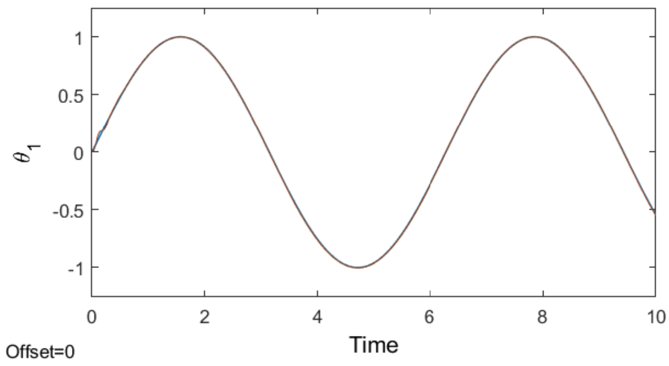


(b)

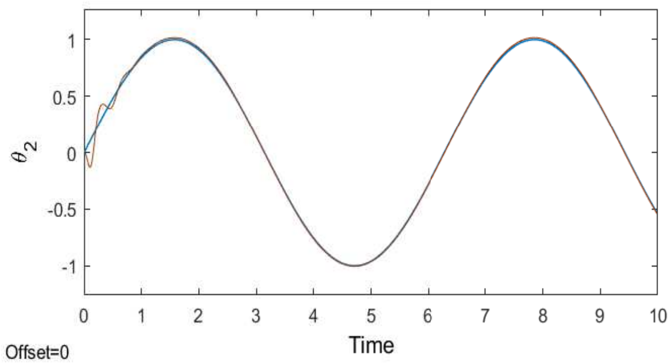


(c)

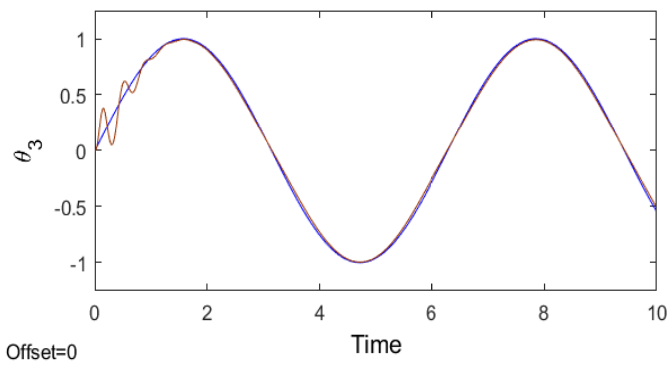
Figure 6.9: Trajectory tracking performance of a) first, b) second joint, and c) third joint, of a $3R$ planar manipulator with the proposed distributed manipulator control scheme. Solid lines show the desired trajectory and the dashed lines show the actual trajectory. Solid lines show the desired trajectories and the dashed lines show the actual trajectories. Angles $(\theta_1, \theta_2, \theta_3)$ are in radians and time is in seconds.



(a)



(b)



(c)

Figure 6.10: Trajectory tracking performance of a) first, b) second joint, and c) third joint, of a $3R$ planar manipulator with the proposed distributed cooperative manipulator control scheme. Solid lines show the desired trajectory and the dashed lines show the actual trajectory. Solid lines show the desired trajectories and the dashed lines show the actual trajectories. Angles $(\theta_1, \theta_2, \theta_3)$ are in radians and time is in seconds.

in Figure 6.11 and 6.12. As expected, the results obtained are identical to that obtained using Matlab-simulink, and hence to avoid duplicity, we skip the results here.

6.4 Distributed vs decentralized schemes:

Now we present an informal discussion comparing the proposed distributed (or centralized) control scheme with the decentralized control scheme in general. The independent joint PID controller is probably the simplest control scheme reported in the literature in the decentralized control architecture. Each of such schemes lead to different trajectory tracking performance, which is expected to be better than that with the independent-joint PID control scheme. However, when the system model is fully available, it has been established theoretically that the trajectory tracking performance with the model-based nonlinear control is superior to that with any other control scheme which do not consider the model fully, particularly the coupled dynamics. In the case of control schemes in decentralized architecture in spite of using adaptive control or other techniques to account for un-modeled dynamics, there is no provision for truly accounting for the dynamic coupling between the links (See Section 3.1.2 in Chapter 3). Apart from this theoretically established fact of leading to inferior trajectory tracking performance compared to that with the model-based control (centralized or distributed), decentralized control schemes proposed in the literature (unlike the simple independent joint PID scheme) involve non-trivial computational overhead. Thus, we may observe that though the decentralized schemes reported in the literature may have marginally lower computational overhead as compared to the proposed model-based control in the distributed architecture, their trajectory performance is expected to be inferior (at least in a theoretical sense and ideal situations) to that with the distributed scheme proposed in this work.

Though we consider the manipulator dynamics is known completely in this work, it may not be the case in reality. When the model is not known exactly, it is possible to use techniques such as adaptive control within the distributed control

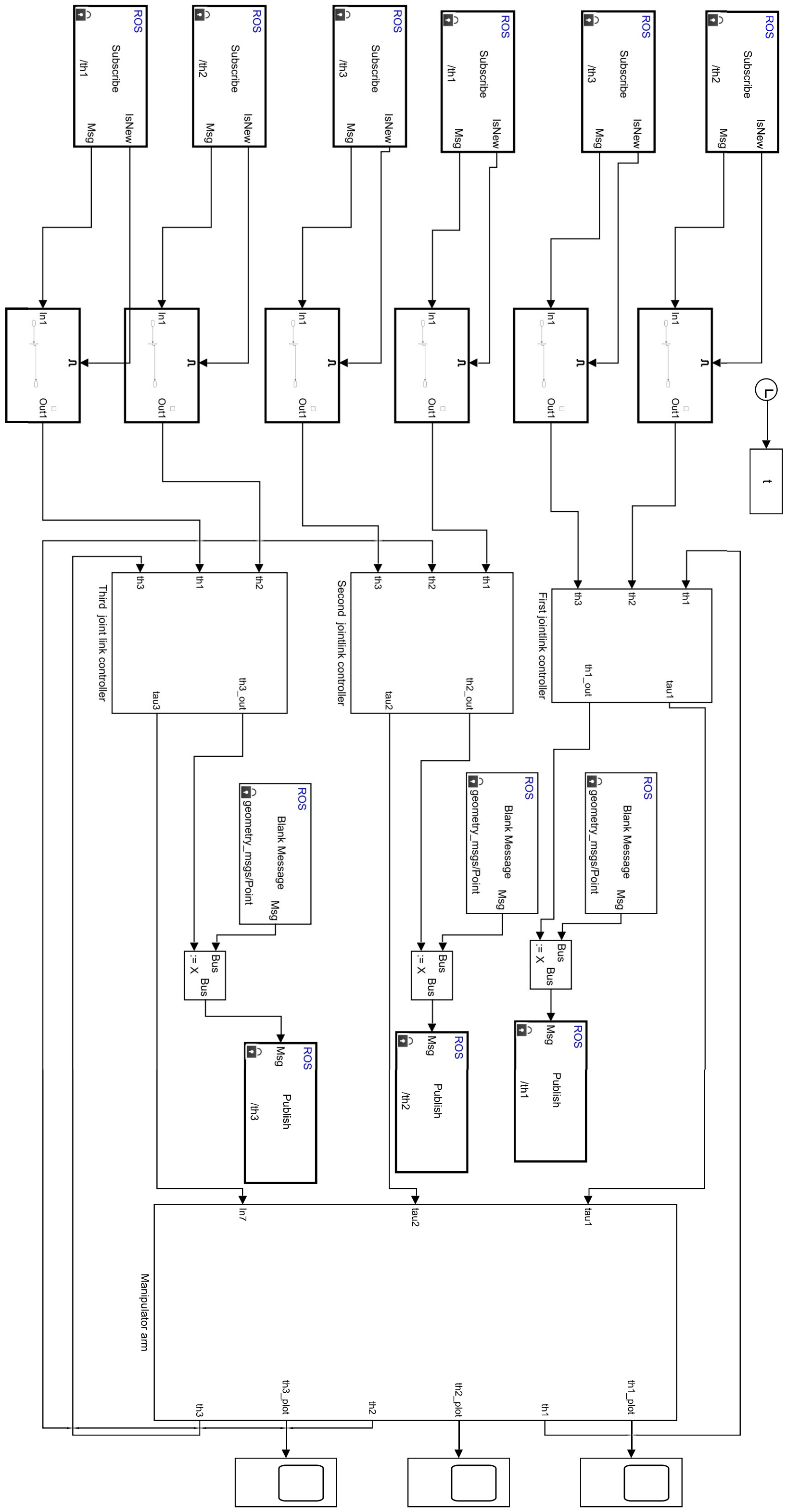


Figure 6.11: ROS-Simulink simulation of distributed control of a 3R manipulator.

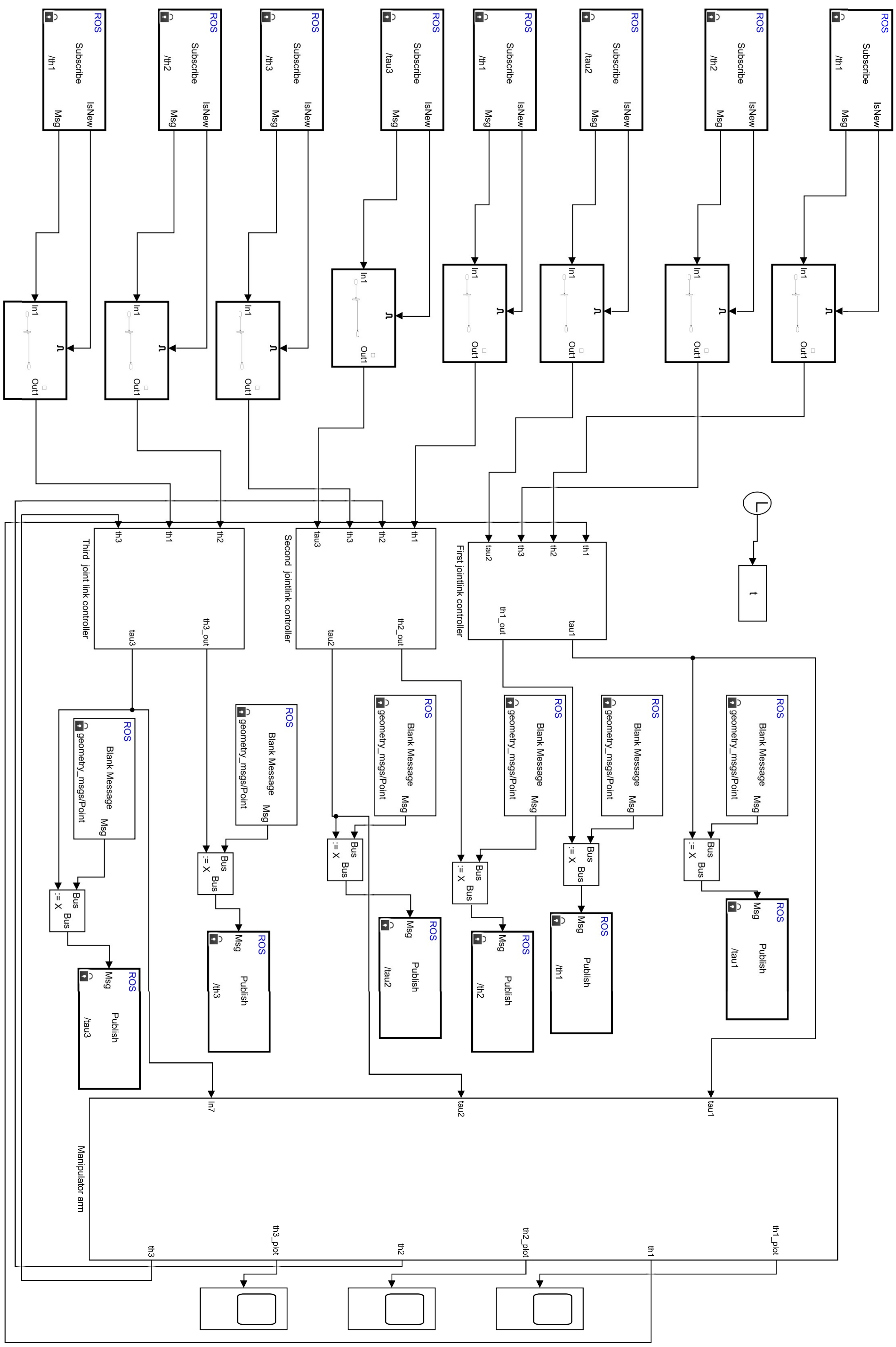


Figure 6.12: ROS-Simulink simulation of cooperative control of a 3R manipulator.

architecture. However, the focus of this paper is on control schemes implemented in a distributed control architecture and establishing equivalence of the centralized and distributed architecture.

CHAPTER 7

Conclusions

In this chapter we summarize the contributions of this thesis and discuss possible directions for the future work.

7.1 Summary of contributions

In this thesis, inspired by multi-agent/robotic systems, first we perceived a manipulator, which is MIMO multi-body system, as a multi-agent system with the joints (or the joint-link pairs) as sub-systems or agents, which interact with each other in a distributed manner. Here, the interaction between the joint-link agents is in the form of interactive forces and moments that lead to dynamic coupling. As the adjacency graph formed by the joint-link agents as nodes and links between two joints as edges is connected, the direct interactions between the immediate neighbors result in interaction (in the form of dynamic coupling) between any two joint-link agents.

We carried out an analysis of the computational cost associated with the model-based control law for planar serial-link manipulators with degrees-of-freedom varying from 2 to 6 using Maple. Using this analysis, we established the fact that the total computational cost associated with the model-based control law increases with the degrees-of-freedom. We proposed a distributed architecture for the control of manipulator now considered as a multi-agent system of joint-link agents, with the primary motivation of reducing computational cost associated with the centralized control scheme. Here, each joint-link agent is controlled by a dedicated controller, and the joint-level controllers communicate and cooperate among themselves. Though one of the primary motivation for the proposed distributed control scheme is to reduce the computational overhead, in this thesis we rely on the natural distributed nature of the manipulator dynamics rather than the program optimization or operation optimization techniques that are used at the algorithmic level.

We proposed a simple distributed control scheme based on the conventional model-based control law and show that it can be implemented using the distributed control architecture. Here, apart from the reduced computational lead time due to distributed computation of the control law at the joint-levels, unlike the decentralized or independent joint control schemes, the proposed control scheme fully utilizes the knowledge of the system dynamics, leading to a feedback linearized closed-loop error dynamics. Though the proposed distributed control scheme is suitable for a general serial-link manipulator, in this thesis, we focussed on planar manipulators with revolute joints. We proved, that the proposed distributed control scheme makes the links of the manipulator, and hence the end-effector, follow the desired trajectory, asymptotically. We defined a quantity called *distribution effectiveness* to quantify how the distributed control schemes share the computational load among the individual joint-level controllers. We also provided a discussion on implication of the discrete-time implementation of the proposed distributed control scheme in contrast to the conventional model-based control scheme. We designed a distributed model-based controller for a planar 3R manipulator, to illustrate the proposed distributed control scheme and the distributed control architecture for a manipulator. For the case of planar manipulators with degrees-of-freedom $2-6$, we provided a method to reduce the computational cost associated with dynamic equations used in the control law by identifying repetitive terms, which may be generalized for any other manipulator in principle.

In an attempt to further improve the distribution effectiveness and reduce the computational lead time, we proposed a cooperative control scheme for a manipulator using the distributed control architecture. While in the basic distributed control scheme proposed, joint-level controllers interact amongst themselves in terms of exchanging desired and measured states (and their derivatives), in the case of the cooperative control scheme the joint-level controller cooperate by exchanging certain computed terms between them. Even in this case, we provided a discussion on implication of the discrete-time implementation. We proved, that the proposed cooperative control law makes the links of the manipulator, and hence the end-effector, follow the desired trajectory, asymptotically. We designed a cooperative distributed model-based controller for a planar 3R manipulator, to illustrate the proposed cooperative manipulator control scheme implemented in the distributed control architecture. We also provided a discussion on computational effectiveness of the proposed cooperative distributed control scheme along with a method to further reduce the computational lead time by identifying repetitive terms in the control law.

We presented a detailed analysis of computational cost associated with the dynamic equation of planar manipulators with degrees-of-freedom from 2 to 6, where we analyze the cost involved in the proposed distributed control schemes in contrast to that in the conventional centralized model-based control scheme, using Maple. We provided results which indicate that the distribution effectiveness of the proposed simple distributed control schemes improves with degrees-of-freedom of the manipulator. We have also provided a detailed discussion on reducing the computational cost by identifying repetitive terms in the dynamic equations at each joint-level, for planar manipulators with degrees-of-freedom from 3 to 6.

We then presented simulation results demonstrating the proposed control schemes. We presented results which show how the trajectory tracking performance of the model-based control law degrades with increase in the sampling time. Then we presented results which demonstrate that with the proposed distributed control schemes every joint tracks the desired trajectory satisfactorily, in comparison with the independent-joint PID control scheme. We presented details of implementation of the proposed distributed manipulator control scheme using Simulink-ROS hybrid platform based on Matlab's Robotics toolbox, which provides a more realistic simulation result and it is also amenable for hardware implementation. Finally, we presented a discussion to compare decentralized control schemes presented in the literature with the distributed control schemes presented in this thesis.

To summarize:

1. We perceived a manipulator, which is MIMO multi-body system, as a multi-agent system with the joints (or the joint-link pair) as sub-systems or agents interacting with each other in a distributed manner.

2. We proposed model-based control schemes in a distributed control architecture, which combine the advantage of the conventional (centralized) model-based scheme, in terms of the trajectory tracking performance and the decentralized control schemes in terms of lower computational lead time due to distribution of computational load among the joint-level controllers.

3. Though one of the primary motivation for the proposed distributed control scheme is to reduce the computational overhead, in this thesis we relied on the natural distributed nature of the manipulator dynamics rather than the program optimization or operation optimization that is used at the algorithmic level.

4. We proposed two simple distributed control schemes both incorporating the dynamics fully, and relying upon the distributed nature of the manipulator dynamics.

5. By a careful choice of the control laws at the joint levels, we achieved a set of linear decoupled closed-loop error dynamics identical to that with the conventional model-based control scheme. Note that, any control law that achieves a true feedback linearization, where the nonlinear and coupling terms get canceled by the feedback, the error dynamics is bound to be similar.

6. We presented details of implementation of the proposed distributed manipulator control scheme using Simulink-ROS hybrid platform based on Matlab's Robotics toolbox, which provides a more realistic simulation result and it is also amenable for hardware implementation.

7. Apart from the reduction in computational overhead due with the natural distribution of the computational effort among the joint-level control, we achieved further reduction in the computational load by identifying repetitive terms in the manipulator dynamics/control law. As demonstrated by the fact that the distribution effectiveness being unaffected with this process, this exercise of reducing computational load is independent of the distributed-ness property of the manipulator dynamics or the proposed distributed control schemes.

7.2 Scope for future work

Some of the interesting and useful directions for future work are as follows:

1. The proposed distributed control schemes may be perceived as a task allocation among individual agents/robots in a multi-agent(robot) system. Here, the task is motion control of the end effector, which is distributed among the individual joint-level controllers which are responsible to control the motion of the respective joints, thereby achieving overall objective of moving the end-effector along the desired trajectory. A truly and completely distributed control is achieved when we incorporate the inverse kinematics along with the manipulator dynamics in a distributed manner. This integration of dynamics and kinematics into the joint-level control schemes is a possible direction for further research.

2. The proposed distributed control architecture opens up scope for future research which could lead to a fully modular and distributed control of a redundant manipulator robust to failure of a few of the joints.

A1: Process of modeling in Maple

Process of obtaining dynamic equation of a 3R planar manipulator in Maple

> $with(codegen):$

> $with(LinearAlgebra): {}^{0}R_1 := \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{1}R_2 := \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{2}R_3 := \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{1}R_0 := \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{2}R_1 := \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{3}R_2 := \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $\omega_1 := \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}:$

> $\omega_{1\,d} := \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}:$

> $\omega_2 := \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}:$

Process of obtaining dynamic equation of a 3R planar manipulator in Maple

> $with(codegen):$

> $with(LinearAlgebra): {}^{0}R_{1} := \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{1}R_{2} := \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{2}R_{3} := \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{1}R_{0} := \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{2}R_{1} := \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $with(LinearAlgebra): {}^{3}R_{2} := \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}:$

> $\omega_1 := \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}:$

> $\omega_{1d} := \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}:$

> $\omega_2 := \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}:$

$$> \omega_{2d} := \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}:$$

$$> \omega_{3} := \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}:$$

$$> \omega_{3d} := \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3 \end{bmatrix}:$$

$$> g_0 := \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}:$$

$$> p_1 := \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}:$$

$$> p_2 := \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}:$$

$$> p_3 := \begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix}:$$

$$> v_1 := MatrixVectorMultiply\left({}^{1}R_0, g_0\right):$$

$$> v_{c1} := CrossProduct\left(\omega_{1d}, p_1\right) + CrossProduct\left(\omega_1, CrossProduct\left(\omega_1, p_1\right)\right) + v_1:$$

$$> F_1 := ScalarMultiply\left(v_{c1}, m_1\right):$$

$$> v_2 := MatrixVectorMultiply\left({}^{2}R_1, \left(CrossProduct\left(\omega_{1d}, p_1\right) + CrossProduct\left(\omega_1, CrossProduct\left(\omega_1, p_1\right)\right) + v_1\right)\right):$$

$$> v_{c2} := CrossProduct\left(\omega_{2d}, p_2\right) + CrossProduct\left(\omega_2, CrossProduct\left(\omega_2, p_2\right)\right) + v_2:$$

$$> F_2 := ScalarMultiply\left(v_{c2}, m_2\right):$$

$$> v_3 := MatrixVectorMultiply\left({}^{3}R_2, \left(CrossProduct\left(\omega_{2d}, p_2\right) + CrossProduct\left(\omega_2, CrossProduct\left(\omega_2, p_2\right)\right) + v_2\right)\right):$$

> $v_{c3} := CrossProduct(\omega_{3d}, p_3) + CrossProduct(\omega_3, CrossProduct(\omega_3, p_3)) + v_3 :$

> $F_3 := ScalarMultiply(v_{c3}, m_3) :$

> $f_3 := F_3 :$

> $\tau_3 := CrossProduct(p_3, F_3) :$

> $f_2 := MatrixVectorMultiply({}^2R_3, f_3) + F_2 :$

> $\tau_2 := \tau_3 + CrossProduct(p_2, F_2) + CrossProduct(p_2, (MatrixVectorMultiply({}^2R_3, f_3))) :$

> $f_1 := MatrixVectorMultiply({}^1R_2, f_2) + F_1 :$

> $\tau_1 := \tau_2 + CrossProduct(p_1, F_2) + CrossProduct(p_1, (MatrixVectorMultiply({}^1R_2, f_2))) :$

