ITERATIVE REGULARIZATION THEORY FOR NONLINEAR ILL-POSED PROBLEMS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

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by

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JULY 2019

Dedicated to

Savithri & Devadas

Big Bro, Sis & Makki

Friends

Teachers

&

My Love

DECLARATION

By the Ph.D. Research Scholar

I hereby **declare** that the research thesis entitled **"ITERATIVE REGULARIZATION THEORY FOR NONLINEAR ILL-POSED PROBLEMS"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this research thesis has not been submitted to any University or Institution for the award of any degree.

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Date : July 19, 2019

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CERTIFICATE

This is to **certify** that the research thesis entitled **"ITERATIVE REGULARIZATION THEORY FOR NONLINEAR ILL-POSED PROBLEMS"** submitted by **Sreedeep C D**, (Register Number MA15F09) as the record of the research work carried out by him, is *accepted as the research thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Prof. Santhosh George
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ABSTRACT

In science and engineering many practical problems can be formulated using mathematical modelling and can be classified as nonlinear ill-posed problems. Here we consider those ill-posed equations involving m-accretive operators in Banach spaces. Using a general Hölder type source condition we were able to obtain an optimal order error estimate. For nonlinear problems, obtaining a closed form solution is possible only in rare cases, so most of the methods considered for approximating the solution of nonlinear problems are iterative. Four different types of iterative schemes are being discussed in this thesis. Firstly, we consider a derivative and inverse free method and obtained second order convergence. Then, we produced an extended Newton-type iterative scheme that converges cubically to the solution which uses assumptions only on the first Fréchet derivative of the operator. Afterwards, we studied Newton-Kantorovich regularization method and obtained second order convergence with weak assumptions. Finally, we examined Secant-type iteration and proved that the proposed iterative scheme has a convergence order at least 2.20557 using assumptions only on first Fréchet derivative of the operator. Through out the work, for choosing the regularization parameter we have taken the adaptive parameter choice strategy given by Pereverzev and Schock (2005).

Keywords: ***Banach space; Nonlinear ill-posed problem; Lavrentiev regularization; m-accretive mappings; Adaptive parameter choice strategy; Extended Newton iterative scheme; Newton-Kantorovich regularization method; Secant-type iterative scheme.***

Mathematics Subject Classification: 47J05; 47H09; 49J30; 47J06; 65J20; 47H06

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Chapter 1

INTRODUCTION

The notion of an Inverse Problem (IP) have acquired a widespread acceptance in modern applied mathematics, science and engineering, although it is improbable that any rigorous formal definition of this concept exists. IP's are the opposites of direct problems. By nature, in a direct problem one finds an effect from a cause, and in an IP one is given with the effect and wants to recover the cause. The most usual situation giving rise to an IP is the need to interpret indirect physical measurements of an unknown object of interest. Keller (1976), a well known American mathematician introduced a general definition for IP with his frequently quoted statement as "We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other." Solving such problems will lead us to a wide range of applications in image processing, radar, communication theory, oceanography, geophysics, computer vision, astronomy, remote sensing, machine learning, natural language processing and many other fields. IP's is a very active field of research in applied sciences. IP's in science and engineering can be formulated mathematically as an operator equation

$$F(u) = f, \tag{1.0.1}$$

where $F : E_1 \to E_2$ is a linear or nonlinear operator between suitable normed spaces E_1 and E_2, f is the observation and u is sought for the solution. IP's most often do not fulfill Hadamard's postulates of well-posedness (see Section 1.1 below) i.e., the equation (1.0.1) might lack a solution in the strict sense, if exists the solution might not be unique and/or might not depend continuously on the data. Therefore, mathematically analyzing these are a bit hard in general. Problems that are not well-posed in the sense of Hadamard (Hadamard (1953)) are termed ill-posed. We will be looking forward only ill-posed IP's.

Throughout the thesis we will be using the following notations.

- The domain of F is denoted by $D(F)$.
- The range of F is denoted by $R(F)$.
- The Fréchet derivative of F (see Definition 1.1.2) is denoted by $F'(.)$.
- $B(u,\rho)$, $\overline{B(u,\rho)}$ stand, respectively for the open and closed balls in E_1, with center $u \in E_1$ and of radius $\rho > 0$.