> $\tau_{31} := collect(expand(\tau_3[3]), \ddot{\theta}_1) :$

> $\tau_{32} := collect(\tau_{31}, \ddot{\theta}_2) :$

> $\tau_{33} := collect(\tau_{32}, \dot{\theta}_1) :$

Collection of terms in τ_3 equation

> $\tau_{34} := collect(\tau_{33}, g)$

τ_3

$$\tau_{34} := (c_1 c_2 c_3 l_3 m_3 - c_1 l_3 m_3 s_2 s_3 - c_2 l_3 m_3 s_1 s_3 - c_3 l_3 m_3 s_1 s_2)\, g + (l_3 m_3 s_3 c_2 l_1 + l_3 m_3 c_3 s_2 l_1 + l_3 m_3 s_3 l_2)\left(\frac{d}{dt}\theta_1(t)\right)^2 + 2\, l_3 m_3 s_3 l_2 \left(\frac{d}{dt}\theta_1(t)\right)\left(\frac{d}{dt}\theta_2(t)\right) + (l_3 m_3 c_3 l_2 + l_3^2 m_3)\left(\frac{d^2}{dt^2}\theta_2(t)\right) + (l_3 m_3 c_3 c_2 l_1 - l_3 m_3 s_3 s_2 l_1 + l_3 m_3 c_3 l_2 + l_3^2 m_3)\left(\frac{d^2}{dt^2}\theta_1(t)\right) + l_3^2 m_3 \left(\frac{d^2}{dt^2}\theta_3(t)\right) + l_3 m_3 s_3 l_2 \left(\frac{d}{dt}\theta_2(t)\right)^2 \quad \textbf{(1)}$$

> $\tau_{21} := collect(expand(\tau_2[3]), \ddot{\theta}_1) :$

> $\tau_{22} := collect(\tau_{21}, \ddot{\theta}_2) :$

> $\tau_{23} := collect(\tau_{22}, \dot{\theta}_1) :$

Collection of terms in τ_2 equation

> $\tau_{24} := collect(\tau_{23}, g)$

τ_2

$$\tau_{24} := (l_2 c_3^2 m_3 c_2 c_1 + l_2 s_3^2 m_3 c_2 c_1 - l_2 c_3^2 m_3 s_2 s_1 - l_2 s_3^2 m_3 s_2 s_1 + c_1 c_2 c_3 l_3 m_3 - c_1 l_3 m_3 s_2 s_3 - c_2 l_3 m_3 s_1 s_3 - c_3 l_3 m_3 s_1 s_2 + l_2 m_2 c_2 c_1 - l_2 m_2 s_2 s_1)\, g + (l_2 c_3^2 m_3 s_2 l_1 + l_2 s_3^2 m_3 s_2 l_1 + l_3 m_3 s_3 c_2 l_1 + l_3 m_3 c_3 s_2 l_1 + l_2 m_2 s_2 l_1)\left(\frac{d}{dt}\theta_1(t)\right)^2 - 2\, l_2 s_3 m_3 l_3 \left(\frac{d}{dt}\theta_1(t)\right)\left(\frac{d}{dt}\theta_3(t)\right) + (c_3^2 m_3 l_2^2 + s_3^2 m_3 l_2^2 + 2\, l_3 m_3 c_3 l_2 + l_2^2 m_2 + l_3^2 m_3)\left(\frac{d^2}{dt^2}\theta_2(t)\right) + l_3^2 m_3 \left(\frac{d^2}{dt^2}\theta_3(t)\right) + (l_2 c_3^2 m_3 c_2 l_1 + l_2 s_3^2 m_3 c_2 l_1 + l_3 m_3 c_3 c_2 l_1 + c_3^2 m_3 l_2^2 - l_3 m_3 s_3 s_2 l_1 + s_3^2 m_3 l_2^2 + l_2 m_2 c_2 l_1 + 2\, l_3 m_3 c_3 l_2 + l_2^2 m_2 + l_3^2 m_3)\left(\frac{d^2}{dt^2}\theta_1(t)\right) - 2\, l_2 s_3 m_3 l_3 \left(\frac{d}{dt}\theta_2(t)\right)\left(\frac{d}{dt} \right. \quad \textbf{(2)}$$

$\theta_3(t)\Big) - l_2\, s_3\, m_3\, l_3 \left(\frac{d}{dt}\,\theta_3(t)\right)^2 + l_2\, c_3\, m_3\, l_3 \left(\frac{d^2}{dt^2}\,\theta_3(t)\right)$

> $\tau_{11} := collect\big(expand(\tau_1[3]), \ddot{\theta}_1\big)$:

> $\tau_{12} := collect\big(\tau_{11}, \ddot{\theta}_2\big)$:

> $\tau_{13} := collect\big(\tau_{12}, g\big)$:

Collection of terms in τ_1 equation

> $\tau_{14} := collect\big(\tau_{13}, \dot{\theta}_1^2\big)$:

> $\tau_{15} := collect\big(\tau_{14}, \ddot{\theta}_3\big)$:

> $\tau_{16} := collect\big(\tau_{15}, \dot{\theta}_2^2\big)$

τ_1

$$\begin{aligned}\tau_{16} := {} & \left(-l_1 s_2 c_3^2 m_3 l_2 - l_1 s_2 s_3^2 m_3 l_2 - l_1 c_2 s_3 m_3 l_3 - l_1 s_2 c_3 m_3 l_3 - l_1 s_2 m_2 l_2\right)\left(\frac{d}{dt}\,\theta_2(t)\right)^2 \\ & + \Big(\left(-2\, l_1 s_2 c_3^2 m_3 l_2 - 2\, l_1 s_2 s_3^2 m_3 l_2 - 2\, l_1 c_2 s_3 m_3 l_3 - 2\, l_1 s_2 c_3 m_3 l_3 - 2\, l_1 s_2 m_2 l_2\right)\left(\frac{d}{dt}\,\theta_1(t)\right) - 2\, l_1 s_2 c_3 m_3 l_3 \left(\frac{d}{dt}\,\theta_3(t)\right) - 2\, l_1 c_2 s_3 m_3 l_3 \left(\frac{d}{dt}\,\theta_3(t)\right) - 2\, l_2 s_3 m_3 l_3 \left(\frac{d}{dt}\,\theta_3(t)\right)\Big)\left(\frac{d}{dt}\,\theta_2(t)\right) + \left(l_3 m_3 c_3 c_2 l_1 - l_3 m_3 s_3 s_2 l_1 + l_3 m_3 c_3 l_2 + l_3^2 m_3\right)\left(\frac{d^2}{dt^2}\,\theta_3(t)\right) + \\ & l_1^2 m_2 s_2 \left(\frac{d}{dt}\,\theta_1(t)\right)^2 + \Big(-2\, l_1 c_2 s_3 m_3 l_3 \left(\frac{d}{dt}\,\theta_3(t)\right) - 2\, l_1 s_2 c_3 m_3 l_3 \left(\frac{d}{dt}\,\theta_3(t)\right) \\ & - 2\, l_2 s_3 m_3 l_3 \left(\frac{d}{dt}\,\theta_3(t)\right)\Big)\left(\frac{d}{dt}\,\theta_1(t)\right) + \big(l_2 c_3^2 m_3 c_2 l_1 + l_2 s_3^2 m_3 c_2 l_1 + l_3 m_3 c_3 c_2 l_1 + \\ & c_3^2 m_3 l_2^2 - l_3 m_3 s_3 s_2 l_1 + s_3^2 m_3 l_2^2 + l_2 m_2 c_2 l_1 + 2\, l_3 m_3 c_3 l_2 + l_1 m_2 l_2 + l_2^2 m_2 + l_3^2 m_3\big)\left(\frac{d^2}{dt^2}\,\theta_2(t)\right) + \big(c_2^2 c_3^2 l_1^2 m_3 + c_2^2 l_1^2 m_3 s_3^2 + c_3^2 l_1^2 m_3 s_2^2 + l_1^2 m_3 s_2^2 s_3^2 + 2\, l_2 c_3^2 m_3 c_2 l_1 + 2\, l_2 s_3^2 m_3 c_2 l_1 \\ & + c_2^2 l_1^2 m_2 + 2\, l_3 m_3 c_3 c_2 l_1 + c_3^2 m_3 l_2^2 + l_1^2 m_2 s_2^2 - 2\, l_3 m_3 s_3 s_2 l_1 + s_3^2 m_3 l_2^2 + c_2 l_1^2 m_2 \\ & + 2\, l_2 m_2 c_2 l_1 + 2\, l_3 m_3 c_3 l_2 + l_1 m_2 l_2 + l_2^2 m_2 + l_3^2 m_3\big)\left(\frac{d^2}{dt^2}\,\theta_1(t)\right) \\ & - l_1 s_2 c_3 m_3 l_3 \left(\frac{d}{dt}\,\theta_3(t)\right)^2 - l_1 c_2 s_3 m_3 l_3 \left(\frac{d}{dt}\,\theta_3(t)\right)^2 + \big(c_1 c_2^2 c_3^2 l_1 m_3 + c_1 c_2^2 l_1 m_3 s_3^2 \\ & + c_1 c_3^2 l_1 m_3 s_2^2 + c_1 l_1 m_3 s_2^2 s_3^2 + c_1 c_2 c_3^2 l_2 m_3 + c_1 c_2 l_2 m_3 s_3^2 - c_3^2 l_2 m_3 s_1 s_2 - l_2 m_3 s_1 s_2 s_3^2 \\ & + c_1 c_2^2 l_1 m_2 + c_1 c_2 c_3 l_3 m_3 + c_1 l_1 m_2 s_2^2 - c_1 l_3 m_3 s_2 s_3 - c_2 l_3 m_3 s_1 s_3 - c_3 l_3 m_3 s_1 s_2 \\ & + c_1 c_2 l_1 m_2 + c_1 c_2 l_2 m_2 - l_1 m_2 s_1 s_2 - l_2 m_2 s_1 s_2\big)\, g - l_2 s_3 m_3 l_3 \left(\frac{d}{dt}\,\theta_3(t)\right)^2\end{aligned} \tag{3}$$

>

A2: Dynamics obtained using Maple

Dynamics of 4R manipulator obtained using Maple (Mass Matrix M, V, and G vectors)

M > $$[[(\cos(\theta_2)\, l_1 l_2 m_2 + \cos(\theta_2)\, l_1 l_2 m_3 + \cos(\theta_2)\, l_1 l_2 m_4 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_3 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_4 + \cos(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4 + l_1^2 m_1 + l_1^2 m_2 + l_1^2 m_3 + l_1^2 m_4), (\cos(\theta_2)\, l_1 l_2 m_2 + \cos(\theta_2)\, l_1 l_2 m_3 + \cos(\theta_2)\, l_1 l_2 m_4 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_3 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_4 + \cos(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4), (\cos(\theta_3 + \theta_2)\, l_1 l_3 m_3 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_4 + \cos(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4), \cos(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4],$$

$$[(\cos(\theta_4 + \theta_3)\, l_2 l_4 m_4 + \cos(\theta_2)\, l_1 l_2 m_2 + \cos(\theta_2)\, l_1 l_2 m_3 + \cos(\theta_2)\, l_1 l_2 m_4 + l_2 m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4 l_3 + l_2^2 m_2 + l_2^2 m_3 + l_2^2 m_4), (\cos(\theta_4 + \theta_3)\, l_2 l_4 m_4 + l_2 m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4 l_3 + l_2^2 m_2 + l_2^2 m_3 + l_2^2 m_4), (\cos(\theta_4 + \theta_3)\, l_2 l_4 m_4 + l_2 m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4 l_3), \cos(\theta_4 + \theta_3) l_2 l_4 m_4],$$

$$[(l_2 m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4 l_3 + \cos(\theta_4)\, l_3 l_4 m_4 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_3 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_4 + l_3^2 m_3 + l_3^2 m_4),\ (l_2 m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4 l_3 + \cos(\theta_4)\, l_3 l_4 m_4 + l_3^2 m_3 + l_3^2 m_4), (\cos(\theta_4)\, l_3 l_4 m_4 + l_3^2 m_3 + l_3^2 m_4), \cos(\theta_4) l_3 l_4 m_4],$$

$$[(\cos(\theta_4)\, l_3 l_4 m_4 + \cos(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4 + \cos(\theta_4 + \theta_3)\, l_2 l_4 m_4 + l_4^2 m_4), (\cos(\theta_4)\, l_3 l_4 m_4 + \cos(\theta_4 + \theta_3)\, l_2 l_4 m_4 + l_4^2 m_4), (\cos(\theta_4)\, l_3 l_4 m_4 + l_4^2 m_4), l_4^2 m_4]]$$

V > $$\Bigg[\Bigg[-\left(\frac{d}{dt}\theta_4(t)\right)^2 \sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4 + \Big(-2\left(\frac{d}{dt}\theta_1(t)\right)\sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4 - 2\left(\frac{d}{dt}\theta_2(t)\right)\sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4 - 2\left(\frac{d}{dt}\theta_3(t)\right)\sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4\Big)\left(\frac{d}{dt}\theta_4(t)\right) + \big(-\sin(\theta_2)\, l_1 l_2 m_2 - \sin(\theta_2)\, l_1 l_2 m_3 - \sin(\theta_2)\, l_1 l_2 m_4 - \sin(\theta_3 + \theta_2)\, l_1 l_3 m_3 - \sin(\theta_3 + \theta_2)\, l_1 l_3 m_4 - \sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4\big)\left(\frac{d}{dt}\theta_1(t)\right)^2 + \Big(\big(-2\sin(\theta_3 + \theta_2)\, l_1 l_3 m_3 - 2\sin(\theta_3 + \theta_2)\, l_1 l_3 m_4 - 2\sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4\big)\left(\frac{d}{dt}\theta_3(t)\right) + \big(-2\sin(\theta_2)\, l_1 l_2 m_2 - 2\sin(\theta_2)\, l_1 l_2 m_3 - 2\sin(\theta_2)\, l_1 l_2 m_4 - 2\sin(\theta_3 + \theta_2)\, l_1 l_3 m_3 - 2\sin(\theta_3 + \theta_2)\, l_1 l_3 m_4 - 2\sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4\big)\left(\frac{d}{dt}\theta_2(t)\right)\Big)\left(\frac{d}{dt}\theta_1(t)\right) + \big(-\sin(\theta_3 + \theta_2)\, l_1 l_3 m_3 - \sin(\theta_3 + \theta_2)\, l_1 l_3 m_4 - \sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4\big)\left(\frac{d}{dt}\theta_3(t)\right)^2 + \big(-2\sin(\theta_3 + \theta_2)\, l_1 l_3 m_3 - 2\sin(\theta_3 + \theta_2)\, l_1 l_3 m_4 - 2\sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4\big)\left(\frac{d}{dt}\theta_2(t)\right)\left(\frac{d}{dt}\theta_3(t)\right) + \big(-\sin(\theta_2)\, l_1 l_2 m_2 - \sin(\theta_2)\, l_1 l_2 m_3 - \sin(\theta_2)\, l_1 l_2 m_4 - \sin(\theta_3 + \theta_2)\, l_1 l_3 m_3 - \sin(\theta_3 + \theta_2)\, l_1 l_3 m_4 - \sin(\theta_4 + \theta_3 + \theta_2)\, l_1 l_4 m_4\big)\left(\frac{d}{dt}\theta_2(t)\right)^2\Bigg],$$

Dynamics of 4R manipulator obtained using Maple (Mass Matrix M, V, and G vectors)

M

$$> \Big[\Big[\big(\cos(\theta_2)\, l_1\, l_2\, m_2 + \cos(\theta_2)\, l_1\, l_2\, m_3 + \cos(\theta_2)\, l_1\, l_2\, m_4 + \cos(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 + \cos(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 + \cos(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4 + l_1^2\, m_1 + l_1^2\, m_2 + l_1^2\, m_3 + l_1^2\, m_4\big), \big(\cos(\theta_2)\, l_1\, l_2\, m_2 + \cos(\theta_2)\, l_1\, l_2\, m_3 + \cos(\theta_2)\, l_1\, l_2\, m_4 + \cos(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 + \cos(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 + \cos(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\big), \big(\cos(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 + \cos(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 + \cos(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\big), \cos(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\Big],$$

$$\Big[\big(\cos(\theta_4 + \theta_3)\, l_2\, l_4\, m_4 + \cos(\theta_2)\, l_1\, l_2\, m_2 + \cos(\theta_2)\, l_1\, l_2\, m_3 + \cos(\theta_2)\, l_1\, l_2\, m_4 + l_2\, m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4\, l_3 + l_2^2\, m_2 + l_2^2\, m_3 + l_2^2\, m_4\big), \big(\cos(\theta_4 + \theta_3)\, l_2\, l_4\, m_4 + l_2\, m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4\, l_3 + l_2^2\, m_2 + l_2^2\, m_3 + l_2^2\, m_4\big), \big(\cos(\theta_4 + \theta_3)\, l_2\, l_4\, m_4 + l_2\, m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4\, l_3\big), \cos(\theta_4 + \theta_3) l_2\, l_4\, m_4\Big],$$

$$\Big[\big(l_2\, m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4\, l_3 + \cos(\theta_4)\, l_3\, l_4\, m_4 + \cos(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 + \cos(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 + l_3^2\, m_3 + l_3^2\, m_4\big), \big(l_2\, m_3 \cos(\theta_3)\, l_3 + l_2 \cos(\theta_3)\, m_4\, l_3 + \cos(\theta_4)\, l_3\, l_4\, m_4 + l_3^2\, m_3 + l_3^2\, m_4\big), \big(\cos(\theta_4)\, l_3\, l_4\, m_4 + l_3^2\, m_3 + l_3^2\, m_4\big), \cos(\theta_4) l_3\, l_4\, m_4\Big],$$

$$\Big[\big(\cos(\theta_4)\, l_3\, l_4\, m_4 + \cos(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4 + \cos(\theta_4 + \theta_3)\, l_2\, l_4\, m_4 + l_4^2\, m_4\big), \big(\cos(\theta_4)\, l_3\, l_4\, m_4 + \cos(\theta_4 + \theta_3)\, l_2\, l_4\, m_4 + l_4^2\, m_4\big), \big(\cos(\theta_4)\, l_3\, l_4\, m_4 + l_4^2\, m_4\big), l_4^2\, m_4\Big]\Big]$$

V

$$> \Big[\Big[-\left(\frac{d}{dt}\theta_4(t)\right)^2 \sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4 + \Big(-2\left(\frac{d}{dt}\theta_1(t)\right)\sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4 - 2\left(\frac{d}{dt}\theta_2(t)\right)\sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4 - 2\left(\frac{d}{dt}\theta_3(t)\right)\sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\Big)\left(\frac{d}{dt}\theta_4(t)\right) + \big(-\sin(\theta_2)\, l_1\, l_2\, m_2 - \sin(\theta_2)\, l_1\, l_2\, m_3 - \sin(\theta_2)\, l_1\, l_2\, m_4 - \sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 - \sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 - \sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\big)\left(\frac{d}{dt}\theta_1(t)\right)^2 + \Big(\big(-2\sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 - 2\sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 - 2\sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\big)\left(\frac{d}{dt}\theta_3(t)\right) + \big(-2\sin(\theta_2)\, l_1\, l_2\, m_2 - 2\sin(\theta_2)\, l_1\, l_2\, m_3 - 2\sin(\theta_2)\, l_1\, l_2\, m_4 - 2\sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 - 2\sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 - 2\sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\big)\left(\frac{d}{dt}\theta_2(t)\right)\Big)\left(\frac{d}{dt}\theta_1(t)\right) + \big(-\sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 - \sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 - \sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\big)\left(\frac{d}{dt}\theta_3(t)\right)^2 + \big(-2\sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 - 2\sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 - 2\sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\big)\left(\frac{d}{dt}\theta_2(t)\right)\left(\frac{d}{dt}\theta_3(t)\right) + \big(-\sin(\theta_2)\, l_1\, l_2\, m_2 - \sin(\theta_2)\, l_1\, l_2\, m_3 - \sin(\theta_2)\, l_1\, l_2\, m_4 - \sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_3 - \sin(\theta_3 + \theta_2)\, l_1\, l_3\, m_4 - \sin(\theta_4 + \theta_3 + \theta_2)\, l_1\, l_4\, m_4\big)\left(\frac{d}{dt}\theta_2(t)\right)^2\Big],$$

$$\Big[\Big(-\sin(\theta_4+\theta_3)\, l_2\, l_4\, m_4 - l_2\, m_3 \sin(\theta_3)\, l_3 - l_2\, m_4 \sin(\theta_3)\, l_3\Big) \left(\frac{d}{dt}\, \theta_2(t)\right)^2 + \Big(-2 \sin(\theta_4$$

$$+ \theta_3) \left(\frac{d}{dt}\, \theta_4(t)\right) l_2\, l_4\, m_4 + \Big(-2 \sin(\theta_4+\theta_3)\, l_2\, l_4\, m_4 - 2\, l_2\, m_3 \sin(\theta_3)\, l_3$$

$$- 2\, l_2\, m_4 \sin(\theta_3)\, l_3\Big) \left(\frac{d}{dt}\, \theta_3(t)\right) + \Big(-2 \sin(\theta_4+\theta_3)\, l_2\, l_4\, m_4 - 2\, l_2\, m_3 \sin(\theta_3)\, l_3$$

$$- 2\, l_2\, m_4 \sin(\theta_3)\, l_3\Big) \left(\frac{d}{dt}\, \theta_1(t)\right)\Big) \left(\frac{d}{dt}\, \theta_2(t)\right) - \sin(\theta_4+\theta_3) \left(\frac{d}{dt}\, \theta_4(t)\right)^2 l_2\, l_4\, m_4 + \Big($$

$$-2 \sin(\theta_4+\theta_3)\, l_2\, l_4\, m_4 \left(\frac{d}{dt}\, \theta_1(t)\right) - 2 \sin(\theta_4+\theta_3) \left(\frac{d}{dt}\, \theta_3(t)\right) l_2\, l_4\, m_4\Big) \left(\frac{d}{dt}\, \theta_4(t)\right)$$

$$+ \Big(-\sin(\theta_4+\theta_3)\, l_2\, l_4\, m_4 - l_2\, m_3 \sin(\theta_3)\, l_3 - l_2\, m_4 \sin(\theta_3)\, l_3\Big) \left(\frac{d}{dt}\, \theta_3(t)\right)^2 + \Big(-2 \sin(\theta_4$$

$$+ \theta_3)\, l_2\, l_4\, m_4 - 2\, l_2\, m_3 \sin(\theta_3)\, l_3 - 2\, l_2\, m_4 \sin(\theta_3)\, l_3\Big) \left(\frac{d}{dt}\, \theta_1(t)\right) \left(\frac{d}{dt}\, \theta_3(t)\right) + \Big($$

$$-\sin(\theta_4+\theta_3)\, l_2\, l_4\, m_4 + \sin(\theta_2)\, l_1\, l_2\, m_2 + \sin(\theta_2)\, l_1\, l_2\, m_3 + \sin(\theta_2)\, l_1\, l_2\, m_4$$

$$- l_2\, m_3 \sin(\theta_3)\, l_3 - l_2\, m_4 \sin(\theta_3)\, l_3\Big) \left(\frac{d}{dt}\, \theta_1(t)\right)^2\Big],$$

$$\Big[-\sin(\theta_4) \left(\frac{d}{dt}\, \theta_4(t)\right)^2 l_3\, l_4\, m_4 + \Big(-2\, l_4 \sin(\theta_4)\, m_4 \left(\frac{d}{dt}\, \theta_1(t)\right) l_3$$

$$- 2 \sin(\theta_4) \left(\frac{d}{dt}\, \theta_2(t)\right) l_3\, l_4\, m_4 - 2\, l_4 \left(\frac{d}{dt}\, \theta_3(t)\right) \sin(\theta_4)\, m_4\, l_3\Big) \left(\frac{d}{dt}\, \theta_4(t)\right) + \Big($$

$$-l_4 \sin(\theta_4)\, m_4\, l_3 + l_2\, m_3 \sin(\theta_3)\, l_3 + l_2\, m_4 \sin(\theta_3)\, l_3\Big) \left(\frac{d}{dt}\, \theta_2(t)\right)^2 + \Big(\Big($$

$$-2\, l_4 \sin(\theta_4)\, m_4\, l_3 + 2\, l_2\, m_3 \sin(\theta_3)\, l_3 + 2\, l_2\, m_4 \sin(\theta_3)\, l_3\Big) \left(\frac{d}{dt}\, \theta_1(t)\right)$$

$$- 2\, l_4 \left(\frac{d}{dt}\, \theta_3(t)\right) \sin(\theta_4)\, m_4\, l_3\Big) \left(\frac{d}{dt}\, \theta_2(t)\right) + \Big(-l_4 \sin(\theta_4)\, m_4\, l_3 + l_2\, m_3 \sin(\theta_3)\, l_3$$

$$+ l_2\, m_4 \sin(\theta_3)\, l_3 + \sin(\theta_3+\theta_2)\, l_1\, l_3\, m_3 + \sin(\theta_3+\theta_2)\, l_1\, l_3\, m_4\Big) \left(\frac{d}{dt}\, \theta_1(t)\right)^2$$

$$- 2 \sin(\theta_4) \left(\frac{d}{dt}\, \theta_1(t)\right) \left(\frac{d}{dt}\, \theta_3(t)\right) l_3\, l_4\, m_4 - \sin(\theta_4) \left(\frac{d}{dt}\, \theta_3(t)\right)^2 l_3\, l_4\, m_4\Big],$$

$$\Big[\sin(\theta_4) \left(\frac{d}{dt}\, \theta_3(t)\right)^2 l_3\, l_4\, m_4 + \Big(2\, l_4 \sin(\theta_4)\, m_4 \left(\frac{d}{dt}\, \theta_1(t)\right) l_3$$

$$+ 2 \sin(\theta_4) \left(\frac{d}{dt}\, \theta_2(t)\right) l_3\, l_4\, m_4\Big) \left(\frac{d}{dt}\, \theta_3(t)\right) + \Big(l_4 \sin(\theta_4)\, m_4\, l_3 + \sin(\theta_4$$

$$+ \theta_3)\, l_2\, l_4\, m_4\Big) \left(\frac{d}{dt}\, \theta_2(t)\right)^2 + \Big(2\, l_4 \sin(\theta_4)\, m_4\, l_3 + 2 \sin(\theta_4$$

$$+ \theta_3)\, l_2\, l_4\, m_4\Big) \left(\frac{d}{dt}\, \theta_1(t)\right) \left(\frac{d}{dt}\, \theta_2(t)\right) + \Big(l_4 \sin(\theta_4)\, m_4\, l_3 + \sin(\theta_4+\theta_3+\theta_2)\, l_1\, l_4\, m_4$$

$$+ \sin(\theta_4+\theta_3)\, l_2\, l_4\, m_4\Big) \left(\frac{d}{dt}\, \theta_1(t)\right)^2\Big]\Big]$$

$$G := \begin{bmatrix} (\cos(\theta_1)\, l_1\, m_1 + \cos(\theta_1)\, l_1\, m_2 + \cos(\theta_1)\, l_1\, m_3 + \cos(\theta_1)\, l_1\, m_4)\, g \\ (l_2\, m_2 \cos(\theta_2 + \theta_1) + l_2\, m_3 \cos(\theta_2 + \theta_1) + l_2\, m_4 \cos(\theta_2 + \theta_1))\, g \\ (\cos(\theta_3 + \theta_2 + \theta_1)\, l_3\, m_3 + \cos(\theta_3 + \theta_2 + \theta_1)\, l_3\, m_4)\, g \\ \cos(\theta_4 + \theta_3 + \theta_2 + \theta_1)\, g\, l_4\, m_4 \end{bmatrix}$$

Dynamics of 5R manipulator obtained using Maple (Mass Matrix M, V, and G vectors)

M >

$$[[(\cos(\theta_2)\, l_2 m_2 l_1 + \cos(\theta_2)\, l_1 l_2 m_3 + \cos(\theta_2)\, l_2 m_4 l_1 + \cos(\theta_2)\, l_2 m_5 l_1 + l_5 m_5 \cos(\theta_5 + \theta_4$$

$$+ \theta_3 + \theta_2)\, l_1 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_3 + l_3 m_4 \cos(\theta_3 + \theta_2)\, l_1 + m_5 l_3 \cos(\theta_3 + \theta_2)\, l_1$$

$$+ \cos(\theta_4 + \theta_3 + \theta_2)\, m_4 l_4 l_1 + m_5 \cos(\theta_4 + \theta_3 + \theta_2)\, l_4 l_1 + l_1^2 m_1 + l_1^2 m_2 + l_1^2 m_3 + l_1^2 m_4 +$$

$$l_1^2 m_5),\ (\cos(\theta_2)\, l_2 m_2 l_1 + \cos(\theta_2)\, l_1 l_2 m_3 + \cos(\theta_2)\, l_2 m_4 l_1 + \cos(\theta_2)\, l_2 m_5 l_1$$

$$+ l_5 m_5 \cos(\theta_5 + \theta_4 + \theta_3 + \theta_2)\, l_1 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_3 + l_3 m_4 \cos(\theta_3 + \theta_2)\, l_1$$

$$+ m_5 l_3 \cos(\theta_3 + \theta_2)\, l_1 + \cos(\theta_4 + \theta_3 + \theta_2)\, m_4 l_4 l_1 + m_5 \cos(\theta_4 + \theta_3 + \theta_2)\, l_4 l_1),$$

$$(l_5 m_5 \cos(\theta_5 + \theta_4 + \theta_3 + \theta_2)\, l_1 + \cos(\theta_3 + \theta_2)\, l_1 l_3 m_3 + l_3 m_4 \cos(\theta_3 + \theta_2)\, l_1$$

$$+ m_5 l_3 \cos(\theta_3 + \theta_2)\, l_1 + \cos(\theta_4 + \theta_3 + \theta_2)\, m_4 l_4 l_1 + m_5 \cos(\theta_4 + \theta_3 + \theta_2)\, l_4 l_1),$$

$$(l_5 m_5 \cos(\theta_5 + \theta_4 + \theta_3 + \theta_2)\, l_1 + \cos(\theta_4 + \theta_3 + \theta_2)\, m_4 l_4 l_1 + m_5 \cos(\theta_4 + \theta_3 + \theta_2)\, l_4 l_1),$$

$$\cos(\theta_5 + \theta_4 + \theta_3 + \theta_2)\, l_1 l_5 m_5],$$

$$[(\cos(\theta_2)\, l_2 m_2 l_1 + \cos(\theta_2)\, l_1 l_2 m_3 + \cos(\theta_2)\, l_2 m_4 l_1 + \cos(\theta_2)\, l_2 m_5 l_1 + \cos(\theta_3)\, l_2 l_3 m_3$$

$$+ l_2 l_3 m_4 \cos(\theta_3) + l_2 l_3 m_5 \cos(\theta_3) + l_5 l_2 m_5 \cos(\theta_5 + \theta_4 + \theta_3) + \cos(\theta_4 + \theta_3)\, l_2 m_4 l_4$$

$$+ \cos(\theta_4 + \theta_3)\, l_2 m_5 l_4 + l_2^2 m_2 + l_2^2 m_3 + l_2^2 m_4 + l_2^2 m_5),\ (\cos(\theta_3)\, l_2 l_3 m_3 + l_2 l_3 m_4 \cos(\theta_3)$$

$$+ l_2 l_3 m_5 \cos(\theta_3) + l_5 l_2 m_5 \cos(\theta_5 + \theta_4 + \theta_3) + \cos(\theta_4 + \theta_3)\, l_2 m_4 l_4 + \cos(\theta_4$$

$$+ \theta_3)\, l_2 m_5 l_4 + l_2^2 m_2 + l_2^2 m_3 + l_2^2 m_4 + l_2^2 m_5),\ (\cos(\theta_3)\, l_2 l_3 m_3 + l_2 l_3 m_4 \cos(\theta_3)$$

$$+ l_2 l_3 m_5 \cos(\theta_3) + l_5 l_2 m_5 \cos(\theta_5 + \theta_4 + \theta_3) + \cos(\theta_4 + \theta_3)\, l_2 m_4 l_4 + \cos(\theta_4$$

$$+ \theta_3)\, l_2 m_5 l_4),\ (l_5 l_2 m_5 \cos(\theta_5 + \theta_4 + \theta_3) + \cos(\theta_4 + \theta_3)\, l_2 m_4 l_4 + \cos(\theta_4$$

$$+ \theta_3)\, l_2 m_5 l_4),\ \cos(\theta_5 + \theta_4 + \theta_3)\, l_2 l_5 m_5],$$

$$[(l_3 m_4 \cos(\theta_4)\, l_4 + m_5 l_3 \cos(\theta_4)\, l_4 + \cos(\theta_3)\, l_2 l_3 m_3 + l_2 l_3 m_4 \cos(\theta_3) + l_2 l_3 m_5 \cos(\theta_3)$$

$$+ \cos(\theta_3 + \theta_2)\, l_1 l_3 m_3 + l_3 m_4 \cos(\theta_3 + \theta_2)\, l_1 + m_5 l_3 \cos(\theta_3 + \theta_2)\, l_1 + l_5 \cos(\theta_5$$

$$+ \theta_4)\, m_5 l_3 + l_3^2 m_3 + l_3^2 m_4 + l_3^2 m_5),\ (l_3 m_4 \cos(\theta_4)\, l_4 + m_5 l_3 \cos(\theta_4)\, l_4 + \cos(\theta_3)\, l_2 l_3 m_3$$

$$+ l_2 l_3 m_4 \cos(\theta_3) + l_2 l_3 m_5 \cos(\theta_3) + l_5 \cos(\theta_5 + \theta_4)\, m_5 l_3 + l_3^2 m_3 + l_3^2 m_4 + l_3^2 m_5),$$

$$(l_3 m_4 \cos(\theta_4)\, l_4 + m_5 l_3 \cos(\theta_4)\, l_4 + l_5 \cos(\theta_5 + \theta_4)\, m_5 l_3 + l_3^2 m_3 + l_3^2 m_4 + l_3^2 m_5),$$

$$(l_3 m_4 \cos(\theta_4)\, l_4 + m_5 l_3 \cos(\theta_4)\, l_4 + l_5 \cos(\theta_5 + \theta_4)\, m_5 l_3),\ \cos(\theta_5 + \theta_4)\, l_3 l_5 m_5],$$

$$[(l_3 m_4 \cos(\theta_4)\, l_4 + m_5 l_3 \cos(\theta_4)\, l_4 + l_5 \cos(\theta_5)\, m_5 l_4 + \cos(\theta_4 + \theta_3)\, l_2 m_4 l_4 + \cos(\theta_4$$

$$+ \theta_3)\, l_2 m_5 l_4 + \cos(\theta_4 + \theta_3 + \theta_2)\, m_4 l_4 l_1 + m_5 \cos(\theta_4 + \theta_3 + \theta_2)\, l_4 l_1 + l_4^2 m_4 + l_4^2 m_5),$$

$$(l_3 m_4 \cos(\theta_4)\, l_4 + m_5 l_3 \cos(\theta_4)\, l_4 + l_5 \cos(\theta_5)\, m_5 l_4 + \cos(\theta_4 + \theta_3)\, l_2 m_4 l_4 + \cos(\theta_4$$

$$+ \theta_3)\, l_2 m_5 l_4 + l_4^2 m_4 + l_4^2 m_5),\ (l_3 m_4 \cos(\theta_4)\, l_4 + m_5 l_3 \cos(\theta_4)\, l_4 + l_5 \cos(\theta_5)\, m_5 l_4 + l_4^2 m_4$$

$$+ l_4^2 m_5),\ (l_5 \cos(\theta_5)\, m_5 l_4 + l_4^2 m_4 + l_4^2 m_5),\ \cos(\theta_5)\, l_4 l_5 m_5],$$

$$[(l_5 \cos(\theta_5)\, m_5 l_4 + l_5 l_2 m_5 \cos(\theta_5 + \theta_4 + \theta_3) + l_5 \cos(\theta_5 + \theta_4)\, m_5 l_3 + l_5 m_5 \cos(\theta_5 + \theta_4$$

$$+ \theta_3 + \theta_2)\, l_1 + l_5^2 m_5),\ (l_5 \cos(\theta_5)\, m_5 l_4 + l_5 l_2 m_5 \cos(\theta_5 + \theta_4 + \theta_3) + l_5 \cos(\theta_5 + \theta_4)\, m_5 l_3$$

$$+ l_5^2 m_5),\ (l_5 \cos(\theta_5)\, m_5 l_4 + l_5 \cos(\theta_5 + \theta_4)\, m_5 l_3 + l_5^2 m_5),\ (l_5 \cos(\theta_5)\, m_5 l_4 + l_5^2 m_5),\ l_5^2 m_5$$

]]

> $$[[-\sin(\theta_5+\theta_4+\theta_3+\theta_2) \left(\frac{d}{dt}\theta_5(t)\right)^2 l_1 l_5 m_5 + (-2 \left(\frac{d}{dt}\theta_1(t)\right) \sin(\theta_5+\theta_4+\theta_3$$

$$+\theta_2) l_1 l_5 m_5 - 2 \left(\frac{d}{dt}\theta_2(t)\right) \sin(\theta_5+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5 - 2 \left(\frac{d}{dt}\theta_3(t)\right) \sin(\theta_5+\theta_4$$

$$+\theta_3+\theta_2) l_1 l_5 m_5 - 2 \left(\frac{d}{dt}\theta_4(t)\right) \sin(\theta_5+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5) \left(\frac{d}{dt}\theta_5(t)\right) + ($$

$$-\sin(\theta_5+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5 - \sin(\theta_4+\theta_3+\theta_2) m_4 l_4 l_1 - \sin(\theta_4+\theta_3+\theta_2) m_5 l_4 l_1)$$

$$\left(\frac{d}{dt}\theta_4(t)\right)^2 + ((-2\sin(\theta_5+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5 - 2\sin(\theta_4+\theta_3+\theta_2) m_4 l_4 l_1$$

$$- 2\sin(\theta_4+\theta_3+\theta_2) m_5 l_4 l_1) \left(\frac{d}{dt}\theta_1(t)\right) + (-2\sin(\theta_5+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5$$

$$- 2\sin(\theta_4+\theta_3+\theta_2) m_4 l_4 l_1 - 2\sin(\theta_4+\theta_3+\theta_2) m_5 l_4 l_1) \left(\frac{d}{dt}\theta_2(t)\right) + (-2\sin(\theta_5$$

$$+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5 - 2\sin(\theta_4+\theta_3+\theta_2) m_4 l_4 l_1 - 2\sin(\theta_4+\theta_3+\theta_2) m_5 l_4 l_1)$$

$$\left(\frac{d}{dt}\theta_3(t)\right)) \left(\frac{d}{dt}\theta_4(t)\right) + (-\sin(\theta_3+\theta_2) l_1 l_3 m_3 - l_3 m_4 l_1 \sin(\theta_3+\theta_2)$$

$$- m_5 l_3 l_1 \sin(\theta_3+\theta_2) - l_2 m_2 \sin(\theta_2) l_1 - \sin(\theta_2) l_1 l_2 m_3 - l_2 m_4 \sin(\theta_2) l_1$$

$$- l_2 m_5 \sin(\theta_2) l_1 - \sin(\theta_5+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5 - \sin(\theta_4+\theta_3+\theta_2) m_4 l_4 l_1 - \sin(\theta_4$$

$$+\theta_3+\theta_2) m_5 l_4 l_1) \left(\frac{d}{dt}\theta_1(t)\right)^2 + ((-2\sin(\theta_3+\theta_2) l_1 l_3 m_3 - 2 l_3 m_4 l_1 \sin(\theta_3+\theta_2)$$

$$- 2 m_5 l_3 l_1 \sin(\theta_3+\theta_2) - 2 l_2 m_2 \sin(\theta_2) l_1 - 2\sin(\theta_2) l_1 l_2 m_3 - 2 l_2 m_4 \sin(\theta_2) l_1$$

$$- 2 l_2 m_5 \sin(\theta_2) l_1 - 2\sin(\theta_5+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5 - 2\sin(\theta_4+\theta_3+\theta_2) m_4 l_4 l_1$$

$$- 2\sin(\theta_4+\theta_3+\theta_2) m_5 l_4 l_1) \left(\frac{d}{dt}\theta_2(t)\right) + (-2\sin(\theta_3+\theta_2) l_1 l_3 m_3 - 2 l_3 m_4 l_1 \sin(\theta_3$$

$$+\theta_2) - 2 m_5 l_3 l_1 \sin(\theta_3+\theta_2) - 2\sin(\theta_5+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5 - 2\sin(\theta_4+\theta_3$$

$$+\theta_2) m_4 l_4 l_1 - 2\sin(\theta_4+\theta_3+\theta_2) m_5 l_4 l_1) \left(\frac{d}{dt}\theta_3(t)\right)) \left(\frac{d}{dt}\theta_1(t)\right) + (-\sin(\theta_3$$

$$+\theta_2) l_1 l_3 m_3 - l_3 m_4 l_1 \sin(\theta_3+\theta_2) - m_5 l_3 l_1 \sin(\theta_3+\theta_2) - l_2 m_2 \sin(\theta_2) l_1$$

$$- \sin(\theta_2) l_1 l_2 m_3 - l_2 m_4 \sin(\theta_2) l_1 - l_2 m_5 \sin(\theta_2) l_1 - \sin(\theta_5+\theta_4+\theta_3+\theta_2) l_1 l_5 m_5$$

$$- \sin(\theta_4+\theta_3+\theta_2) m_4 l_4 l_1 - \sin(\theta_4+\theta_3+\theta_2) m_5 l_4 l_1) \left(\frac{d}{dt}\theta_2(t)\right)^2 + (-\sin(\theta_3$$

$$+\theta_2) l_1 l_3 m_3 - l_3 m_4 l_1 \sin(\theta_3+\theta_2) - m_5 l_3 l_1 \sin(\theta_3+\theta_2) - \sin(\theta_5+\theta_4+\theta_3$$

$$+\theta_2)\, l_1\, l_5\, m_5-\sin(\theta_4+\theta_3+\theta_2)\, m_4\, l_4\, l_1-\sin(\theta_4+\theta_3+\theta_2)\, m_5\, l_4\, l_1)\, \left(\frac{d}{dt}\,\theta_3(t)\right)^2+($$

$$-2\sin(\theta_3+\theta_2)\, l_1\, l_3\, m_3-2\, l_3\, m_4\, l_1\sin(\theta_3+\theta_2)-2\, m_5\, l_3\, l_1\sin(\theta_3+\theta_2)-2\sin(\theta_5+\theta_4$$

$$+\theta_3+\theta_2)\, l_1\, l_5\, m_5-2\sin(\theta_4+\theta_3+\theta_2)\, m_4\, l_4\, l_1-2\sin(\theta_4+\theta_3+\theta_2)\, m_5\, l_4\, l_1)$$

$$\left(\frac{d}{dt}\,\theta_2(t)\right)\left(\frac{d}{dt}\,\theta_3(t)\right)],$$

$$[-\left(\frac{d}{dt}\,\theta_5(t)\right)^2\sin(\theta_5+\theta_4+\theta_3)\, l_2\, l_5\, m_5+(-2\, l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3)\left(\frac{d}{dt}\,\theta_1(t)\right)$$

$$-2\, l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3)\left(\frac{d}{dt}\,\theta_2(t)\right)-2\, l_5\, l_2\, m_5\left(\frac{d}{dt}\,\theta_3(t)\right)\sin(\theta_5+\theta_4+\theta_3)$$

$$-2\left(\frac{d}{dt}\,\theta_4(t)\right)\sin(\theta_5+\theta_4+\theta_3)\, l_2\, l_5\, m_5)\left(\frac{d}{dt}\,\theta_5(t)\right)+(-\sin(\theta_3)\, l_2\, l_3\, m_3$$

$$-l_2\, l_3\, m_4\sin(\theta_3)-l_2\, l_3\, m_5\sin(\theta_3)-\sin(\theta_4+\theta_3)\, l_2\, m_4\, l_4-\sin(\theta_4+\theta_3)\, l_2\, m_5\, l_4$$

$$-l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3))\left(\frac{d}{dt}\,\theta_2(t)\right)^2+((-2\sin(\theta_4+\theta_3)\, l_2\, m_4\, l_4-2\sin(\theta_4$$

$$+\theta_3)\, l_2\, m_5\, l_4-2\, l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3))\left(\frac{d}{dt}\,\theta_4(t)\right)+(-2\sin(\theta_3)\, l_2\, l_3\, m_3$$

$$-2\, l_2\, l_3\, m_4\sin(\theta_3)-2\, l_2\, l_3\, m_5\sin(\theta_3)-2\sin(\theta_4+\theta_3)\, l_2\, m_4\, l_4-2\sin(\theta_4+\theta_3)\, l_2\, m_5\, l_4$$

$$-2\, l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3))\left(\frac{d}{dt}\,\theta_3(t)\right)+(-2\sin(\theta_3)\, l_2\, l_3\, m_3-2\, l_2\, l_3\, m_4\sin(\theta_3)$$

$$-2\, l_2\, l_3\, m_5\sin(\theta_3)-2\sin(\theta_4+\theta_3)\, l_2\, m_4\, l_4-2\sin(\theta_4+\theta_3)\, l_2\, m_5\, l_4-2\, l_5\, l_2\, m_5\sin(\theta_5$$

$$+\theta_4+\theta_3))\left(\frac{d}{dt}\,\theta_1(t)\right))\left(\frac{d}{dt}\,\theta_2(t)\right)+(-\sin(\theta_4+\theta_3)\, l_2\, m_4\, l_4-\sin(\theta_4+\theta_3)\, l_2\, m_5\, l_4$$

$$-l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3))\left(\frac{d}{dt}\,\theta_4(t)\right)^2+((-2\sin(\theta_4+\theta_3)\, l_2\, m_4\, l_4-2\sin(\theta_4$$

$$+\theta_3)\, l_2\, m_5\, l_4-2\, l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3))\left(\frac{d}{dt}\,\theta_3(t)\right)+(-2\sin(\theta_4+\theta_3)\, l_2\, m_4\, l_4$$

$$-2\sin(\theta_4+\theta_3)\, l_2\, m_5\, l_4-2\, l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3))\left(\frac{d}{dt}\,\theta_1(t)\right))\left(\frac{d}{dt}\,\theta_4(t)\right)+($$

$$-\sin(\theta_3)\, l_2\, l_3\, m_3-l_2\, l_3\, m_4\sin(\theta_3)-l_2\, l_3\, m_5\sin(\theta_3)-\sin(\theta_4+\theta_3)\, l_2\, m_4\, l_4-\sin(\theta_4$$