1.1 Ill-posed problem

According to Hadamard (1953), a French mathematician, the problem of solving the operator equation (1.0.1) is said to be well-posed if the following three conditions are fulfilled:

(1.) **Existence**: For each $f \in E_2$, there is a solution $u \in E_1$ of (1.0.1) ;

(2.) **Uniqueness**: The solution u is unique ;

(3.) **Stability**: The dependence of u upon F is continuous .

For operator equations of the form (1.0.1) this criteria of Hadamard's well-posedness can be rewritten as follows:

(1.) $F(E_1) = E_2$;

(2.) F is one-to-one;

(3.) F^{-1} is continuous.

We can spot that first two criteria are of algebraic in nature, while the third one depends mostly on the topologies chosen for E_1 and E_2. As the theory of linear ill-posed problems are well furnished (Engl et al. (1996); Groetsch (1984); Nashed and Rall (1976)), we are looking forward in studying nonlinear ill-posed problems.

1.1.1 Nonlinear ill-posed problem

Let $E = E_1 = E_2$ be a Banach space, let the dual space E^* of E is the set of all linear continuous functionals on E and F be a nonlinear operator from $D(F) \subseteq E$ into E. We denote $\langle u, J\rangle$ instead of $J(u)$ for $J \in E^*$ and $u \in E$ and we denote $\|.\|$ for norm on both E and E^*. As for linear case the theory is not so well developed in nonlinear case (George and Nair (1993); Engl et al. (1996); Tautenhahn (1996); Nair (2009)). If F is not surjective, then the operator equation (1.0.1) is not solvable. We use the concept of quasi-solution $\hat{u}$.

DEFINITION 1.1.1 (Alber and Ryazantseva (2006)). *An element $\hat{u} \in E$ is called a quasi-solution of equation (1.0.1) if it minimizes the residual $\| F(u) - f \|$ on the set E.*

DEFINITION 1.1.2 (Alber and Ryazantseva (2006)). *Let F be an operator mapping a Banach space E into itself. If there exists a bounded linear operator $L : E \to E$ such that for $u_0 \in E$*

$$\lim_{\|h\|\to 0} \frac{\|F(u_0 + h) - F(u_0) - L(h)\|}{\|h\|} = 0,$$

then F is said to be Fréchet-differentiable at u_0 and the bounded linear operator $F'(u_0) := L$ is called the first Fréchet-derivative of F at u_0.

We need the following definitions in the sequel. We assume that E is Banach, reflexive and strictly convex together with its dual space E^*.

DEFINITION 1.1.3 (Alber and Ryazantseva (2006)). *Let E be a reflexive space, E^* its dual space and $F : E \to 2^{E^*}$ (An operator $A : E_1 \to 2^{E_2}$, we mean that A is multiple-valued, i.e., the mapping need not be necessarily having a one-to-one correspondence). The set of pairs $(u, f) \in E \times E^*$ such that $f \in F(u)$ is called the graph of an operator F and is denoted by grF. A set $G \subseteq E \times E^*$ is called monotone if the inequality*

$$\langle f - g, u - v\rangle \geq 0 \quad (1.1.1)$$

holds for all pairs (u, f) and (v, g) from G.

DEFINITION 1.1.4 (Alber and Ryazantseva (2006)). *An operator $F : E \to 2^{E^*}$ is monotone if its graph is a monotone set, i.e., if for all $u, v \in D(F)$ and for all $f \in F(u)$ and $g \in F(v)$,*

$$\langle f - g, u - v \rangle \geq 0.$$

DEFINITION 1.1.5 (Alber and Ryazantseva (2006)). *An operator $j : E \to 2^{E^*}$ is called normalized duality mapping in E if the following equalities are satisfied:*

$$\langle J, u \rangle = \|J\| \|u\| = \|u\|^2, \quad \forall J \in j(u), \quad \forall u \in E.$$

DEFINITION 1.1.6 (Alber and Ryazantseva (2006)). *An operator $F : E \to 2^E$ is called accretive if for all $u_1, u_2 \in D(F)$ with $v_1 \in F(u_1)$ and $v_2 \in F(u_2)$,*

$$\langle J(u_1 - u_2), v_1 - v_2 \rangle \geq 0 \tag{1.1.2}$$

where J is the single valued normalized duality mapping on E.