$$+\theta_3)\, l_2\, m_5\, l_4-l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3))\left(\frac{d}{dt}\,\theta_3(t)\right)^2+(-2\sin(\theta_3)\, l_2\, l_3\, m_3$$

$$-2\, l_2\, l_3\, m_4\sin(\theta_3)-2\, l_2\, l_3\, m_5\sin(\theta_3)-2\sin(\theta_4+\theta_3)\, l_2\, m_4\, l_4-2\sin(\theta_4+\theta_3)\, l_2\, m_5\, l_4$$

$$-2\, l_5\, l_2\, m_5\sin(\theta_5+\theta_4+\theta_3))\left(\frac{d}{dt}\,\theta_1(t)\right)\left(\frac{d}{dt}\,\theta_3(t)\right)+(l_2\, m_2\sin(\theta_2)\, l_1$$

$$+\sin(\theta_2)\, l_1\, l_2\, m_3+l_2\, m_4\sin(\theta_2)\, l_1+l_2\, m_5\sin(\theta_2)\, l_1-\sin(\theta_3)\, l_2\, l_3\, m_3-l_2\, l_3\, m_4\sin(\theta_3)$$

$$- l_2 l_3 m_5 \sin(\theta_3) - \sin(\theta_4 + \theta_3) l_2 m_4 l_4 - \sin(\theta_4 + \theta_3) l_2 m_5 l_4 - l_5 l_2 m_5 \sin(\theta_5 + \theta_4$$

$$+ \theta_3)) \left(\frac{d}{dt} \theta_1(t)\right)^2 \Big],$$

$$\Big[\left(-\sin(\theta_4) l_3 m_4 l_4 - \sin(\theta_4) m_5 l_3 l_4 - l_5 m_5 l_3 \sin(\theta_5 + \theta_4)\right) \left(\frac{d}{dt} \theta_4(t)\right)^2 + \Big(\Big($$

$$-2 \sin(\theta_4) l_3 m_4 l_4 - 2 \sin(\theta_4) m_5 l_3 l_4 - 2 l_5 m_5 l_3 \sin(\theta_5 + \theta_4)) \left(\frac{d}{dt} \theta_2(t)\right)$$

$$- 2 \sin(\theta_4) l_3 m_4 \left(\frac{d}{dt} \theta_3(t)\right) l_4 - 2 \sin(\theta_4) m_5 l_3 \left(\frac{d}{dt} \theta_3(t)\right) l_4$$

$$- 2 l_5 m_5 l_3 \left(\frac{d}{dt} \theta_3(t)\right) \sin(\theta_5 + \theta_4) - 2 l_5 \left(\frac{d}{dt} \theta_5(t)\right) m_5 l_3 \sin(\theta_5 + \theta_4) + \Big($$

$$-2 \sin(\theta_4) l_3 m_4 l_4 - 2 \sin(\theta_4) m_5 l_3 l_4 - 2 l_5 m_5 l_3 \sin(\theta_5 + \theta_4)) \left(\frac{d}{dt} \theta_1(t)\right) \Big)$$

$$\left(\frac{d}{dt} \theta_4(t)\right) + \Big(\sin(\theta_3) l_2 l_3 m_3 + l_2 l_3 m_4 \sin(\theta_3) + l_2 l_3 m_5 \sin(\theta_3) - \sin(\theta_4) l_3 m_4 l_4$$

$$- \sin(\theta_4) m_5 l_3 l_4 - l_5 m_5 l_3 \sin(\theta_5 + \theta_4)\Big) \left(\frac{d}{dt} \theta_2(t)\right)^2 + \Big($$

$$-2 \sin(\theta_4) l_3 m_4 \left(\frac{d}{dt} \theta_3(t)\right) l_4 - 2 l_5 m_5 l_3 \left(\frac{d}{dt} \theta_3(t)\right) \sin(\theta_5 + \theta_4)$$

$$- 2 l_5 \left(\frac{d}{dt} \theta_5(t)\right) m_5 l_3 \sin(\theta_5 + \theta_4) - 2 \sin(\theta_4) m_5 l_3 \left(\frac{d}{dt} \theta_3(t)\right) l_4 + \Big(2 \sin(\theta_3) l_2 l_3 m_3$$

$$+ 2 l_2 l_3 m_4 \sin(\theta_3) + 2 l_2 l_3 m_5 \sin(\theta_3) - 2 \sin(\theta_4) l_3 m_4 l_4 - 2 \sin(\theta_4) m_5 l_3 l_4$$

$$- 2 l_5 m_5 l_3 \sin(\theta_5 + \theta_4)\Big) \left(\frac{d}{dt} \theta_1(t)\right) \Big) \left(\frac{d}{dt} \theta_2(t)\right) - 2 \left(\frac{d}{dt} \theta_3(t)\right) \left(\frac{d}{dt} \theta_5(t)\right) \sin(\theta_5$$

$$+ \theta_4) l_3 l_5 m_5 + \Big(\sin(\theta_3 + \theta_2) l_1 l_3 m_3 + l_3 m_4 l_1 \sin(\theta_3 + \theta_2) + m_5 l_3 l_1 \sin(\theta_3 + \theta_2)$$

$$+ \sin(\theta_3) l_2 l_3 m_3 + l_2 l_3 m_4 \sin(\theta_3) + l_2 l_3 m_5 \sin(\theta_3) - \sin(\theta_4) l_3 m_4 l_4 - \sin(\theta_4) m_5 l_3 l_4$$

$$- l_5 m_5 l_3 \sin(\theta_5 + \theta_4)\Big) \left(\frac{d}{dt} \theta_1(t)\right)^2 + \Big(-2 \sin(\theta_4) l_3 m_4 \left(\frac{d}{dt} \theta_3(t)\right) l_4$$

$$- 2 \sin(\theta_4) m_5 l_3 \left(\frac{d}{dt} \theta_3(t)\right) l_4 - 2 l_5 m_5 l_3 \left(\frac{d}{dt} \theta_3(t)\right) \sin(\theta_5 + \theta_4)$$

$$- 2 l_5 \left(\frac{d}{dt} \theta_5(t)\right) m_5 l_3 \sin(\theta_5 + \theta_4)\Big) \left(\frac{d}{dt} \theta_1(t)\right) - \left(\frac{d}{dt} \theta_3(t)\right)^2 \sin(\theta_4) l_3 l_4 m_5$$

$$- \left(\frac{d}{dt} \theta_3(t)\right)^2 \sin(\theta_4) l_3 l_4 m_4 - \left(\frac{d}{dt} \theta_3(t)\right)^2 \sin(\theta_5 + \theta_4) l_3 l_5 m_5 - \left(\frac{d}{dt} \theta_5(t)\right)^2 \sin(\theta_5$$

$$+ \theta_4) l_3 l_5 m_5 \Big],$$

$$\Big[\left(-l_5 m_5 \sin(\theta_5) l_4 + \sin(\theta_4) l_3 m_4 l_4 + \sin(\theta_4) m_5 l_3 l_4\right) \left(\frac{d}{dt} \theta_3(t)\right)^2 + \Big(\Big($$

$$-2 l_5 m_5 \sin(\theta_5) l_4 + 2 \sin(\theta_4) l_3 m_4 l_4 + 2 \sin(\theta_4) m_5 l_3 l_4) \left(\frac{d}{dt} \theta_2(t)\right)$$

$$
\begin{aligned}
&-2\sin(\theta_5)\left(\frac{d}{dt}\theta_4(t)\right) l_4 l_5 m_5 - 2 l_5\left(\frac{d}{dt}\theta_5(t)\right) m_5 \sin(\theta_5)\, l_4 + \big(-2 l_5 m_5 \sin(\theta_5)\, l_4 \\
&+ 2\sin(\theta_4)\, l_3 m_4 l_4 + 2\sin(\theta_4)\, m_5 l_3 l_4\big)\left(\frac{d}{dt}\theta_1(t)\right)\Bigg)\left(\frac{d}{dt}\theta_3(t)\right) + \big(-l_5 m_5 \sin(\theta_5)\, l_4 \\
&+ \sin(\theta_4)\, l_3 m_4 l_4 + \sin(\theta_4)\, m_5 l_3 l_4 + \sin(\theta_4+\theta_3)\, l_2 m_4 l_4 + \sin(\theta_4+\theta_3)\, l_2 m_5 l_4\big) \\
&\left(\frac{d}{dt}\theta_2(t)\right)^2 + \Bigg(-2\sin(\theta_5)\left(\frac{d}{dt}\theta_4(t)\right) l_4 l_5 m_5 - 2 l_5\left(\frac{d}{dt}\theta_5(t)\right) m_5 \sin(\theta_5)\, l_4 + \big(\\
&-2 l_5 m_5 \sin(\theta_5)\, l_4 + 2\sin(\theta_4)\, l_3 m_4 l_4 + 2\sin(\theta_4)\, m_5 l_3 l_4 + 2\sin(\theta_4+\theta_3)\, l_2 m_4 l_4 \\
&+ 2\sin(\theta_4+\theta_3)\, l_2 m_5 l_4\big)\left(\frac{d}{dt}\theta_1(t)\right)\Bigg)\left(\frac{d}{dt}\theta_2(t)\right) - \sin(\theta_5)\left(\frac{d}{dt}\theta_4(t)\right)^2 l_4 l_5 m_5 \\
&-2\sin(\theta_5)\left(\frac{d}{dt}\theta_4(t)\right)\left(\frac{d}{dt}\theta_5(t)\right) l_4 l_5 m_5 + \big(-l_5 m_5 \sin(\theta_5)\, l_4 + \sin(\theta_4)\, l_3 m_4 l_4 \\
&+ \sin(\theta_4)\, m_5 l_3 l_4 + \sin(\theta_4+\theta_3+\theta_2)\, m_4 l_4 l_1 + \sin(\theta_4+\theta_3+\theta_2)\, m_5 l_4 l_1 + \sin(\theta_4 \\
&+\theta_3)\, l_2 m_4 l_4 + \sin(\theta_4+\theta_3)\, l_2 m_5 l_4\big)\left(\frac{d}{dt}\theta_1(t)\right)^2 + \Bigg(-2\sin(\theta_5)\left(\frac{d}{dt}\theta_4(t)\right) l_4 l_5 m_5 \\
&-2 l_5\left(\frac{d}{dt}\theta_5(t)\right) m_5 \sin(\theta_5)\, l_4\Bigg)\left(\frac{d}{dt}\theta_1(t)\right) - \sin(\theta_5)\left(\frac{d}{dt}\theta_5(t)\right)^2 l_4 l_5 m_5\Bigg], \\
&\Bigg[\big(l_5 m_5 \sin(\theta_5)\, l_4 + l_5 m_5 l_3 \sin(\theta_5+\theta_4)\big)\left(\frac{d}{dt}\theta_3(t)\right)^2 + \Bigg(\Big(2 l_5 m_5 \sin(\theta_5)\, l_4 \\
&+ 2 l_5 m_5 l_3 \sin(\theta_5+\theta_4)\Big)\left(\frac{d}{dt}\theta_2(t)\right) + \Big(2 l_5 m_5 \sin(\theta_5)\, l_4 + 2 l_5 m_5 l_3 \sin(\theta_5 \\
&+\theta_4)\Big)\left(\frac{d}{dt}\theta_1(t)\right) + 2\sin(\theta_5)\left(\frac{d}{dt}\theta_4(t)\right) l_4 l_5 m_5\Bigg)\left(\frac{d}{dt}\theta_3(t)\right) + \big(l_5 m_5 \sin(\theta_5)\, l_4 \\
&+ l_5 l_2 m_5 \sin(\theta_5+\theta_4+\theta_3) + l_5 m_5 l_3 \sin(\theta_5+\theta_4)\big)\left(\frac{d}{dt}\theta_2(t)\right)^2 + \Bigg(\Big(2 l_5 m_5 \sin(\theta_5)\, l_4 \\
&+ 2 l_5 l_2 m_5 \sin(\theta_5+\theta_4+\theta_3) + 2 l_5 m_5 l_3 \sin(\theta_5+\theta_4)\Big)\left(\frac{d}{dt}\theta_1(t)\right) \\
&+ 2\sin(\theta_5)\left(\frac{d}{dt}\theta_4(t)\right) l_4 l_5 m_5\Bigg)\left(\frac{d}{dt}\theta_2(t)\right) + \big(l_5 m_5 \sin(\theta_5)\, l_4 + l_5 l_2 m_5 \sin(\theta_5+\theta_4 \\
&+\theta_3) + \sin(\theta_5+\theta_4+\theta_3+\theta_2)\, l_1 l_5 m_5 + l_5 m_5 l_3 \sin(\theta_5+\theta_4)\big)\left(\frac{d}{dt}\theta_1(t)\right)^2 \\
&+ 2\sin(\theta_5)\left(\frac{d}{dt}\theta_1(t)\right)\left(\frac{d}{dt}\theta_4(t)\right) l_4 l_5 m_5 + \sin(\theta_5)\left(\frac{d}{dt}\theta_4(t)\right)^2 l_4 l_5 m_5\Bigg]\Bigg]
\end{aligned}
$$

$$G := \begin{bmatrix} \left(\cos(\theta_1)\, l_1\, m_1 + \cos(\theta_1)\, l_1\, m_2 + \cos(\theta_1)\, l_1\, m_3 + \cos(\theta_1)\, l_1\, m_4 + \cos(\theta_1)\, l_1\, m_5\right) g \\ \left(\cos(\theta_2 + \theta_1)\, l_2\, m_2 + \cos(\theta_2 + \theta_1)\, l_2\, m_3 + \cos(\theta_2 + \theta_1)\, l_2\, m_4 + \cos(\theta_2 + \theta_1)\, l_2\, m_5\right) g \\ \left(\cos(\theta_3 + \theta_2 + \theta_1)\, l_3\, m_3 + \cos(\theta_3 + \theta_2 + \theta_1)\, l_3\, m_4 + \cos(\theta_3 + \theta_2 + \theta_1)\, l_3\, m_5\right) g \\ \left(\cos(\theta_3 + \theta_2 + \theta_1 + \theta_4)\, l_4\, m_4 + \cos(\theta_3 + \theta_2 + \theta_1 + \theta_4)\, l_4\, m_5\right) g \\ \cos(\theta_5 + \theta_3 + \theta_2 + \theta_1 + \theta_4)\, g\, l_5\, m_5 \end{bmatrix}$$

A3: Process of identifying repetitive terms with the help of Maple

Identifying repetitive terms τ_1 equation for a 2R planar manipulator using Maple

$$
\begin{aligned}
n1 := \Bigg[& [0], \\
& [0], \\
& \Bigg[l2\, m2 \left(l2 \left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x) \right) - s2 \left(-l1 \left(\frac{d}{dx} th1(x) \right)^2 + s1\, g \right) \right. \\
& \left. + c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) + c1\, g \right) \right) + l1\, m1 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) + c1\, g \right) + \Bigg(s2\, m2 \Bigg(\\
& -l2 \left(\frac{d}{dx} th1(x) + \frac{d}{dx} th2(x) \right)^2 + c2 \left(-l1 \left(\frac{d}{dx} th1(x) \right)^2 + s1\, g \right) \\
& + s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) + c1\, g \right) \Bigg) + c2\, m2 \left(l2 \left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x) \right) - s2 \Bigg(\right. \\
& \left. -l1 \left(\frac{d}{dx} th1(x) \right)^2 + s1\, g \Bigg) + c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) + c1\, g \right) \right) \Bigg) l1 \Bigg] \Bigg]
\end{aligned}
\tag{1}
$$

> $cost(optimise(n1))$

$$35\ functions + 19\ additions + 36\ multiplications \tag{2}$$

Identifying repetitive terms τ_1 equation for a 2R planar manipulator using Maple

$$n1 := \left[\left[0\right],\ \left[0\right],\ \left[l2\, m2 \left(l2 \left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x) \right) - s2 \left(-l1 \left(\frac{d}{dx} th1(x) \right)^2 + s1\, g \right) + c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) + c1\, g \right) \right) + l1\, m1 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) + c1\, g \right) + \left(s2\, m2 \left(-l2 \left(\frac{d}{dx} th1(x) + \frac{d}{dx} th2(x) \right)^2 + c2 \left(-l1 \left(\frac{d}{dx} th1(x) \right)^2 + s1\, g \right) + s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) + c1\, g \right) \right) + c2\, m2 \left(l2 \left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x) \right) - s2 \left(-l1 \left(\frac{d}{dx} th1(x) \right)^2 + s1\, g \right) + c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) + c1\, g \right) \right) \right) l1 \right]\right] \tag{1}$$

> $cost(optimise(n1))$

$$35\ functions + 19\ additions + 36\ multiplications \tag{2}$$

Identifying repetitive terms τ_1 equation for a 3R planar manipulator using Maple

$$n1 := \left[\left[0 \right], \right. \tag{1}$$

$$\left[0 \right],$$

$$\begin{aligned}
&[m3 (l3 (\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \mathrm{C} \frac{d^2}{dx^2} th3(x)) \mathrm{K}\, s3 (\mathrm{K}\, l2 (\frac{d}{dx} th1(x) \\
&\mathrm{C} \frac{d}{dx} th2(x))^2 \mathrm{C}\, c2 (\mathrm{K}\, l1 (\frac{d}{dx} th1(x))^2 \mathrm{C}\, s1\, g) \mathrm{C}\, s2 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g)) \\
&\mathrm{C}\, c3 (l2 (\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)) \mathrm{K}\, s2 (\mathrm{K}\, l1 (\frac{d}{dx} th1(x))^2 \mathrm{C}\, s1\, g) \\
&\mathrm{C}\, c2 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g)))) l3 \mathrm{C}\, m2 (l2 (\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)) \mathrm{K}\, s2 (\\
&\mathrm{K}\, l1 (\frac{d}{dx} th1(x))^2 \mathrm{C}\, s1\, g) \mathrm{C}\, c2 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g)) l2 \mathrm{C} (s3\, m3 (\\
&\mathrm{K}\, l3 (\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x))^2 \mathrm{C}\, c3 (\mathrm{K}\, l2 (\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x))^2 \\
&\mathrm{C}\, c2 (\mathrm{K}\, l1 (\frac{d}{dx} th1(x))^2 \mathrm{C}\, s1\, g) \mathrm{C}\, s2 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g)) \\
&\mathrm{C}\, s3 (l2 (\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)) \mathrm{K}\, s2 (\mathrm{K}\, l1 (\frac{d}{dx} th1(x))^2 \mathrm{C}\, s1\, g) \\
&\mathrm{C}\, c2 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g))) \mathrm{C}\, c3\, m3 (l3 (\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \\
&\mathrm{C} \frac{d^2}{dx^2} th3(x)) \mathrm{K}\, s3 (\mathrm{K}\, l2 (\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x))^2 \mathrm{C}\, c2 (\mathrm{K}\, l1 (\frac{d}{dx} th1(x))^2 \\
&\mathrm{C}\, s1\, g) \mathrm{C}\, s2 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g)) \mathrm{C}\, c3 (l2 (\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)) \\
&\mathrm{K}\, s2 (\mathrm{K}\, l1 (\frac{d}{dx} th1(x))^2 \mathrm{C}\, s1\, g) \mathrm{C}\, c2 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g))))) l2 \\
&\mathrm{C}\, m1 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g) l1 \mathrm{C} (s2\, m2 (\mathrm{K}\, l2 (\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x))^2 \mathrm{C}\, c2 (\\
&\mathrm{K}\, l1 (\frac{d}{dx} th1(x))^2 \mathrm{C}\, s1\, g) \mathrm{C}\, s2 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g)) \mathrm{C}\, c2\, m2 (l2 (\frac{d^2}{dx^2} th1(x) \\
&\mathrm{C} \frac{d^2}{dx^2} th2(x)) \mathrm{K}\, s2 (\mathrm{K}\, l1 (\frac{d}{dx} th1(x))^2 \mathrm{C}\, s1\, g) \mathrm{C}\, c2 (l1 (\frac{d^2}{dx^2} th1(x)) \mathrm{C}\, c1\, g))) \\
&l1]]
\end{aligned}$$

O $cost(optimise(n1))$

$$119\ \textit{functions} \subset 64\ \textit{additions} \subset 104\ \textit{multiplications} \quad \textbf{(2)}$$

Identifying repetitive terms τ_1 equation for a 4R planar manipulator using Maple

$$n1 := \left[\left[0\right],\right. \tag{1}$$

$$\left[0\right],$$

$$\left[l4\, m4\left(l4\left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x) + \frac{d^2}{dx^2} th3(x) + \frac{d^2}{dx^2} th4(x)\right) - s4\left(\right.\right.\right.$$

$$-l3\left(\frac{d}{dx} th1(x) + \frac{d}{dx} th2(x) + \frac{d}{dx} th3(x)\right)^2 + c3\left(-l2\left(\frac{d}{dx} th1(x) + \frac{d}{dx} th2(x)\right)^2\right.$$

$$\left.\left. + c2\left(-l1\left(\frac{d}{dx} th1(x)\right)^2 + s1\, g\right) + s2\left(l1\left(\frac{d^2}{dx^2} th1(x)\right) + c1\, g\right)\right)\right)$$

$$+ s3\left(l2\left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x)\right) - s2\left(-l1\left(\frac{d}{dx} th1(x)\right)^2 + s1\, g\right)\right.$$

$$\left.\left.\left. + c2\left(l1\left(\frac{d^2}{dx^2} th1(x)\right) + c1\, g\right)\right)\right)\right) + c4\left(l3\left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x)\right.\right.$$

$$\left. + \frac{d^2}{dx^2} th3(x)\right) - s3\left(-l2\left(\frac{d}{dx} th1(x) + \frac{d}{dx} th2(x)\right)^2 + c2\left(-l1\left(\frac{d}{dx} th1(x)\right)^2\right.\right.$$

$$\left.\left. + s1\, g\right) + s2\left(l1\left(\frac{d^2}{dx^2} th1(x)\right) + c1\, g\right)\right) + c3\left(l2\left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x)\right)\right.$$

$$\left.\left.\left.\left. - s2\left(-l1\left(\frac{d}{dx} th1(x)\right)^2 + s1\, g\right) + c2\left(l1\left(\frac{d^2}{dx^2} th1(x)\right) + c1\, g\right)\right)\right)\right)\right)$$

$$+ l3\, m3\left(l3\left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x) + \frac{d^2}{dx^2} th3(x)\right) - s3\left(-l2\left(\frac{d}{dx} th1(x)\right.\right.\right.$$

$$\left.\left. + \frac{d}{dx} th2(x)\right)^2 + c2\left(-l1\left(\frac{d}{dx} th1(x)\right)^2 + s1\, g\right) + s2\left(l1\left(\frac{d^2}{dx^2} th1(x)\right) + c1\, g\right)\right)$$

$$+ c3\left(l2\left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x)\right) - s2\left(-l1\left(\frac{d}{dx} th1(x)\right)^2 + s1\, g\right)\right.$$

$$\left.\left.\left. + c2\left(l1\left(\frac{d^2}{dx^2} th1(x)\right) + c1\, g\right)\right)\right)\right) + l3\left(s4\, m4\left(-l3\left(\frac{d}{dx} th1(x) + \frac{d}{dx} th2(x)\right.\right.\right.$$

$$\left. + \frac{d}{dx} th3(x) + \frac{d}{dx} th4(x)\right)^2 + c4\left(-l3\left(\frac{d}{dx} th1(x) + \frac{d}{dx} th2(x) + \frac{d}{dx} th3(x)\right)^2\right.$$

$$+ c3\left(-l2\left(\frac{d}{dx} th1(x) + \frac{d}{dx} th2(x)\right)^2 + c2\left(-l1\left(\frac{d}{dx} th1(x)\right)^2 + s1\, g\right)\right.$$

$$\left.\left. + s2\left(l1\left(\frac{d^2}{dx^2} th1(x)\right) + c1\, g\right)\right)\right) + s3\left(l2\left(\frac{d^2}{dx^2} th1(x) + \frac{d^2}{dx^2} th2(x)\right) - s2\left(\right.\right.$$

$$\left.\left.\left. -l1\left(\frac{d}{dx} th1(x)\right)^2 + s1\, g\right) + c2\left(l1\left(\frac{d^2}{dx^2} th1(x)\right) + c1\, g\right)\right)\right) + s4\left(l3\left(\frac{d^2}{dx^2} th1(x)\right.\right.$$

$$\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\Big)\,\mathrm{K}\,s3\,\Big(kl2\,\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\Big)^2\,\mathrm{C}\,c2\,\Big($$

$$kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2\,\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)\,\mathrm{C}\,c3\,\Big(l2\,\Big(\frac{d^2}{dx^2}\,th1(x)$$

$$\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\Big)\,\mathrm{K}\,s2\,\Big(kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2\,\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,c2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)\Big)\Big)$$

$$\mathrm{C}\,c4\,m4\,\Big(l4\,\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th4(x)\Big)\,\mathrm{K}\,s4\,\Big($$

$$kl3\,\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\,\mathrm{C}\,\frac{d}{dx}\,th3(x)\Big)^2\,\mathrm{C}\,c3\,\Big(kl2\,\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\Big)^2$$

$$\mathrm{C}\,c2\,\Big(kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2\,\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)$$

$$\mathrm{C}\,s3\,\Big(l2\,\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\Big)\,\mathrm{K}\,s2\,\Big(kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2\,\mathrm{C}\,s1\,g\Big)$$

$$\mathrm{C}\,c2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)\Big)\,\mathrm{C}\,c4\,\Big(l3\,\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)$$

$$\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\Big)\,\mathrm{K}\,s3\,\Big(kl2\,\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\Big)^2\,\mathrm{C}\,c2\,\Big(kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2$$

$$\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)\,\mathrm{C}\,c3\,\Big(l2\,\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\Big)$$

$$\mathrm{K}\,s2\,\Big(kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2\,\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,c2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)\Big)\Big)\Big)$$

$$\mathrm{C}\,l2\,m2\,\Big(l2\,\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\Big)\,\mathrm{K}\,s2\,\Big(kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2\,\mathrm{C}\,s1\,g\Big)$$

$$\mathrm{C}\,c2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)\,\mathrm{C}\,l2\,\Big(s3\,m3\,\Big(kl3\,\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)$$

$$\mathrm{C}\,\frac{d}{dx}\,th3(x)\Big)^2\,\mathrm{C}\,c3\,\Big(kl2\,\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\Big)^2\,\mathrm{C}\,c2\,\Big(kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2$$

$$\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)\,\mathrm{C}\,s3\,\Big(l2\,\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\Big)$$

$$\mathrm{K}\,s2\,\Big(kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2\,\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,c2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)\Big)$$

$$\mathrm{C}\,c3\,m3\,\Big(l3\,\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\Big)\,\mathrm{K}\,s3\,\Big(kl2\,\Big(\frac{d}{dx}\,th1(x)$$

$$\mathrm{C}\,\frac{d}{dx}\,th2(x)\Big)^2\,\mathrm{C}\,c2\,\Big(kl1\,\Big(\frac{d}{dx}\,th1(x)\Big)^2\,\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\,\Big(l1\,\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\,\mathrm{C}\,c1\,g\Big)\Big)$$

$$\mathrm{C}\, c3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \right) \mathrm{K}\, s2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \right.$$

$$\left. \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \right) \Bigg) \Bigg) \Bigg) \mathrm{C}\, l1\, m1 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \mathrm{C}\, l1 \Bigg(s2\, m2 \Bigg($$

$$\mathrm{K}\, l2 \left(\frac{d}{dx} th1(x) \,\mathrm{C}\, \frac{d}{dx} th2(x) \right)^2 \mathrm{C}\, c2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right)$$

$$\mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \mathrm{C}\, c2\, m2 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \right) \mathrm{K}\, s2 \right. \Bigg($$

$$\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \Bigg) \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \Bigg) \Bigg) \Bigg] \Bigg]$$

O *cost*(*optimise*(*n1*))

316 *functions* C 166 *additions* C 244 *multiplications* **(2)**

Identifying repetitive terms τ_1 equation for a 5R planar manipulator using Maple

$$n1 := \Big[\Big[\,[0], \qquad (1)$$

$$[0],$$

$$\Big[\, l5\; m5 \Big(l5 \Big(\frac{d^2}{dx^2} th1(x) \,C\, \frac{d^2}{dx^2} th2(x) \,C\, \frac{d^2}{dx^2} th3(x) \,C\, \frac{d^2}{dx^2} th4(x) \,C\, \frac{d^2}{dx^2} th5(x) \Big)$$

$$K\; s5 \Big(l4 \Big(\frac{d}{dx} th1(x) \,C\, \frac{d}{dx} th2(x) \,C\, \frac{d}{dx} th3(x) \,C\, \frac{d}{dx} th4(x) \Big)^2 C\; c4 \Big($$

$$l3 \Big(\frac{d}{dx} th1(x) \,C\, \frac{d}{dx} th2(x) \,C\, \frac{d}{dx} th3(x) \Big)^2 C\; c3 \Big(l2 \Big(\frac{d}{dx} th1(x) \,C\, \frac{d}{dx} th2(x) \Big)^2$$

$$C\; c2 \Big(l1 \Big(\frac{d}{dx} th1(x) \Big)^2 C\; s1\; g \Big) C\; s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) C\; c1\; g \Big) \Big)$$

$$C\; s3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \,C\, \frac{d^2}{dx^2} th2(x) \Big) K\; s2 \Big(l1 \Big(\frac{d}{dx} th1(x) \Big)^2 C\; s1\; g \Big)$$

$$C\; c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) C\; c1\; g \Big) \Big) \Big) C\; s4 \Big(l3 \Big(\frac{d^2}{dx^2} th1(x) \,C\, \frac{d^2}{dx^2} th2(x)$$

$$C\, \frac{d^2}{dx^2} th3(x) \Big) K\; s3 \Big(l2 \Big(\frac{d}{dx} th1(x) \,C\, \frac{d}{dx} th2(x) \Big)^2 C\; c2 \Big(l1 \Big(\frac{d}{dx} th1(x) \Big)^2$$

$$C\; s1\; g \Big) C\; s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) C\; c1\; g \Big) \Big) C\; c3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \,C\, \frac{d^2}{dx^2} th2(x) \Big)$$

$$K\; s2 \Big(l1 \Big(\frac{d}{dx} th1(x) \Big)^2 C\; s1\; g \Big) C\; c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) C\; c1\; g \Big) \Big) \Big) \Big)$$

$$C\; c5 \Big(l4 \Big(\frac{d^2}{dx^2} th1(x) \,C\, \frac{d^2}{dx^2} th2(x) \,C\, \frac{d^2}{dx^2} th3(x) \,C\, \frac{d^2}{dx^2} th4(x) \Big) K\; s4 \Big($$

$$l3 \Big(\frac{d}{dx} th1(x) \,C\, \frac{d}{dx} th2(x) \,C\, \frac{d}{dx} th3(x) \Big)^2 C\; c3 \Big(l2 \Big(\frac{d}{dx} th1(x) \,C\, \frac{d}{dx} th2(x) \Big)^2$$

$$C\; c2 \Big(l1 \Big(\frac{d}{dx} th1(x) \Big)^2 C\; s1\; g \Big) C\; s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) C\; c1\; g \Big) \Big)$$

$$C\; s3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \,C\, \frac{d^2}{dx^2} th2(x) \Big) K\; s2 \Big(l1 \Big(\frac{d}{dx} th1(x) \Big)^2 C\; s1\; g \Big)$$

$$C\; c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) C\; c1\; g \Big) \Big) \Big) C\; c4 \Big(l3 \Big(\frac{d^2}{dx^2} th1(x) \,C\, \frac{d^2}{dx^2} th2(x)$$

$$C\, \frac{d^2}{dx^2} th3(x) \Big) K\; s3 \Big(l2 \Big(\frac{d}{dx} th1(x) \,C\, \frac{d}{dx} th2(x) \Big)^2 C\; c2 \Big(l1 \Big(\frac{d}{dx} th1(x) \Big)^2$$

$$C\; s1\; g \Big) C\; s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) C\; c1\; g \Big) \Big) C\; c3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \,C\, \frac{d^2}{dx^2} th2(x) \Big)$$

$$-s2\left(-l1\left(\frac{d}{dx}\,th1(x)\right)^{2}+s1\,g\right)+c2\left(l1\left(\frac{d^{2}}{dx^{2}}\,th1(x)\right)+c1\,g\right)\Big)\Big)\Big)\Big)\Big)$$

$$+l4\,m4\Big(l4\left(\frac{d^{2}}{dx^{2}}\,th1(x)+\frac{d^{2}}{dx^{2}}\,th2(x)+\frac{d^{2}}{dx^{2}}\,th3(x)+\frac{d^{2}}{dx^{2}}\,th4(x)\right)-s4\Big($$

$$-l3\left(\frac{d}{dx}\,th1(x)+\frac{d}{dx}\,th2(x)+\frac{d}{dx}\,th3(x)\right)^{2}+c3\Big(-l2\left(\frac{d}{dx}\,th1(x)+\frac{d}{dx}\,th2(x)\right)^{2}$$

$$+c2\left(-l1\left(\frac{d}{dx}\,th1(x)\right)^{2}+s1\,g\right)+s2\left(l1\left(\frac{d^{2}}{dx^{2}}\,th1(x)\right)+c1\,g\right)\Big)$$

$$+s3\Big(l2\left(\frac{d^{2}}{dx^{2}}\,th1(x)+\frac{d^{2}}{dx^{2}}\,th2(x)\right)-s2\Big(-l1\left(\frac{d}{dx}\,th1(x)\right)^{2}+s1\,g\Big)$$

$$+c2\left(l1\left(\frac{d^{2}}{dx^{2}}\,th1(x)\right)+c1\,g\right)\Big)\Big)\Big)+c4\Big(l3\Big(\frac{d^{2}}{dx^{2}}\,th1(x)+\frac{d^{2}}{dx^{2}}\,th2(x)$$

$$+\frac{d^{2}}{dx^{2}}\,th3(x)\Big)-s3\Big(-l2\left(\frac{d}{dx}\,th1(x)+\frac{d}{dx}\,th2(x)\right)^{2}+c2\Big(-l1\left(\frac{d}{dx}\,th1(x)\right)^{2}$$

$$+s1\,g\Big)+s2\left(l1\left(\frac{d^{2}}{dx^{2}}\,th1(x)\right)+c1\,g\right)\Big)+c3\Big(l2\left(\frac{d^{2}}{dx^{2}}\,th1(x)+\frac{d^{2}}{dx^{2}}\,th2(x)\right)$$

$$-s2\left(-l1\left(\frac{d}{dx}\,th1(x)\right)^{2}+s1\,g\right)+c2\left(l1\left(\frac{d^{2}}{dx^{2}}\,th1(x)\right)+c1\,g\right)\Big)\Big)\Big)\Big)+l4\Big(s5\,m5\Big($$

$$-l5\left(\frac{d}{dx}\,th1(x)+\frac{d}{dx}\,th2(x)+\frac{d}{dx}\,th3(x)+\frac{d}{dx}\,th4(x)+\frac{d}{dx}\,th5(x)\right)^{2}+c5\Big($$

$$-l4\left(\frac{d}{dx}\,th1(x)+\frac{d}{dx}\,th2(x)+\frac{d}{dx}\,th3(x)+\frac{d}{dx}\,th4(x)\right)^{2}+c4\Big(-l3\Big(\frac{d}{dx}\,th1(x)$$

$$+\frac{d}{dx}\,th2(x)+\frac{d}{dx}\,th3(x)\Big)^{2}+c3\Big(-l2\left(\frac{d}{dx}\,th1(x)+\frac{d}{dx}\,th2(x)\right)^{2}+c2\Big($$

$$-l1\left(\frac{d}{dx}\,th1(x)\right)^{2}+s1\,g\Big)+s2\left(l1\left(\frac{d^{2}}{dx^{2}}\,th1(x)\right)+c1\,g\right)\Big)+s3\Big(l2\Big(\frac{d^{2}}{dx^{2}}\,th1(x)$$

$$+\frac{d^{2}}{dx^{2}}\,th2(x)\Big)-s2\left(-l1\left(\frac{d}{dx}\,th1(x)\right)^{2}+s1\,g\right)+c2\left(l1\left(\frac{d^{2}}{dx^{2}}\,th1(x)\right)+c1\,g\right)\Big)\Big)\Big)$$

$$+s4\Big(l3\left(\frac{d^{2}}{dx^{2}}\,th1(x)+\frac{d^{2}}{dx^{2}}\,th2(x)+\frac{d^{2}}{dx^{2}}\,th3(x)\right)-s3\Big(-l2\Big(\frac{d}{dx}\,th1(x)$$

$$+\frac{d}{dx}\,th2(x)\Big)^{2}+c2\left(-l1\left(\frac{d}{dx}\,th1(x)\right)^{2}+s1\,g\right)+s2\left(l1\left(\frac{d^{2}}{dx^{2}}\,th1(x)\right)+c1\,g\right)\Big)$$

$$+c3\Big(l2\left(\frac{d^{2}}{dx^{2}}\,th1(x)+\frac{d^{2}}{dx^{2}}\,th2(x)\right)-s2\Big(-l1\left(\frac{d}{dx}\,th1(x)\right)^{2}+s1\,g\Big)$$

$$+c2\left(l1\left(\frac{d^{2}}{dx^{2}}\,th1(x)\right)+c1\,g\right)\Big)\Big)\Big)\Big)+s5\Big(l4\Big(\frac{d^{2}}{dx^{2}}\,th1(x)+\frac{d^{2}}{dx^{2}}\,th2(x)$$

$$+\frac{d^2}{dx^2}th3(x)+\frac{d^2}{dx^2}th4(x)\Big)-s4\Big(-l3\Big(\frac{d}{dx}th1(x)+\frac{d}{dx}th2(x)+\frac{d}{dx}th3(x)\Big)^2$$

$$+c3\Big(-l2\Big(\frac{d}{dx}th1(x)+\frac{d}{dx}th2(x)\Big)^2+c2\Big(-l1\Big(\frac{d}{dx}th1(x)\Big)^2+s1\,g\Big)$$

$$+s2\Big(l1\Big(\frac{d^2}{dx^2}th1(x)\Big)+c1\,g\Big)\Big)+s3\Big(l2\Big(\frac{d^2}{dx^2}th1(x)+\frac{d^2}{dx^2}th2(x)\Big)-s2\Big($$

$$-l1\Big(\frac{d}{dx}th1(x)\Big)^2+s1\,g\Big)+c2\Big(l1\Big(\frac{d^2}{dx^2}th1(x)\Big)+c1\,g\Big)\Big)\Big)\Big)+c4\Big(l3\Big(\frac{d^2}{dx^2}th1(x)$$

$$+\frac{d^2}{dx^2}th2(x)+\frac{d^2}{dx^2}th3(x)\Big)-s3\Big(-l2\Big(\frac{d}{dx}th1(x)+\frac{d}{dx}th2(x)\Big)^2+c2\Big($$

$$-l1\Big(\frac{d}{dx}th1(x)\Big)^2+s1\,g\Big)+s2\Big(l1\Big(\frac{d^2}{dx^2}th1(x)\Big)+c1\,g\Big)\Big)+c3\Big(l2\Big(\frac{d^2}{dx^2}th1(x)$$

$$+\frac{d^2}{dx^2}th2(x)\Big)-s2\Big(-l1\Big(\frac{d}{dx}th1(x)\Big)^2+s1\,g\Big)+c2\Big(l1\Big(\frac{d^2}{dx^2}th1(x)\Big)+c1\,g\Big)\Big)\Big)\Big)\Big)$$

$$\Big)+c5\,m5\Big(l5\Big(\frac{d^2}{dx^2}th1(x)+\frac{d^2}{dx^2}th2(x)+\frac{d^2}{dx^2}th3(x)+\frac{d^2}{dx^2}th4(x)+\frac{d^2}{dx^2}th5(x)\Big)$$

$$-s5\Big(-l4\Big(\frac{d}{dx}th1(x)+\frac{d}{dx}th2(x)+\frac{d}{dx}th3(x)+\frac{d}{dx}th4(x)\Big)^2+c4\Big($$

$$-l3\Big(\frac{d}{dx}th1(x)+\frac{d}{dx}th2(x)+\frac{d}{dx}th3(x)\Big)^2+c3\Big(-l2\Big(\frac{d}{dx}th1(x)+\frac{d}{dx}th2(x)\Big)^2$$

$$+c2\Big(-l1\Big(\frac{d}{dx}th1(x)\Big)^2+s1\,g\Big)+s2\Big(l1\Big(\frac{d^2}{dx^2}th1(x)\Big)+c1\,g\Big)\Big)$$

$$+s3\Big(l2\Big(\frac{d^2}{dx^2}th1(x)+\frac{d^2}{dx^2}th2(x)\Big)-s2\Big(-l1\Big(\frac{d}{dx}th1(x)\Big)^2+s1\,g\Big)$$

$$+c2\Big(l1\Big(\frac{d^2}{dx^2}th1(x)\Big)+c1\,g\Big)\Big)\Big)+s4\Big(l3\Big(\frac{d^2}{dx^2}th1(x)+\frac{d^2}{dx^2}th2(x)$$

$$+\frac{d^2}{dx^2}th3(x)\Big)-s3\Big(-l2\Big(\frac{d}{dx}th1(x)+\frac{d}{dx}th2(x)\Big)^2+c2\Big(-l1\Big(\frac{d}{dx}th1(x)\Big)^2$$

$$+s1\,g\Big)+s2\Big(l1\Big(\frac{d^2}{dx^2}th1(x)\Big)+c1\,g\Big)\Big)+c3\Big(l2\Big(\frac{d^2}{dx^2}th1(x)+\frac{d^2}{dx^2}th2(x)\Big)$$

$$-s2\Big(-l1\Big(\frac{d}{dx}th1(x)\Big)^2+s1\,g\Big)+c2\Big(l1\Big(\frac{d^2}{dx^2}th1(x)\Big)+c1\,g\Big)\Big)\Big)\Big)\Big)$$

$$+c5\Big(l4\Big(\frac{d^2}{dx^2}th1(x)+\frac{d^2}{dx^2}th2(x)+\frac{d^2}{dx^2}th3(x)+\frac{d^2}{dx^2}th4(x)\Big)-s4\Big($$

$$-l3\Big(\frac{d}{dx}th1(x)+\frac{d}{dx}th2(x)+\frac{d}{dx}th3(x)\Big)^2+c3\Big(-l2\Big(\frac{d}{dx}th1(x)+\frac{d}{dx}th2(x)\Big)^2$$

$$\mathrm{C}\, c2 \left(kl1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big)$$

$$\mathrm{C}\, s3 \Big(l2 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right) \mathrm{K}\, s2 \left(kl1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right)$$

$$\mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big) \Big) \Big) \mathrm{C}\, c4 \Big(l3 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)$$

$$\mathrm{C} \frac{d^2}{dx^2} th3(x) \Big) \mathrm{K}\, s3 \Big(kl2 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \right)^2 \mathrm{C}\, c2 \Big(kl1 \left(\frac{d}{dx} th1(x) \right)^2$$

$$\mathrm{C}\, s1\, g \Big) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big) \mathrm{C}\, c3 \Big(l2 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right)$$

$$\mathrm{K}\, s2 \left(kl1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big) \Big) \Big) \Big) \Big)$$

$$\mathrm{C}\, l3\, m3 \Big(l3 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \mathrm{C} \frac{d^2}{dx^2} th3(x) \right) \mathrm{K}\, s3 \Big(kl2 \Big(\frac{d}{dx} th1(x)$$

$$\mathrm{C} \frac{d}{dx} th2(x) \Big)^2 \mathrm{C}\, c2 \left(kl1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big)$$

$$\mathrm{C}\, c3 \Big(l2 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right) \mathrm{K}\, s2 \left(kl1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right)$$

$$\mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big) \Big) \Big) \mathrm{C}\, l3 \Big(s4\, m4 \Big(kl4 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x)$$

$$\mathrm{C} \frac{d}{dx} th3(x) \mathrm{C} \frac{d}{dx} th4(x) \Big)^2 \mathrm{C}\, c4 \Big(kl3 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \right)^2$$

$$\mathrm{C}\, c3 \Big(kl2 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \right)^2 \mathrm{C}\, c2 \left(kl1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right)$$

$$\mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big) \mathrm{C}\, s3 \Big(l2 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right) \mathrm{K}\, s2 \Big($$

$$kl1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \Big) \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big) \Big) \mathrm{C}\, s4 \Big(l3 \Big(\frac{d^2}{dx^2} th1(x)$$

$$\mathrm{C} \frac{d^2}{dx^2} th2(x) \mathrm{C} \frac{d^2}{dx^2} th3(x) \Big) \mathrm{K}\, s3 \Big(kl2 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \right)^2 \mathrm{C}\, c2 \Big($$

$$kl1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \Big) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big) \mathrm{C}\, c3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x)$$

$$\mathrm{C} \frac{d^2}{dx^2} th2(x) \Big) \mathrm{K}\, s2 \left(kl1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Big) \Big) \Big) \Big)$$

$$\mathrm{C}\, c4\, m4 \Big(l4 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \mathrm{C} \frac{d^2}{dx^2} th3(x) \mathrm{C} \frac{d^2}{dx^2} th4(x) \right) \mathrm{K}\, s4 \Big($$

$$K l3 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x) C \frac{d}{dx} th3(x)\right)^2 C c3 \left(K l2 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right)^2\right.$$

$$C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1 g\right) C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big)\Big)$$

$$C s3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right) K s2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1 g\right)\right.$$

$$C c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big)\Big)\Big) C c4 \left(l3 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right.\right.$$

$$C \frac{d^2}{dx^2} th3(x)\Big) K s3 \left(K l2 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right)^2 C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2\right.\right.$$

$$C s1 g\Big) C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big) C c3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right)\right.$$

$$K s2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1 g\right) C c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big)\Big)\Big)\Big)$$

$$C l2\, m2 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right) K s2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1 g\right)\right.$$

$$C c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big) C l2 \left(s3\, m3 \left(K l3 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right.\right.\right.$$

$$C \frac{d}{dx} th3(x)\Big)^2 C c3 \left(K l2 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right)^2 C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2\right.\right.$$

$$C s1 g\Big) C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big) C s3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right)\right.$$

$$K s2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1 g\right) C c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big)\Big)$$

$$C c3\, m3 \left(l3 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x) C \frac{d^2}{dx^2} th3(x)\right) K s3 \left(K l2 \left(\frac{d}{dx} th1(x)\right.\right.\right.$$

$$C \frac{d}{dx} th2(x)\Big)^2 C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1 g\right) C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big)\Big)$$

$$C c3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right) K s2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1 g\right)\right.$$

$$C c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big)\Big)\Big)\Big) C l1\, m1 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right) C l1 \left(s2\, m2 \left(\right.\right.$$

$$K l2 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right)^2 C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1 g\right)$$

$$C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1 g\right)\Big) C c2\, m2 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right) K s2 \left(\right.\right.$$

$$\mathrm{K}\,l1\left(\frac{\mathrm{d}}{\mathrm{d}x}\,th1(x)\right)^{2}\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,c2\,\left(l1\left(\frac{\mathrm{d}^{2}}{\mathrm{d}x^{2}}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Big)\Big)\Big]\Big]$$