DEFINITION 1.1.7 (Alber and Ryazantseva (2006)). *An operator $F : E \to 2^E$ is called accretive if for all $u_1, u_2 \in D(F)$ with $v_1 \in F(u_1)$ and $v_2 \in F(u_2)$,*

$$\|u_1 - u_2\| \leq \|u_1 - u_2 + \lambda(F(u_1) - F(u_2))\|, \qquad \lambda > 0. \tag{1.1.3}$$

It can be verified easily that Definitions 1.1.6 and 1.1.7 are equivalent (Alber and Ryazantseva (2006)).

DEFINITION 1.1.8 (Alber and Ryazantseva (2006)). *An accretive operator $F : E \to 2^E$ is called m-accretive if*

$$R(F + \alpha I) = E \tag{1.1.4}$$

for all $\alpha > 0$, where I denote the identity operator on E.

DEFINITION 1.1.9 (Krasnosel'skii et al. (1976)). *A closed linear operator $F : E \to E$ is called of positive type if the operators $(F + \alpha I)^{-1}$ exist for all $\alpha \geq 0$ and if*

$$\| (F + \alpha I)^{-1} \| \leq \frac{c}{1 + \alpha},$$

where $c > 0$ is a constant.

Next we give two examples of nonlinear ill-posed problems.

EXAMPLE 1.1.10. *Parameter identification problem [Hofmann et al. (2016)] : Consider parameter identification problem in an elliptic PDE ; i.e.,to find the source term u in the elliptic boundary value problem*

$$\begin{aligned} -\Delta f + \xi(f) &= u \text{ in } \Omega \\ f &= 0 \text{ on } \partial\Omega \end{aligned}$$

from measurement of f in Ω. Here $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz continuously differentiable monotonically increasing function and $\Omega \subseteq \mathbb{R}^3$ is a smooth domain. The corresponding forward operator in this case is $F : H^2(\Omega) \longrightarrow H^2(\Omega)$ defined by

$$F(u) = f$$

is monotone. This can be seen as follows:

$$\begin{aligned} \langle F(u_1) - F(u_2), u_1 - u_2 \rangle &= \int_\Omega (f_1 - f_2)(u_1 - u_2) dx \\ &= \int_\Omega (f_1 - f_2)(-\Delta(f_1 - f_2) + \xi(f_1) - \xi(f_2)) dx \\ &= \int_\Omega (| \bigtriangledown (f_1 - f_2)|^2 + (\xi(f_1) - \xi(f_2))(f_1 - f_2)) dx \\ &\geq \| \bigtriangledown (f_1 - f_2)\|^2_{L^2(\Omega)} \geq 0. \end{aligned}$$

EXAMPLE 1.1.11. ***Geological Prospecting:****(cf. Vasin and George (2014)).*

In the structural inverse gravimetry problem for a two-layer medium, the required solution is a function $x_3 = u(x_1, x_2)$, which describes the interface between *media with different constant densities σ_1, σ_2. In a Cartesian coordinate system with the vertical axis x_3 directed downward, the equation with respect to the unknown function $x_3 = u(x_1, x_2)$ has the form*

$$\Gamma\Delta\sigma \int\int_D \frac{1}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + H^2]^{1/2}} dx_1' dx_2'$$
$$- \int\int_D \frac{1}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + u^2(x_1', x_2')]^{1/2}} dx_1' dx_2' = \Delta g(x_1, x_2); \tag{1.1.5}$$

here Γ is gravity constant, $\Delta\sigma = \sigma_1 - \sigma_2$ is the jump in density at the interface H, detailed by the function $u(x_1, x_2)$ to be calculated. $\Delta g(x_1, x_2)$ is the unknown gravitational field caused by some deviation in the interface H from horizontal asymptotic plane $x_3 = S$, i.e., for the sought for solution $\hat{u}(x_1, x_2)$ the following equality holds

$$\lim_{|x_1|,|x_2| \to \infty} |\hat{u}(x_1, x_2) - H| = 0,$$

$g(x_1, x_2)$ is given on the domain D.