O $cost(optimise(n1))$

$$746\ functions\ \mathrm{C}\ 385\ additions\ \mathrm{C}\ 528\ multiplications \qquad (2)$$

Identifying repetitive terms τ_1 equation for a 6R planar manipulator using Maple

O

$$n1 := \Big[\big[0\big], \qquad \textbf{(1)}$$

$$\big[0\big],$$

$$\Big[l6\, m6 \Big(l6 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \mathrm{C} \frac{d^2}{dx^2} th3(x) \mathrm{C} \frac{d^2}{dx^2} th4(x) \mathrm{C} \frac{d^2}{dx^2} th5(x)$$

$$\mathrm{C} \frac{d^2}{dx^2} th6(x)\Big) \mathrm{K}\, s6 \Big(\mathrm{K} l5 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \mathrm{C} \frac{d}{dx} th4(x)$$

$$\mathrm{C} \frac{d}{dx} th5(x)\Big)^2 \mathrm{C}\, c5 \Big(\mathrm{K} l4 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \mathrm{C} \frac{d}{dx} th4(x)\Big)^2$$

$$\mathrm{C}\, c4 \Big(\mathrm{K} l3 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x)\Big)^2 \mathrm{C}\, c3 \Big(\mathrm{K} l2 \Big(\frac{d}{dx} th1(x)$$

$$\mathrm{C} \frac{d}{dx} th2(x)\Big)^2 \mathrm{C}\, c2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x)\Big)^2 \mathrm{C}\, s1\, g\Big) \mathrm{C}\, s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x)\Big) \mathrm{C}\, c1\, g\Big)\Big)$$

$$\mathrm{C}\, s3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)\Big) \mathrm{K}\, s2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x)\Big)^2 \mathrm{C}\, s1\, g\Big)$$

$$\mathrm{C}\, c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x)\Big) \mathrm{C}\, c1\, g\Big)\Big)\Big) \mathrm{C}\, s4 \Big(l3 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)$$

$$\mathrm{C} \frac{d^2}{dx^2} th3(x)\Big) \mathrm{K}\, s3 \Big(\mathrm{K} l2 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x)\Big)^2 \mathrm{C}\, c2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x)\Big)^2$$

$$\mathrm{C}\, s1\, g\Big) \mathrm{C}\, s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x)\Big) \mathrm{C}\, c1\, g\Big)\Big) \mathrm{C}\, c3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)\Big)$$

$$\mathrm{K}\, s2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x)\Big)^2 \mathrm{C}\, s1\, g\Big) \mathrm{C}\, c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x)\Big) \mathrm{C}\, c1\, g\Big)\Big)\Big)\Big)$$

$$\mathrm{C}\, s5 \Big(l4 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \mathrm{C} \frac{d^2}{dx^2} th3(x) \mathrm{C} \frac{d^2}{dx^2} th4(x)\Big) \mathrm{K}\, s4 \Big($$

$$\mathrm{K} l3 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x)\Big)^2 \mathrm{C}\, c3 \Big(\mathrm{K} l2 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x)\Big)^2$$

$$\mathrm{C}\, c2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x)\Big)^2 \mathrm{C}\, s1\, g\Big) \mathrm{C}\, s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x)\Big) \mathrm{C}\, c1\, g\Big)\Big)$$

$$\mathrm{C}\, s3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)\Big) \mathrm{K}\, s2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x)\Big)^2 \mathrm{C}\, s1\, g\Big)$$

$$\mathrm{C}\, c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x)\Big) \mathrm{C}\, c1\, g\Big)\Big)\Big) \mathrm{C}\, c4 \Big(l3 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)$$

$$\mathrm{C} \frac{d^2}{dx^2} th3(x)\Big) \mathrm{K}\, s3 \Big(\mathrm{K} l2 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x)\Big)^2 \mathrm{C}\, c2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x)\Big)^2$$

$$\mathrm{C}\, s1\, g\Big)\, \mathrm{C}\, s2\, \Big(l1\, \Big(\frac{d^2}{dx^2}\, th1(x)\Big)\, \mathrm{C}\, c1\, g\Big)\Big)\, \mathrm{C}\, c3\, \Big(l2\, \Big(\frac{d^2}{dx^2}\, th1(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th2(x)\Big)$$

$$\mathrm{K}\, s2\, \Big(\mathrm{k}l1\, \Big(\frac{d}{dx}\, th1(x)\Big)^2\, \mathrm{C}\, s1\, g\Big)\, \mathrm{C}\, c2\, \Big(l1\, \Big(\frac{d^2}{dx^2}\, th1(x)\Big)\, \mathrm{C}\, c1\, g\Big)\Big)\Big)\Big)\Big)$$

$$\mathrm{C}\, c6\, \Big(l5\, \Big(\frac{d^2}{dx^2}\, th1(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th2(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th3(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th4(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th5(x)\Big)$$

$$\mathrm{K}\, s5\, \Big(\mathrm{k}l4\, \Big(\frac{d}{dx}\, th1(x)\, \mathrm{C}\, \frac{d}{dx}\, th2(x)\, \mathrm{C}\, \frac{d}{dx}\, th3(x)\, \mathrm{C}\, \frac{d}{dx}\, th4(x)\Big)^2\, \mathrm{C}\, c4\, \Big($$

$$\mathrm{k}l3\, \Big(\frac{d}{dx}\, th1(x)\, \mathrm{C}\, \frac{d}{dx}\, th2(x)\, \mathrm{C}\, \frac{d}{dx}\, th3(x)\Big)^2\, \mathrm{C}\, c3\, \Big(\mathrm{k}l2\, \Big(\frac{d}{dx}\, th1(x)\, \mathrm{C}\, \frac{d}{dx}\, th2(x)\Big)^2$$

$$\mathrm{C}\, c2\, \Big(\mathrm{k}l1\, \Big(\frac{d}{dx}\, th1(x)\Big)^2\, \mathrm{C}\, s1\, g\Big)\, \mathrm{C}\, s2\, \Big(l1\, \Big(\frac{d^2}{dx^2}\, th1(x)\Big)\, \mathrm{C}\, c1\, g\Big)\Big)$$

$$\mathrm{C}\, s3\, \Big(l2\, \Big(\frac{d^2}{dx^2}\, th1(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th2(x)\Big)\, \mathrm{K}\, s2\, \Big(\mathrm{k}l1\, \Big(\frac{d}{dx}\, th1(x)\Big)^2\, \mathrm{C}\, s1\, g\Big)$$

$$\mathrm{C}\, c2\, \Big(l1\, \Big(\frac{d^2}{dx^2}\, th1(x)\Big)\, \mathrm{C}\, c1\, g\Big)\Big)\Big)\, \mathrm{C}\, s4\, \Big(l3\, \Big(\frac{d^2}{dx^2}\, th1(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th2(x)$$

$$\mathrm{C}\, \frac{d^2}{dx^2}\, th3(x)\Big)\, \mathrm{K}\, s3\, \Big(\mathrm{k}l2\, \Big(\frac{d}{dx}\, th1(x)\, \mathrm{C}\, \frac{d}{dx}\, th2(x)\Big)^2\, \mathrm{C}\, c2\, \Big(\mathrm{k}l1\, \Big(\frac{d}{dx}\, th1(x)\Big)^2$$

$$\mathrm{C}\, s1\, g\Big)\, \mathrm{C}\, s2\, \Big(l1\, \Big(\frac{d^2}{dx^2}\, th1(x)\Big)\, \mathrm{C}\, c1\, g\Big)\Big)\, \mathrm{C}\, c3\, \Big(l2\, \Big(\frac{d^2}{dx^2}\, th1(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th2(x)\Big)$$

$$\mathrm{K}\, s2\, \Big(\mathrm{k}l1\, \Big(\frac{d}{dx}\, th1(x)\Big)^2\, \mathrm{C}\, s1\, g\Big)\, \mathrm{C}\, c2\, \Big(l1\, \Big(\frac{d^2}{dx^2}\, th1(x)\Big)\, \mathrm{C}\, c1\, g\Big)\Big)\Big)\Big)$$

$$\mathrm{C}\, c5\, \Big(l4\, \Big(\frac{d^2}{dx^2}\, th1(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th2(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th3(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th4(x)\Big)\, \mathrm{K}\, s4\, \Big($$

$$\mathrm{k}l3\, \Big(\frac{d}{dx}\, th1(x)\, \mathrm{C}\, \frac{d}{dx}\, th2(x)\, \mathrm{C}\, \frac{d}{dx}\, th3(x)\Big)^2\, \mathrm{C}\, c3\, \Big(\mathrm{k}l2\, \Big(\frac{d}{dx}\, th1(x)\, \mathrm{C}\, \frac{d}{dx}\, th2(x)\Big)^2$$

$$\mathrm{C}\, c2\, \Big(\mathrm{k}l1\, \Big(\frac{d}{dx}\, th1(x)\Big)^2\, \mathrm{C}\, s1\, g\Big)\, \mathrm{C}\, s2\, \Big(l1\, \Big(\frac{d^2}{dx^2}\, th1(x)\Big)\, \mathrm{C}\, c1\, g\Big)\Big)$$

$$\mathrm{C}\, s3\, \Big(l2\, \Big(\frac{d^2}{dx^2}\, th1(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th2(x)\Big)\, \mathrm{K}\, s2\, \Big(\mathrm{k}l1\, \Big(\frac{d}{dx}\, th1(x)\Big)^2\, \mathrm{C}\, s1\, g\Big)$$

$$\mathrm{C}\, c2\, \Big(l1\, \Big(\frac{d^2}{dx^2}\, th1(x)\Big)\, \mathrm{C}\, c1\, g\Big)\Big)\Big)\, \mathrm{C}\, c4\, \Big(l3\, \Big(\frac{d^2}{dx^2}\, th1(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th2(x)$$

$$\mathrm{C}\, \frac{d^2}{dx^2}\, th3(x)\Big)\, \mathrm{K}\, s3\, \Big(\mathrm{k}l2\, \Big(\frac{d}{dx}\, th1(x)\, \mathrm{C}\, \frac{d}{dx}\, th2(x)\Big)^2\, \mathrm{C}\, c2\, \Big(\mathrm{k}l1\, \Big(\frac{d}{dx}\, th1(x)\Big)^2$$

$$\mathrm{C}\, s1\, g\Big)\, \mathrm{C}\, s2\, \Big(l1\, \Big(\frac{d^2}{dx^2}\, th1(x)\Big)\, \mathrm{C}\, c1\, g\Big)\Big)\, \mathrm{C}\, c3\, \Big(l2\, \Big(\frac{d^2}{dx^2}\, th1(x)\, \mathrm{C}\, \frac{d^2}{dx^2}\, th2(x)\Big)$$

$$\mathrm{K}\, s2 \left(\mathrm{k}l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \Bigg) \Bigg) \Bigg) \Bigg)$$

$$\mathrm{C}\, l5\, m5 \left(l5 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \mathrm{C} \frac{d^2}{dx^2} th3(x) \mathrm{C} \frac{d^2}{dx^2} th4(x) \mathrm{C} \frac{d^2}{dx^2} th5(x) \right) \right.$$

$$\mathrm{K}\, s5 \left(\mathrm{k}l4 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \mathrm{C} \frac{d}{dx} th4(x) \right)^2 \mathrm{C}\, c4 \Bigg(\right.$$

$$\mathrm{k}l3 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \right)^2 \mathrm{C}\, c3 \left(\mathrm{k}l2 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \right)^2 \right.$$

$$\mathrm{C}\, c2 \left(\mathrm{k}l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg)$$

$$\mathrm{C}\, s3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right) \mathrm{K}\, s2 \left(\mathrm{k}l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \right.$$

$$\mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \Bigg) \mathrm{C}\, s4 \left(l3 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right. \right.$$

$$\mathrm{C} \frac{d^2}{dx^2} th3(x) \Bigg) \mathrm{K}\, s3 \left(\mathrm{k}l2 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \right)^2 \mathrm{C}\, c2 \left(\mathrm{k}l1 \left(\frac{d}{dx} th1(x) \right)^2 \right. \right.$$

$$\mathrm{C}\, s1\, g \Bigg) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \mathrm{C}\, c3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right) \right.$$

$$\mathrm{K}\, s2 \left(\mathrm{k}l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \Bigg) \Bigg)$$

$$\mathrm{C}\, c5 \left(l4 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \mathrm{C} \frac{d^2}{dx^2} th3(x) \mathrm{C} \frac{d^2}{dx^2} th4(x) \right) \mathrm{K}\, s4 \Bigg(\right.$$

$$\mathrm{k}l3 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \right)^2 \mathrm{C}\, c3 \left(\mathrm{k}l2 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \right)^2 \right.$$

$$\mathrm{C}\, c2 \left(\mathrm{k}l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg)$$

$$\mathrm{C}\, s3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right) \mathrm{K}\, s2 \left(\mathrm{k}l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \right.$$

$$\mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \Bigg) \mathrm{C}\, c4 \left(l3 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right. \right.$$

$$\mathrm{C} \frac{d^2}{dx^2} th3(x) \Bigg) \mathrm{K}\, s3 \left(\mathrm{k}l2 \left(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \right)^2 \mathrm{C}\, c2 \left(\mathrm{k}l1 \left(\frac{d}{dx} th1(x) \right)^2 \right. \right.$$

$$\mathrm{C}\, s1\, g \Bigg) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \mathrm{C}\, c3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \right) \right.$$

$$\mathrm{K}\, s2 \left(\mathrm{k}l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \Bigg) \Bigg) \Bigg) \mathrm{C}\, l5 \left(s6\, m6 \Bigg(\right.$$

$$\mathrm{K}\,l6\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\,\mathrm{C}\,\frac{d}{dx}\,th3(x)\,\mathrm{C}\,\frac{d}{dx}\,th4(x)\,\mathrm{C}\,\frac{d}{dx}\,th5(x)\,\mathrm{C}\,\frac{d}{dx}\,th6(x)\right)^2$$

$$\mathrm{C}\,c6\left(\mathrm{K}\,l5\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\,\mathrm{C}\,\frac{d}{dx}\,th3(x)\,\mathrm{C}\,\frac{d}{dx}\,th4(x)\,\mathrm{C}\,\frac{d}{dx}\,th5(x)\right)^2\mathrm{C}\,c5\left(\right.\right.$$

$$\mathrm{K}\,l4\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\,\mathrm{C}\,\frac{d}{dx}\,th3(x)\,\mathrm{C}\,\frac{d}{dx}\,th4(x)\right)^2\mathrm{C}\,c4\left(\mathrm{K}\,l3\left(\frac{d}{dx}\,th1(x)\right.\right.$$

$$\left.\mathrm{C}\,\frac{d}{dx}\,th2(x)\,\mathrm{C}\,\frac{d}{dx}\,th3(x)\right)^2\mathrm{C}\,c3\left(\mathrm{K}\,l2\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\right)^2\mathrm{C}\,c2\left(\right.\right.$$

$$\left.\mathrm{K}\,l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,s2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Bigg)\,\mathrm{C}\,s3\left(l2\left(\frac{d^2}{dx^2}\,th1(x)\right.\right.$$

$$\left.\left.\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right)\mathrm{K}\,s2\left(\mathrm{K}\,l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,c2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\right)\Bigg)\Bigg)$$

$$\mathrm{C}\,s4\left(l3\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\right)\mathrm{K}\,s3\left(\mathrm{K}\,l2\left(\frac{d}{dx}\,th1(x)\right.\right.\right.$$

$$\left.\mathrm{C}\,\frac{d}{dx}\,th2(x)\right)^2\mathrm{C}\,c2\left(\mathrm{K}\,l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,s2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Bigg)$$

$$\mathrm{C}\,c3\left(l2\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right)\mathrm{K}\,s2\left(\mathrm{K}\,l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\right.$$

$$\left.\mathrm{C}\,c2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\right)\Bigg)\Bigg)\,\mathrm{C}\,s5\left(l4\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right.\right.$$

$$\left.\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th4(x)\right)\mathrm{K}\,s4\left(\mathrm{K}\,l3\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\,\mathrm{C}\,\frac{d}{dx}\,th3(x)\right)^2\right.$$

$$\mathrm{C}\,c3\left(\mathrm{K}\,l2\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\right)^2\mathrm{C}\,c2\left(\mathrm{K}\,l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\right.$$

$$\left.\mathrm{C}\,s2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\right)\mathrm{C}\,s3\left(l2\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right)\mathrm{K}\,s2\left(\right.\right.$$

$$\left.\mathrm{K}\,l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,c2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Bigg)\Bigg)\,\mathrm{C}\,c4\left(l3\left(\frac{d^2}{dx^2}\,th1(x)\right.\right.$$

$$\left.\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\right)\mathrm{K}\,s3\left(\mathrm{K}\,l2\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\right)^2\mathrm{C}\,c2\left(\right.\right.$$

$$\left.\mathrm{K}\,l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,s2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Bigg)\,\mathrm{C}\,c3\left(l2\left(\frac{d^2}{dx^2}\,th1(x)\right.\right.$$

$$\left.\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right)\mathrm{K}\,s2\left(\mathrm{K}\,l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,c2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Bigg)\Bigg)\Bigg)\Bigg)$$

$$\Bigg)\,\mathrm{C}\,s6\left(l5\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th4(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th5(x)\right)\right.$$

$$\mathrm{K}\, s5 \left(\mathrm{K}\, l4 \left(\frac{d}{dx} th1(x) \,\mathrm{C}\, \frac{d}{dx} th2(x) \,\mathrm{C}\, \frac{d}{dx} th3(x) \,\mathrm{C}\, \frac{d}{dx} th4(x) \right)^2 \mathrm{C}\, c4 \right($$

$$\mathrm{K}\, l3 \left(\frac{d}{dx} th1(x) \,\mathrm{C}\, \frac{d}{dx} th2(x) \,\mathrm{C}\, \frac{d}{dx} th3(x) \right)^2 \mathrm{C}\, c3 \left(\mathrm{K}\, l2 \left(\frac{d}{dx} th1(x) \,\mathrm{C}\, \frac{d}{dx} th2(x) \right)^2 \right.$$

$$\left. \mathrm{C}\, c2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \right)$$

$$\mathrm{C}\, s3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \right) \mathrm{K}\, s2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \right.$$

$$\left. \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \right) \Bigg) \mathrm{C}\, s4 \left(l3 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \right. \right.$$

$$\left. \mathrm{C}\, \frac{d^2}{dx^2} th3(x) \right) \mathrm{K}\, s3 \left(\mathrm{K}\, l2 \left(\frac{d}{dx} th1(x) \,\mathrm{C}\, \frac{d}{dx} th2(x) \right)^2 \mathrm{C}\, c2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \right. \right.$$

$$\left. \mathrm{C}\, s1\, g \right) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \mathrm{C}\, c3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \right) \right.$$

$$\left. \mathrm{K}\, s2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \right) \Bigg) \Bigg) \Bigg)$$

$$\mathrm{C}\, c5 \left(l4 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th3(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th4(x) \right) \mathrm{K}\, s4 \right($$

$$\mathrm{K}\, l3 \left(\frac{d}{dx} th1(x) \,\mathrm{C}\, \frac{d}{dx} th2(x) \,\mathrm{C}\, \frac{d}{dx} th3(x) \right)^2 \mathrm{C}\, c3 \left(\mathrm{K}\, l2 \left(\frac{d}{dx} th1(x) \,\mathrm{C}\, \frac{d}{dx} th2(x) \right)^2 \right.$$

$$\left. \mathrm{C}\, c2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \right)$$

$$\mathrm{C}\, s3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \right) \mathrm{K}\, s2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \right.$$

$$\left. \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \right) \Bigg) \mathrm{C}\, c4 \left(l3 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \right. \right.$$

$$\left. \mathrm{C}\, \frac{d^2}{dx^2} th3(x) \right) \mathrm{K}\, s3 \left(\mathrm{K}\, l2 \left(\frac{d}{dx} th1(x) \,\mathrm{C}\, \frac{d}{dx} th2(x) \right)^2 \mathrm{C}\, c2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \right. \right.$$

$$\left. \mathrm{C}\, s1\, g \right) \mathrm{C}\, s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \Bigg) \mathrm{C}\, c3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \right) \right.$$

$$\left. \mathrm{K}\, s2 \left(\mathrm{K}\, l1 \left(\frac{d}{dx} th1(x) \right)^2 \mathrm{C}\, s1\, g \right) \mathrm{C}\, c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x) \right) \mathrm{C}\, c1\, g \right) \right) \Bigg) \Bigg) \Bigg) \Bigg) \Bigg)$$

$$\mathrm{C}\, c6\, m6 \left(l6 \left(\frac{d^2}{dx^2} th1(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th2(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th3(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th4(x) \,\mathrm{C}\, \frac{d^2}{dx^2} th5(x) \right. \right.$$

$$\left. \mathrm{C}\, \frac{d^2}{dx^2} th6(x) \right) \mathrm{K}\, s6 \left(\mathrm{K}\, l5 \left(\frac{d}{dx} th1(x) \,\mathrm{C}\, \frac{d}{dx} th2(x) \,\mathrm{C}\, \frac{d}{dx} th3(x) \,\mathrm{C}\, \frac{d}{dx} th4(x) \right. \right.$$

$$+\frac{d}{dx}\mathit{th5}(x)\Big)^2+c5\Big(-l4\Big(\frac{d}{dx}\mathit{th1}(x)+\frac{d}{dx}\mathit{th2}(x)+\frac{d}{dx}\mathit{th3}(x)+\frac{d}{dx}\mathit{th4}(x)\Big)^2$$

$$+c4\Big(-l3\Big(\frac{d}{dx}\mathit{th1}(x)+\frac{d}{dx}\mathit{th2}(x)+\frac{d}{dx}\mathit{th3}(x)\Big)^2+c3\Big(-l2\Big(\frac{d}{dx}\mathit{th1}(x)$$

$$+\frac{d}{dx}\mathit{th2}(x)\Big)^2+c2\Big(-l1\Big(\frac{d}{dx}\mathit{th1}(x)\Big)^2+s1\,g\Big)+s2\Big(l1\Big(\frac{d^2}{dx^2}\mathit{th1}(x)\Big)+c1\,g\Big)\Big)$$

$$+s3\Big(l2\Big(\frac{d^2}{dx^2}\mathit{th1}(x)+\frac{d^2}{dx^2}\mathit{th2}(x)\Big)-s2\Big(-l1\Big(\frac{d}{dx}\mathit{th1}(x)\Big)^2+s1\,g\Big)$$

$$+c2\Big(l1\Big(\frac{d^2}{dx^2}\mathit{th1}(x)\Big)+c1\,g\Big)\Big)\Big)+s4\Big(l3\Big(\frac{d^2}{dx^2}\mathit{th1}(x)+\frac{d^2}{dx^2}\mathit{th2}(x)$$

$$+\frac{d^2}{dx^2}\mathit{th3}(x)\Big)-s3\Big(-l2\Big(\frac{d}{dx}\mathit{th1}(x)+\frac{d}{dx}\mathit{th2}(x)\Big)^2+c2\Big(-l1\Big(\frac{d}{dx}\mathit{th1}(x)\Big)^2$$

$$+s1\,g\Big)+s2\Big(l1\Big(\frac{d^2}{dx^2}\mathit{th1}(x)\Big)+c1\,g\Big)\Big)+c3\Big(l2\Big(\frac{d^2}{dx^2}\mathit{th1}(x)+\frac{d^2}{dx^2}\mathit{th2}(x)\Big)$$

$$-s2\Big(-l1\Big(\frac{d}{dx}\mathit{th1}(x)\Big)^2+s1\,g\Big)+c2\Big(l1\Big(\frac{d^2}{dx^2}\mathit{th1}(x)\Big)+c1\,g\Big)\Big)\Big)\Big)\Big)$$

$$+s5\Big(l4\Big(\frac{d^2}{dx^2}\mathit{th1}(x)+\frac{d^2}{dx^2}\mathit{th2}(x)+\frac{d^2}{dx^2}\mathit{th3}(x)+\frac{d^2}{dx^2}\mathit{th4}(x)\Big)-s4\Big($$

$$-l3\Big(\frac{d}{dx}\mathit{th1}(x)+\frac{d}{dx}\mathit{th2}(x)+\frac{d}{dx}\mathit{th3}(x)\Big)^2+c3\Big(-l2\Big(\frac{d}{dx}\mathit{th1}(x)+\frac{d}{dx}\mathit{th2}(x)\Big)^2$$

$$+c2\Big(-l1\Big(\frac{d}{dx}\mathit{th1}(x)\Big)^2+s1\,g\Big)+s2\Big(l1\Big(\frac{d^2}{dx^2}\mathit{th1}(x)\Big)+c1\,g\Big)\Big)$$

$$+s3\Big(l2\Big(\frac{d^2}{dx^2}\mathit{th1}(x)+\frac{d^2}{dx^2}\mathit{th2}(x)\Big)-s2\Big(-l1\Big(\frac{d}{dx}\mathit{th1}(x)\Big)^2+s1\,g\Big)$$

$$+c2\Big(l1\Big(\frac{d^2}{dx^2}\mathit{th1}(x)\Big)+c1\,g\Big)\Big)\Big)+c4\Big(l3\Big(\frac{d^2}{dx^2}\mathit{th1}(x)+\frac{d^2}{dx^2}\mathit{th2}(x)$$

$$+\frac{d^2}{dx^2}\mathit{th3}(x)\Big)-s3\Big(-l2\Big(\frac{d}{dx}\mathit{th1}(x)+\frac{d}{dx}\mathit{th2}(x)\Big)^2+c2\Big(-l1\Big(\frac{d}{dx}\mathit{th1}(x)\Big)^2$$

$$+s1\,g\Big)+s2\Big(l1\Big(\frac{d^2}{dx^2}\mathit{th1}(x)\Big)+c1\,g\Big)\Big)+c3\Big(l2\Big(\frac{d^2}{dx^2}\mathit{th1}(x)+\frac{d^2}{dx^2}\mathit{th2}(x)\Big)$$

$$-s2\Big(-l1\Big(\frac{d}{dx}\mathit{th1}(x)\Big)^2+s1\,g\Big)+c2\Big(l1\Big(\frac{d^2}{dx^2}\mathit{th1}(x)\Big)+c1\,g\Big)\Big)\Big)\Big)\Big)\Big)$$

$$+c6\Big(l5\Big(\frac{d^2}{dx^2}\mathit{th1}(x)+\frac{d^2}{dx^2}\mathit{th2}(x)+\frac{d^2}{dx^2}\mathit{th3}(x)+\frac{d^2}{dx^2}\mathit{th4}(x)+\frac{d^2}{dx^2}\mathit{th5}(x)\Big)$$

$$-s5\Big(-l4\Big(\frac{d}{dx}\mathit{th1}(x)+\frac{d}{dx}\mathit{th2}(x)+\frac{d}{dx}\mathit{th3}(x)+\frac{d}{dx}\mathit{th4}(x)\Big)^2+c4\Big($$

$$K l3 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x) C \frac{d}{dx} th3(x)\right)^2 C c3 \left(K l2 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right)^2\right.$$

$$C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1\, g\right) C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1\, g\right)\Big)$$

$$C s3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right) K s2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1\, g\right)\right.$$

$$C c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1\, g\right)\Big)\Big) C s4 \left(l3 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right.\right.$$

$$C \frac{d^2}{dx^2} th3(x)\Big) K s3 \left(K l2 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right)^2 C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2\right.\right.$$

$$C s1\, g\Big) C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1\, g\right)\Big) C c3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right)\right.$$

$$K s2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1\, g\right) C c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1\, g\right)\Big)\Big)\Big)\Big)$$

$$C c5 \left(l4 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x) C \frac{d^2}{dx^2} th3(x) C \frac{d^2}{dx^2} th4(x)\right) K s4 \Big(\right.$$

$$K l3 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x) C \frac{d}{dx} th3(x)\right)^2 C c3 \left(K l2 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right)^2\right.$$

$$C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1\, g\right) C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1\, g\right)\Big)$$

$$C s3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right) K s2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1\, g\right)\right.$$

$$C c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1\, g\right)\Big)\Big) C c4 \left(l3 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right.\right.$$

$$C \frac{d^2}{dx^2} th3(x)\Big) K s3 \left(K l2 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right)^2 C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2\right.\right.$$

$$C s1\, g\Big) C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1\, g\right)\Big) C c3 \left(l2 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x)\right)\right.$$

$$K s2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1\, g\right) C c2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1\, g\right)\Big)\Big)\Big)\Big)\Big)\Big)\Big)$$

$$C l4\, m4 \left(l4 \left(\frac{d^2}{dx^2} th1(x) C \frac{d^2}{dx^2} th2(x) C \frac{d^2}{dx^2} th3(x) C \frac{d^2}{dx^2} th4(x)\right) K s4 \Big(\right.$$

$$K l3 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x) C \frac{d}{dx} th3(x)\right)^2 C c3 \left(K l2 \left(\frac{d}{dx} th1(x) C \frac{d}{dx} th2(x)\right)^2\right.$$

$$C c2 \left(K l1 \left(\frac{d}{dx} th1(x)\right)^2 C s1\, g\right) C s2 \left(l1 \left(\frac{d^2}{dx^2} th1(x)\right) C c1\, g\right)\Big)$$

$$\mathrm{C}\, s3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \Big) \mathrm{K}\, s2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x) \Big)^2 \mathrm{C}\, s1\, g \Big)$$

$$\mathrm{C}\, c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) \mathrm{C}\, c1\, g \Big) \Big) \Big) \mathrm{C}\, c4 \Big(l3 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)$$

$$\mathrm{C} \frac{d^2}{dx^2} th3(x) \Big) \mathrm{K}\, s3 \Big(\mathrm{K} l2 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \Big)^2 \mathrm{C}\, c2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x) \Big)^2$$

$$\mathrm{C}\, s1\, g \Big) \mathrm{C}\, s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) \mathrm{C}\, c1\, g \Big) \Big) \mathrm{C}\, c3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \Big)$$

$$\mathrm{K}\, s2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x) \Big)^2 \mathrm{C}\, s1\, g \Big) \mathrm{C}\, c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) \mathrm{C}\, c1\, g \Big) \Big) \Big) \Big) \mathrm{C}\, l4 \Big(s5\, m5 \Big($$

$$\mathrm{K} l5 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \mathrm{C} \frac{d}{dx} th4(x) \mathrm{C} \frac{d}{dx} th5(x) \Big)^2 \mathrm{C}\, c5 \Big($$

$$\mathrm{K} l4 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \mathrm{C} \frac{d}{dx} th4(x) \Big)^2 \mathrm{C}\, c4 \Big(\mathrm{K} l3 \Big(\frac{d}{dx} th1(x)$$

$$\mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \Big)^2 \mathrm{C}\, c3 \Big(\mathrm{K} l2 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \Big)^2 \mathrm{C}\, c2 \Big($$

$$\mathrm{K} l1 \Big(\frac{d}{dx} th1(x) \Big)^2 \mathrm{C}\, s1\, g \Big) \mathrm{C}\, s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) \mathrm{C}\, c1\, g \Big) \Big) \mathrm{C}\, s3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x)$$

$$\mathrm{C} \frac{d^2}{dx^2} th2(x) \Big) \mathrm{K}\, s2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x) \Big)^2 \mathrm{C}\, s1\, g \Big) \mathrm{C}\, c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) \mathrm{C}\, c1\, g \Big) \Big) \Big)$$

$$\mathrm{C}\, s4 \Big(l3 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \mathrm{C} \frac{d^2}{dx^2} th3(x) \Big) \mathrm{K}\, s3 \Big(\mathrm{K} l2 \Big(\frac{d}{dx} th1(x)$$

$$\mathrm{C} \frac{d}{dx} th2(x) \Big)^2 \mathrm{C}\, c2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x) \Big)^2 \mathrm{C}\, s1\, g \Big) \mathrm{C}\, s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) \mathrm{C}\, c1\, g \Big) \Big)$$

$$\mathrm{C}\, c3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \Big) \mathrm{K}\, s2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x) \Big)^2 \mathrm{C}\, s1\, g \Big)$$

$$\mathrm{C}\, c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) \mathrm{C}\, c1\, g \Big) \Big) \Big) \Big) \mathrm{C}\, s5 \Big(l4 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x)$$

$$\mathrm{C} \frac{d^2}{dx^2} th3(x) \mathrm{C} \frac{d^2}{dx^2} th4(x) \Big) \mathrm{K}\, s4 \Big(\mathrm{K} l3 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \mathrm{C} \frac{d}{dx} th3(x) \Big)^2$$

$$\mathrm{C}\, c3 \Big(\mathrm{K} l2 \Big(\frac{d}{dx} th1(x) \mathrm{C} \frac{d}{dx} th2(x) \Big)^2 \mathrm{C}\, c2 \Big(\mathrm{K} l1 \Big(\frac{d}{dx} th1(x) \Big)^2 \mathrm{C}\, s1\, g \Big)$$

$$\mathrm{C}\, s2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) \mathrm{C}\, c1\, g \Big) \Big) \mathrm{C}\, s3 \Big(l2 \Big(\frac{d^2}{dx^2} th1(x) \mathrm{C} \frac{d^2}{dx^2} th2(x) \Big) \mathrm{K}\, s2 \Big($$

$$\mathrm{K} l1 \Big(\frac{d}{dx} th1(x) \Big)^2 \mathrm{C}\, s1\, g \Big) \mathrm{C}\, c2 \Big(l1 \Big(\frac{d^2}{dx^2} th1(x) \Big) \mathrm{C}\, c1\, g \Big) \Big) \Big) \mathrm{C}\, c4 \Big(l3 \Big(\frac{d^2}{dx^2} th1(x)$$

$$\mathrm{C}\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th3(x)\Big)\,\mathrm{K}\,s3\Big(kl2\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\frac{d}{dx}\,th2(x)\Big)^2\mathrm{C}\,c2\Big($$

$$kl1\Big(\frac{d}{dx}\,th1(x)\Big)^2\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\mathrm{C}\,c1\,g\Big)\Big)\,\mathrm{C}\,c3\Big(l2\Big(\frac{d^2}{dx^2}\,th1(x)$$

$$\mathrm{C}\frac{d^2}{dx^2}\,th2(x)\Big)\,\mathrm{K}\,s2\Big(kl1\Big(\frac{d}{dx}\,th1(x)\Big)^2\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,c2\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\mathrm{C}\,c1\,g\Big)\Big)\Big)\Big)$$

$$\Big)\,\mathrm{C}\,c5\,m5\Big(l5\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th3(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th4(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th5(x)\Big)$$

$$\mathrm{K}\,s5\Big(kl4\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\frac{d}{dx}\,th2(x)\,\mathrm{C}\frac{d}{dx}\,th3(x)\,\mathrm{C}\frac{d}{dx}\,th4(x)\Big)^2\mathrm{C}\,c4\Big($$

$$kl3\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\frac{d}{dx}\,th2(x)\,\mathrm{C}\frac{d}{dx}\,th3(x)\Big)^2\mathrm{C}\,c3\Big(kl2\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\frac{d}{dx}\,th2(x)\Big)^2$$

$$\mathrm{C}\,c2\Big(kl1\Big(\frac{d}{dx}\,th1(x)\Big)^2\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\mathrm{C}\,c1\,g\Big)\Big)$$

$$\mathrm{C}\,s3\Big(l2\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th2(x)\Big)\,\mathrm{K}\,s2\Big(kl1\Big(\frac{d}{dx}\,th1(x)\Big)^2\mathrm{C}\,s1\,g\Big)$$

$$\mathrm{C}\,c2\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\mathrm{C}\,c1\,g\Big)\Big)\Big)\,\mathrm{C}\,s4\Big(l3\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th2(x)$$

$$\mathrm{C}\frac{d^2}{dx^2}\,th3(x)\Big)\,\mathrm{K}\,s3\Big(kl2\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\frac{d}{dx}\,th2(x)\Big)^2\mathrm{C}\,c2\Big(kl1\Big(\frac{d}{dx}\,th1(x)\Big)^2$$

$$\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\mathrm{C}\,c1\,g\Big)\Big)\,\mathrm{C}\,c3\Big(l2\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th2(x)\Big)$$

$$\mathrm{K}\,s2\Big(kl1\Big(\frac{d}{dx}\,th1(x)\Big)^2\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,c2\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\mathrm{C}\,c1\,g\Big)\Big)\Big)\Big)$$

$$\mathrm{C}\,c5\Big(l4\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th3(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th4(x)\Big)\,\mathrm{K}\,s4\Big($$

$$kl3\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\frac{d}{dx}\,th2(x)\,\mathrm{C}\frac{d}{dx}\,th3(x)\Big)^2\mathrm{C}\,c3\Big(kl2\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\frac{d}{dx}\,th2(x)\Big)^2$$

$$\mathrm{C}\,c2\Big(kl1\Big(\frac{d}{dx}\,th1(x)\Big)^2\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\mathrm{C}\,c1\,g\Big)\Big)$$

$$\mathrm{C}\,s3\Big(l2\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th2(x)\Big)\,\mathrm{K}\,s2\Big(kl1\Big(\frac{d}{dx}\,th1(x)\Big)^2\mathrm{C}\,s1\,g\Big)$$

$$\mathrm{C}\,c2\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\mathrm{C}\,c1\,g\Big)\Big)\Big)\,\mathrm{C}\,c4\Big(l3\Big(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\frac{d^2}{dx^2}\,th2(x)$$

$$\mathrm{C}\frac{d^2}{dx^2}\,th3(x)\Big)\,\mathrm{K}\,s3\Big(kl2\Big(\frac{d}{dx}\,th1(x)\,\mathrm{C}\frac{d}{dx}\,th2(x)\Big)^2\mathrm{C}\,c2\Big(kl1\Big(\frac{d}{dx}\,th1(x)\Big)^2$$

$$\mathrm{C}\,s1\,g\Big)\,\mathrm{C}\,s2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Big)\,\mathrm{C}\,c3\left(l2\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right)\right.$$

$$\left.\mathrm{K}\,s2\left(\mathrm{K}l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,c2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\right)\Big)\Big)\Big)\Big)$$

$$\mathrm{C}\,l3\,m3\left(l3\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\right)\mathrm{K}\,s3\left(\mathrm{K}l2\left(\frac{d}{dx}\,th1(x)\right.\right.\right.$$

$$\left.\left.\mathrm{C}\,\frac{d}{dx}\,th2(x)\right)^2\mathrm{C}\,c2\left(\mathrm{K}l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,s2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\right)$$

$$\mathrm{C}\,c3\left(l2\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right)\mathrm{K}\,s2\left(\mathrm{K}l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\right.$$

$$\left.\mathrm{C}\,c2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\right)\Big)\Big)\,\mathrm{C}\,l3\left(s4\,m4\left(\mathrm{K}l4\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\right.\right.\right.$$

$$\left.\mathrm{C}\,\frac{d}{dx}\,th3(x)\,\mathrm{C}\,\frac{d}{dx}\,th4(x)\right)^2\mathrm{C}\,c4\left(\mathrm{K}l3\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\,\mathrm{C}\,\frac{d}{dx}\,th3(x)\right)^2\right.$$

$$\mathrm{C}\,c3\left(\mathrm{K}l2\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\right)^2\mathrm{C}\,c2\left(\mathrm{K}l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\right.$$

$$\left.\mathrm{C}\,s2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\right)\mathrm{C}\,s3\left(l2\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right)\mathrm{K}\,s2\right($$

$$\left.\mathrm{K}l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,c2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Big)\Big)\,\mathrm{C}\,s4\left(l3\left(\frac{d^2}{dx^2}\,th1(x)\right.\right.$$

$$\left.\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\right)\mathrm{K}\,s3\left(\mathrm{K}l2\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\right)^2\mathrm{C}\,c2\right($$

$$\left.\mathrm{K}l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,s2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Big)\,\mathrm{C}\,c3\left(l2\left(\frac{d^2}{dx^2}\,th1(x)\right.\right.$$

$$\left.\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right)\mathrm{K}\,s2\left(\mathrm{K}l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,c2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\Big)\Big)\Big)\Big)$$

$$\mathrm{C}\,c4\,m4\left(l4\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th4(x)\right)\mathrm{K}\,s4\right($$

$$\mathrm{K}l3\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\,\mathrm{C}\,\frac{d}{dx}\,th3(x)\right)^2\mathrm{C}\,c3\left(\mathrm{K}l2\left(\frac{d}{dx}\,th1(x)\,\mathrm{C}\,\frac{d}{dx}\,th2(x)\right)^2\right.$$

$$\left.\mathrm{C}\,c2\left(\mathrm{K}l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\mathrm{C}\,s2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\right)$$

$$\mathrm{C}\,s3\left(l2\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right)\mathrm{K}\,s2\left(\mathrm{K}l1\left(\frac{d}{dx}\,th1(x)\right)^2\mathrm{C}\,s1\,g\right)\right.$$

$$\left.\mathrm{C}\,c2\left(l1\left(\frac{d^2}{dx^2}\,th1(x)\right)\mathrm{C}\,c1\,g\right)\right)\Big)\,\mathrm{C}\,c4\left(l3\left(\frac{d^2}{dx^2}\,th1(x)\,\mathrm{C}\,\frac{d^2}{dx^2}\,th2(x)\right.\right.$$

$$\mathrm{C}\,\frac{d^2}{dx^2}\,th3(x)\Big)\;\mathrm{K}\;s3\,\Big(\mathrm{K}\,l2\Big(\frac{d}{dx}\,th1(x)\;\mathrm{C}\;\frac{d}{dx}\,th2(x)\Big)^2\;\mathrm{C}\;c2\,\Big(\mathrm{K}\,l1\Big(\frac{d}{dx}\,th1(x)\Big)^2$$

$$\mathrm{C}\;s1\,g\Big)\;\mathrm{C}\;s2\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\Big)\;\mathrm{C}\;c3\,\Big(l2\Big(\frac{d^2}{dx^2}\,th1(x)\;\mathrm{C}\;\frac{d^2}{dx^2}\,th2(x)\Big)$$

$$\mathrm{K}\;s2\,\Big(\mathrm{K}\,l1\Big(\frac{d}{dx}\,th1(x)\Big)^2\;\mathrm{C}\;s1\,g\Big)\;\mathrm{C}\;c2\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\Big)\Big)\Big)\Big)$$

$$\mathrm{C}\;l2\,m2\,\Big(l2\Big(\frac{d^2}{dx^2}\,th1(x)\;\mathrm{C}\;\frac{d^2}{dx^2}\,th2(x)\Big)\;\mathrm{K}\;s2\,\Big(\mathrm{K}\,l1\Big(\frac{d}{dx}\,th1(x)\Big)^2\;\mathrm{C}\;s1\,g\Big)$$

$$\mathrm{C}\;c2\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\Big)\;\mathrm{C}\;l2\,\Big(s3\,m3\,\Big(\mathrm{K}\,l3\Big(\frac{d}{dx}\,th1(x)\;\mathrm{C}\;\frac{d}{dx}\,th2(x)$$

$$\mathrm{C}\;\frac{d}{dx}\,th3(x)\Big)^2\;\mathrm{C}\;c3\,\Big(\mathrm{K}\,l2\Big(\frac{d}{dx}\,th1(x)\;\mathrm{C}\;\frac{d}{dx}\,th2(x)\Big)^2\;\mathrm{C}\;c2\,\Big(\mathrm{K}\,l1\Big(\frac{d}{dx}\,th1(x)\Big)^2$$

$$\mathrm{C}\;s1\,g\Big)\;\mathrm{C}\;s2\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\Big)\;\mathrm{C}\;s3\,\Big(l2\Big(\frac{d^2}{dx^2}\,th1(x)\;\mathrm{C}\;\frac{d^2}{dx^2}\,th2(x)\Big)$$

$$\mathrm{K}\;s2\,\Big(\mathrm{K}\,l1\Big(\frac{d}{dx}\,th1(x)\Big)^2\;\mathrm{C}\;s1\,g\Big)\;\mathrm{C}\;c2\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\Big)\Big)$$

$$\mathrm{C}\;c3\,m3\,\Big(l3\Big(\frac{d^2}{dx^2}\,th1(x)\;\mathrm{C}\;\frac{d^2}{dx^2}\,th2(x)\;\mathrm{C}\;\frac{d^2}{dx^2}\,th3(x)\Big)\;\mathrm{K}\;s3\,\Big(\mathrm{K}\,l2\Big(\frac{d}{dx}\,th1(x)$$

$$\mathrm{C}\;\frac{d}{dx}\,th2(x)\Big)^2\;\mathrm{C}\;c2\,\Big(\mathrm{K}\,l1\Big(\frac{d}{dx}\,th1(x)\Big)^2\;\mathrm{C}\;s1\,g\Big)\;\mathrm{C}\;s2\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\Big)$$

$$\mathrm{C}\;c3\,\Big(l2\Big(\frac{d^2}{dx^2}\,th1(x)\;\mathrm{C}\;\frac{d^2}{dx^2}\,th2(x)\Big)\;\mathrm{K}\;s2\,\Big(\mathrm{K}\,l1\Big(\frac{d}{dx}\,th1(x)\Big)^2\;\mathrm{C}\;s1\,g\Big)$$

$$\mathrm{C}\;c2\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\Big)\Big)\Big)\;\mathrm{C}\;l1\,m1\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\;\mathrm{C}\;l1\,\Big(s2\,m2\,\Big($$

$$\mathrm{K}\,l2\Big(\frac{d}{dx}\,th1(x)\;\mathrm{C}\;\frac{d}{dx}\,th2(x)\Big)^2\;\mathrm{C}\;c2\,\Big(\mathrm{K}\,l1\Big(\frac{d}{dx}\,th1(x)\Big)^2\;\mathrm{C}\;s1\,g\Big)$$

$$\mathrm{C}\;s2\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\Big)\;\mathrm{C}\;c2\,m2\,\Big(l2\Big(\frac{d^2}{dx^2}\,th1(x)\;\mathrm{C}\;\frac{d^2}{dx^2}\,th2(x)\Big)\;\mathrm{K}\;s2\,\Big($$

$$\mathrm{K}\,l1\Big(\frac{d}{dx}\,th1(x)\Big)^2\;\mathrm{C}\;s1\,g\Big)\;\mathrm{C}\;c2\,\Big(l1\Big(\frac{d^2}{dx^2}\,th1(x)\Big)\;\mathrm{C}\;c1\,g\Big)\Big)\Big)\Big)\Big]\Big]$$

O *cost*(*optimise*(*n1*))

1649 *functions* C 841 *additions* C 1100 *multiplications* **(2)**

O

A4: Process of identifying $\tilde{\tau}$ aided by Maple

$$
\begin{aligned}
> \tau_3 := {} & l_3\, m_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t) \Big) l_3 - s_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2 \\
& + c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + c_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \\
& + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big) :
\end{aligned}
$$

$$
\begin{aligned}
> \tau_2 := {} & l_3\, m_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t) \Big) l_3 - s_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2 \\
& + c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + c_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \\
& + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big) \\
& + l_2\, m_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) \\
& + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + l_2 \Big(s_3\, m_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) + \frac{d}{dt}\theta_3(t) \Big)^2 l_3 \\
& + c_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2 + c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 \\
& + c_1\, g \Big) \Big) + s_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) \\
& + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big) + c_3\, m_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t) \Big) l_3 \\
& - s_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2 + c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 \\
& + c_1\, g \Big) \Big) + c_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) \\
& + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big) \Big) :
\end{aligned}
$$

$$
\begin{aligned}
> \tau_1 := {} & l_3\, m_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t) \Big) l_3 - s_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2 \\
& + c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + c_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \\
& + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big) \\
& + l_2\, m_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) \\
& + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + l_2 \Big(s_3\, m_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) + \frac{d}{dt}\theta_3(t) \Big)^2 l_3
\end{aligned}
$$

$$> \tau_3 := l_3\, m_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t) \Big) l_3 - s_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2$$

$$+ c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + c_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t)$$

$$+ \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big) :$$

$$> \tau_2 := l_3\, m_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t) \Big) l_3 - s_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2$$

$$+ c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + c_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t)$$

$$+ \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big)$$

$$+ l_2\, m_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big)$$

$$+ c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + l_2 \Big(s_3\, m_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) + \frac{d}{dt}\theta_3(t) \Big)^2 l_3$$

$$+ c_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2 + c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1$$

$$+ c_1\, g \Big) \Big) + s_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big)$$

$$+ c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big) + c_3\, m_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t) \Big) l_3$$

$$- s_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2 + c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1$$

$$+ c_1\, g \Big) \Big) + c_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big)$$

$$+ c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big) \Big) :$$

$$> \tau_1 := l_3\, m_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t) \Big) l_3 - s_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) \Big)^2 l_2$$

$$+ c_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + s_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + c_3 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t)$$

$$+ \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big) + c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) \Big)$$

$$+ l_2\, m_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) \Big) l_2 - s_2 \Big(-\Big(\frac{d}{dt}\theta_1(t) \Big)^2 l_1 + s_1\, g \Big)$$

$$+ c_2 \Big(\Big(\frac{d^2}{dt^2}\theta_1(t) \Big) l_1 + c_1\, g \Big) \Big) + l_2 \Big(s_3\, m_3 \Big(-\Big(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) + \frac{d}{dt}\theta_3(t) \Big)^2 l_3$$

$$+ c_3 \left(-\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t)\right)^2 l_2 + c_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + s_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1\right.\right.$$

$$\left.\left.+ c_1 g\right)\right) + s_3 \left(\left(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t)\right) l_2 - s_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right)\right.$$

$$\left.+ c_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 + c_1 g\right)\right)\Bigg) + c_3 m_3 \left(\left(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t)\right) l_3\right.$$

$$- s_3 \left(-\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t)\right)^2 l_2 + c_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + s_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1\right.\right.$$

$$\left.\left.+ c_1 g\right)\right) + c_3 \left(\left(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t)\right) l_2 - s_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right)\right.$$

$$\left.+ c_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 + c_1 g\right)\right)\Bigg)\Bigg) + l_1 m_1 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 + c_1 g\right) + l_1 \Bigg(s_2 \Bigg(c_3 m_3 \Bigg($$

$$-\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) + \frac{d}{dt}\theta_3(t)\right)^2 l_3 + c_3 \Bigg(-\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t)\right)^2 l_2 + c_2 \Bigg($$

$$-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\Bigg) + s_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 + c_1 g\right)\Bigg) + s_3 \Bigg(\Bigg(\frac{d^2}{dt^2}\theta_1(t)$$

$$+ \frac{d^2}{dt^2}\theta_2(t)\Bigg) l_2 - s_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + c_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 + c_1 g\right)\Bigg)\Bigg)$$

$$- s_3 m_3 \Bigg(\left(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t)\right) l_3 - s_3 \Bigg(-\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t)\right)^2 l_2$$

$$+ c_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + s_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 + c_1 g\right)\Bigg) + c_3 \Bigg(\Bigg(\frac{d^2}{dt^2}\theta_1(t)$$

$$+ \frac{d^2}{dt^2}\theta_2(t)\Bigg) l_2 - s_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + c_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 + c_1 g\right)\Bigg)\Bigg) + m_2 \Bigg($$

$$-\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t)\right)^2 l_2 + c_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + s_2 \Bigg(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1$$

$$+ c_1 g\Bigg)\Bigg)\Bigg) + c_2 \Bigg(s_3 m_3 \Bigg(-\left(\frac{d}{dt}\theta_1(t) + \frac{d}{dt}\theta_2(t) + \frac{d}{dt}\theta_3(t)\right)^2 l_3 + c_3 \Bigg(-\Bigg(\frac{d}{dt}\theta_1(t)$$

$$+ \frac{d}{dt}\theta_2(t)\Bigg)^2 l_2 + c_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + s_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 + c_1 g\right)\Bigg)$$

$$+ s_3 \Bigg(\left(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t)\right) l_2 - s_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + c_2 \Bigg(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1$$

$$+ c_1 g\Bigg)\Bigg)\Bigg) + c_3 m_3 \Bigg(\left(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t) + \frac{d^2}{dt^2}\theta_3(t)\right) l_3 - s_3 \Bigg(-\Bigg(\frac{d}{dt}\theta_1(t)$$

$$+ \frac{d}{dt}\theta_2(t)\Bigg)^2 l_2 + c_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + s_2 \left(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1 + c_1 g\right)\Bigg)$$

$$+ c_3 \Bigg(\left(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t)\right) l_2 - s_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right) + c_2 \Bigg(\left(\frac{d^2}{dt^2}\theta_1(t)\right) l_1$$

$$+ c_1 g\Bigg)\Bigg)\Bigg) + m_2 \Bigg(\left(\frac{d^2}{dt^2}\theta_1(t) + \frac{d^2}{dt^2}\theta_2(t)\right) l_2 - s_2 \left(-\left(\frac{d}{dt}\theta_1(t)\right)^2 l_1 + s_1 g\right)$$

$$+ c_2 \left(\left(\left(\frac{d^2}{dt^2} \theta_1(t) \right) l_1 + c_1 g \right) \right) \right) \right)$$

Bibliography

Aghili, F. (2007), Torque control of electric motors without using torque sensor, *in* 'Proc of the IEEE International Conference on Intelligent Robots and Systems', pp. 3604–3609.

Andreev, A. & Peregudova, O. (2017), 'Trajectory tracking control for robot manipulators using only position measurements', *International Journal of COntrol* .

Asada, H. & Slotline, J.J.E. (1986), *Robot Analysis and Control*, John Wiley and Sons.

Bohner, P. & Luppen, R. (1997), Reactive multi- agent based control of redundant manipulators, *in* 'Proceedings of the IEEE Intemational Conference on Robotics and Automation', pp. 1067–1072.

Cho, S.-J., Lee, J. S., Kim, J., Kuc, T.-Y., Chang, P.-H. & Jin, M. (2016), 'Adaptive time-delay control with a supervising switching technique for robot manipulators', *Transactions of Institute of Measurement and Control* **39**, 1374–1382.

Cohen, W., Sirotin & Rave (2008), Lawn mowing system for known areas, *in* '2008 International Conference on Computational Intelligence for Modelling Control Automation', pp. 539–544.

Craig, J. J. (1986), Adaptive Control of Mechanical Manipulators, PhD thesis.

Craig, J. J. (2005), *Introduction to robotics: mechanics and control*, Vol. 3, Pearson Prentice Hall, Upper Saddle River.

Dasgupta, P., Baca, J., Guruprasad, K.R., Munoz-Melendez, A. & Jumadinova, J. (2015), 'The COMRADE system for multi-robot autonomous landmine detection in post-conflict regions', *Journal of Robotics* **2015**.

Doty, K. L. & Harrison, R. R. (1993), Sweep strategies for a sensory-driven, behavior-based vacuum cleaning agent, *in* 'AAAI 1993 Fall Symposium Series', pp. 1–6.

Essakki, B., Bhat, R. & Su, C.-Y. (2013), 'Robust control of collaborative manipulators-flexible object system', *International Journal of Advanced Robotic systems* .

Franklin, G. F., Powell, J. D. & Emami-Naeini, A. (2002), *Feedback control of dynamic systems*, 4 edn, Pearson Education Asia.

Ghosal, A. (2013), *Robotics: Fundamental Concepts and Analysis*, Oxford University Press.

Guruprasad, K.R. & Ghosal, A. (1999), Model reference learning control for rigid robots, *in* 'Proceedings of ASME DETC'.

Guruprasad, K.R. & Ghose, D. (2011), 'Automated multi-agent search using centroidal voronoi configuration', *IEEE Transactions on Automation Science and Engineering* **1**(8), 420–423.

Guruprasad, K.R., Wilson, Z. & Dasgupta, P. (2012), 'Complete coverage of an initially unknown environment by multiple robots using voronoi partition', *Proceedings of the Advances in Control and Optimization of Dynamic Systems* pp. 1–8.

Hashimoti, M. (1989), Robot motion control based on joint torque sensing, *in* 'Proc of the IEEE International Conference on Robotics and Automation'.

Hsia, T. & Gao, L. (1990), Nonlinear model predictive control of a robot manipulator, *in* 'Proceedings of the IEEE International Conference on Robotics and Automation', pp. 2070–2075.

Hsu, L.-C. (2006), 'A fully adaptive decentralized control of robot manipulators', *Automatica* **42**(10).

Huang, Y. J. & Wang, Y. J. (2000), 'Robust pid tuning strategy for uncertain plants based on the kharitonovs theorem', *ISA Transactions* **39**(4), 419431.

Jia, H., Zhuang, W., Bai, Y., Fan, P. & Huang, Q. (2007), The distributed control system of a light space manipulator, *in* 'Proceedings of the IEEE International Conference on Mechatronics and Automation', pp. 3525–3530.

Jin, L., Li, S., Yuc, J. & inbo He (n.d.), 'Robot manipulator control using neural networks: A survey'.

Kosuge, K., Takeuchi, H. & Furuta, K. (1990), 'Motion control of a robot arm using joint torque sensors', *IEEE Transactions on RObotics and Automation* **6**(2).

Leena, G. & ray, G. (2012), 'A set of decentralized pid controllers for an n-link robot manipulator', **37**, 405423.

Li, S., Wang, H. & Rafique, M. U. (2018), 'A novel recurrent neural network for manipulator control with improved noise tolerance', *IEEE Transactions on Neural Networks and Learning Systems* **29**(5), 1908–1918.

Liu, M. (1997), 'Decentralized PD and robust nonlinear control for robot manipulators', *Journal of Intelligent and Robotic Systems* **20**(2-4), 319–332.

Liu, M. (1999), 'Decentralized control of robot manipulators: nonlinear and adaptive approaches', *IEEE Transactions on Automatic Control* **44**(2).

Markiewicz, R. (1973), Analysis of computed torque drive method and comparision with conventional position servo for a computed-controlled manipulator, Technical report.

Morit, A., Nayat, F., Osatot, N. & Kawaokatt, T. (1996), *in* 'Proceedings of the IEEElSICElRSJ International Conference on Multisensor Fusion and Integration for Intelligent Systems Multiagent-Based Distributed Manipulator Control', pp. 289–296.

Narendra, K.S. & Oleng, N.O. (2002), 'Exact output tracking in decentralized adaptive control systems'.

Paul, R. P. (1972), Modelling, trajectory calculations, and servoing of a computer controlled arm, Technical Report AIM-177.

Poigent, P. & Gautier, M. (2000), Nonlinear model predictive control of a robot manipulator, *in* 'Proceedings of 6th International Workshop on Advanced Motion Control', IEEE, pp. 401–406.

Preffer, L. E., Khatib, O. & Hake, J. (1989), 'Joint torque sensory feedback in the control of a puma manipulator', *IEEE Transactions on Robotics and Automation* **5**(4), 418–425.

Saha, S. (2008), *Introduction to Robotics*, McGraw Hill.

Sastry, S. S. (1984), 'Model-reference adaptive control - stability, parameter covergence, and robustness', *IMA Journal of Math. Control and Information* pp. 27–66.

Senda, K., Nishibu, S. & Mano, S. (2003), Fault tolerant distributed control of manipulator with visual serving, *in* 'Proceedings of the SICE Annual Conference in Fukui', pp. 304–308.

Seraji, H. (1989), 'Decentralized adaptive control of manipulators: Theory, simulation, and experimentation', *IEEE Transactions on Robotics and Automation* .

Slotine, J.-J. E. & Weiping, L. (1988), 'Adaptive manipulator control: A case study', *IEEE Transactions on Automatic Control* **33**(11), 995–1003.

Spong, M. W. & Vidyasagar, M. (2008), *Robot dynamics and control*, John Wiley & Sons.

Tang, Y. & Guerrero, G. (1998), Decentralized robust control of robot manipulators, *in* 'in Proceedings of the American Control Conference', Vol. 2, p. 922926.

Tarokh, M. (1996), 'Decentralized adaptive tracking control of robot manipulators'.

Tsuji, T., Nakayama, S. & Ito, K. (1993), Distributed feedback control for redundant manipulators based on virtual arms, *in* 'Proceedings of the International Symposium on Autonomous Decentralized Systems', pp. 143–149.

Wang, J-Q. & Wend, H.D. (1999), 'Robust decentralized control of robot manipulators', *International Journal of System Science* **30**, 323330.

Yang, Z.-J., Fukushima, Y. & Qin, P. (2012), 'Decentralized adaptive robust control of robot manipulators using disturbance observers', *IEEE Transactions on Control Systems Technology* **20**(5).

Yim, J. & Park, J. H. (1999), Nonlinear control of robotic manipulator, *in* 'Proceedings of IEEE International Conference on Systems, Man, and Cybernetics', Vol. 2, IEEE, pp. 866–871.

Yuan, J. S. (1988), 'Closed-loop manipulator control using quaternion feedback', *IEEE Journal of Robotics and Automation* **4**(4).

Yun, X. (1988), Dynamic state feedback control of constrained robot manipulators, *in* 'Proceedings of the 27th IEEE Conference on Decision and Control'.

Table 7.1: List of Publications based on PhD Research Work

Sl.No	Title of the paper	Authors(in the same order as in the paper.Underline the Research scholars name)	Name of the Journal	Month, Year of publication	Category
1	Modelbased manipulator control and the computational cost	S.Soumya, K R Guruprasad	Journal of advance research in Dynamical and control systems	Accepted	1
2	Model based cooperative control for a serial link planar manipulator	S.Soumya, K R Guruprasad	Robotica	Under review	1
3	Multi agent system inspired distributed control of a serial link robot	S.Soumya, K R Guruprasad	Journal of Automation,Mobile robotics and Intelligent Systems	Under review	1
4	Model based distributed cooperative control of a robotic manipulator	S.Soumya, K R Guruprasad	IEEE International WIE conference (WIECON-ECE) 2015	December,2015	3
5	Multi agent system inspired distributed control of manipulator	S Soumya, K R Guruprasad	International Engineering symposium (IES 2015) Japan	March,2015	4
6	Modelbased Distributed control of manipulator	S.Soumya, K R Guruprasad	Summer school on Multi robot systems, Poster presentation at NUS Singapore	June, 2016	5

Category:

1. Journal paper, full paper reviewed
2. Journal paper, Abstract reviews
3. Conference/Symposium paper, full paper reviewed
4. Conference/Symposium paper, abstract reviewed
5. others (including papers in Workshops, NITK Research Bulletins,Short notes etc.)

Soumya S

Research scholar

Name & signature, with date

Dr.K R Guruprasad

Research Guide

Name & signature, with date

RESUME

SOUMYA S
Ph.D Student, N.I.T K Surathkal
Mobile: +91-919446146797
E-mail: soumya5.subbu@gmail.com
soumya.me12f06@nitk.edu.in

ACADEMIC DETAILS

Qualification	Specialization	University	Year	CGPA/%
Ph.D(Course work)	Mechatronics Engineering	NITK,Surathkal, India	2017	8.00
M-Tech	Mechatronics Engineering	NITK, Surathkal, India	2012	7.54
B-Tech	Electronics & Communication Engg:	B.C.E.T, India	2009	7.75

AREAS OF INTEREST

- Nonlinear dynamic control.
- Robotics and control.

TECHNICAL SKILLS

Software/Tools	:	MATLAB/Simulink, ROS, LaTeX, Maple17.0, LabVIEW
Programming Languages	:	C

PhD Work

- **Multi-agent System Inspired Distributed Co-operative Dynamic Control of Serial link Manipulator**

(Supervisor: **Prof. K R Guruprasad**, Asst. Professor, National Institute of Technology Karnataka, India,)

Abstract

In this work we consider a situation where the manipulator model is known completely. In such a situation a centralized nonlinear controller such as model based control law is most suitable from the perspective of performance. However, the computational overhead makes it difficult to implement it in realtime. This disadvantage of high computational cost involved in the computer implementation of such a centralized control law has motivated a large quantum of research into manipulator control, including the decentralized control schemes. Though decentralized control laws typically lead to lower computational lead time, they do not consider the dynamic coupling between links, instead, use either robust control techniques or adaptive control techniques. In fact purposefully neglecting the available nonlinear dynamics, particularly the dynamic coupling effects, and using additional techniques to account for them, may lead to additional computational overhead. We propose a distributed control architecture for manipulator control. Here, unlike a decentralized scheme, the joint level controllers communicate and cooperate among themselves and fully utilize the knowledge of dynamic equations. The control architecture proposed achieves full feedback liberalization without any approximations. Further, the computational load gets distributed among the joint level controllers, hence reducing the lead time and enabling much efficient computer implementation of the manipulator control in real time in this work we focus on planar manipulators with revolute joints.simulations were conducted to demonstrate that the manipulator successfully tracks the desired trajectory. .

CURRENT MEMBERSHIP IN PROFESSIONAL SOCIETIES AND ORGANISATIONS

Society/Organisation	rank	Member Since
IEEE Control System Society & RAS	Student Member	2013
ACDOS(Automatic Control and Dynamic Optimisation Society)	Student Member	2014
RSI(Robotic Society of India)	Student Member	2018

HONORS AND AWARDS

- Received MHRD Scholarship for fulltime Research scholar in the year 2012dec-2017 dec
- Received TEQUIP funding to attend IEEE RAS Summer school on Multi-Robot systems
- Received Institute Alumini Association fund to attend IEEE WIECON-ECE 2015
- Received JASSO scholarship during (4-11th 2015).
- Received GATE (Graduate Aptitude Test in Engineering) Scholarship in the year 2010July-2012July.

PRINCIPLE PUBLICATIONS/ PRESENTATION

- Soumya.S, K.R.Guruprasad, Multiagent inspired distributed control of manipulator, *IES2015 Conference proceedings,* International symposium IES2015,4-11 March 2015, Kumamoto University, Japan.
- Soumya.S, K.R.Guruprasad, Model-based Distributed Cooperative Control of a Robotic Manipulator, in *IEEE International Conference on Electrical and computer engineering (WIECON-ECE),* ,19-20 December 2015, BUET, Dhaka,Bangladesh.
- Soumya.S, K.R.Guruprasad, Multi agent system inspired distributed control of manipulator, in *IEEE RAS Summer school on Multi-Robot systems, 4-12 June 2016*, National University of Singapore, Singapore.
- Soumya.S, K.R.Guruprasad, Model-based manipulator control and the computational cost, IJET, accepted
- Soumya.S, K.R.Guruprasad, Distributed Cooperative Control of Serial link Manipulator Communicated to ASME,
- Finalist in ICRA-17 DJI Robomasters Mobile Manipulation challenge during (29thMay-3rd June 2017) at Marina bay sands, Singapore.

PROFESSIONAL DEVELOPMENT ACTIVITIES(Workshops,training,etc..)

- Undergone an industrial training on Mass communication and Information technology, power electronics in KELTRON(Kerala state electronics Development Corporation LTD)During 2nd - 7th July 2007
- Undergone an Internship and project on SMI540 Based Safety/Integrity monitoring in ALM in Robert Bosch Business and Engineering solutions Bangalore, India
- Made robot and took part in Perfect Machine (robotics event in Engineer 2011 (technical symposium of NITK).
- Short term training program on Practical approaches in Robotics and Smart Material Technologies during (24-28june2013) held at Indian Institute of Technology Design and Manufacturing, Kancheepuram, Tamilnadu, India
- Workshop on Differential Equations and its Applications during (18-21Dec2013) at Indian Institute of Space Science and Technology (IIST), Thiruvananthapuram, Kerala, India.
- International workshop on Small satellite and sensor technology for disaster management during (30th March-2nd April 2014) at Indian Institute of Science(IISc),Bangalore, Karnataka, India.
- Participated in Indian control conference during (5-7 Jan2015) at IITM, Chennai, Tamil Nadu.
- Attended spring seminar and internship during (4-11th March2015) at Kumamoto University, Japan.
- Attended Advance in Robotics Conferenceduring (2-4th July2015) at BITS Pilani KK Birla Goa campus.
- Attended one day workshop on Mission make in India: Synergy with technical education system on 3rd october2015 at NITK.

References

Dr. K.R.Guruprasad,	Dr. Prasad Krishna,	K K Appukuttan
Assistant Professor,	Professor,	Professor
Department of Mechanical Eng:,	——-,	——,
N.I.T.K Surathkal,	——-,	——,
Mangalore, India.	——,	——,
Email:krgprao@gmail.com.	Email:krishnprasad@gmail.com	Email:pp_kuttan@Yahoo.com
Ph:+919449378018.	Ph:+919481263296.	Ph:+919425609059

Personal Details

Languages :English-fluent, Hindi-basic, Malayalam-fluent and Tamil-basic