By considering (1.1.5) the first term does not depend on $u(x_1, x_2)$ so equation can be modified as

$$F(u) \equiv - \int\int_D \frac{1}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + u^2(x_1', x_2')]^{1/2}} dx_1' dx_2' = f(x_1, x_2),$$

where $f(x_1, x_2) = \Delta g(x_1, x_2) + F(H)$.

1.2 Regularization Method

Generally, regularization is the approximation of a given ill-posed problem by a family of nearby well-posed problems. Procedures that head to stable approximations to an ill-posed problem are called regularization methods(Nair (2009)). Since (1.0.1) is ill-posed in general, the strong convergence and stability of approximate solutions can be attained only by exercising some regularization procedure. Practically, the available data will be f^δ instead of f in (1.0.1) with

$$\| f - f^\delta \| \leq \delta. \tag{1.2.1}$$

Throughout this thesis, we assume that $f^\delta \in E_2$ satisfies (1.2.1).

DEFINITION 1.2.1. *(Alber and Ryazantseva (2006)). An operator $R_\alpha f^\delta : E_2 \to E_1$ is called a regularization operator of equation (1.0.1) if it satisfies the two requirements:*

(1) $R_\alpha f^\delta$ is defined for all $\alpha \geq 0$ and for all $f^\delta \in E_2$ satisfying (1.2.1);

(2) There exist a function $\alpha = \alpha(\delta)$ such that $R_{\alpha(\delta)} f^\delta =: u_\alpha^\delta \to \hat{u}$ as $\delta \to 0$, where $\hat{u}$ is the solution of (1.0.1).

Operators $R_\alpha f^\delta$ leads to a variety of regularization methods. An element u_α^δ is called the regularized solution and α is called the regularization parameter. So a regularization method involves:

(1) construction of a regularization operator;

(2) choosing the regularization parameter $\alpha = \alpha(\delta)$ which ensure convergence of u_α^δ to some $\hat{u}$ as $\delta \to 0$.

The most generally used regularization methods for (1.0.1) with nonlinear F and approximate data f^δ are:

1. Tikhonov regularization method in which the solution u_α^δ of the equation

$$F'(u)^*(F(u) - f^\delta) + \alpha(u - u_0) = 0$$

is taken as the approximate solution of (1.0.1) (Tautenhahn and Jin (2003)).

2. If F is monotone operator and if domain and range are same for F, in this case we can consider Lavrentiev regularization method, in which the solution u_α^δ of the equation

$$F(u) + \alpha(u - u_0) = f^\delta \tag{1.2.2}$$

is taken as an approximate for $\hat{u}$ (Tautenhahn (2002)).

In our study we will be using Lavrentiev regularization method for obtaining stable approximation for $\hat{u}$.

1.2.1 Source Conditions

Suppose there exist a function $\varphi : (0,p] \to (0,\infty)$ with $p \geq\| F'(\hat{u}) \|$ and $v \in E_1$ such that

$$u_0 - \hat{u} = \varphi(F'(\hat{u}))v, \tag{1.2.3}$$

where u_0 is an initial guess, $\hat{u}$ is the solution of (1.0.1) and $F'(\hat{u})$ is the Fréchet derivative of F at $\hat{u}$ and

$$\| \hat{u} - R_\alpha f \| \leq \varphi(\alpha), \tag{1.2.4}$$

then φ is called a source function and the condition (1.2.3) is called a source condition. To obtain error bounds on the distance $\|u_\alpha^\delta - \hat{u}\|$ one needs some additional smoothness assumptions of the form (1.2.4) (known as "a priori assumptions") on the unknown $\hat{u}$, with respect to the operator $F'(\hat{u})$ or $F'(u_0)$. In literature, various source conditions are used. For example, Hölder-type source condition (Tautenhahn (2002, 2004)), i.e., $\hat{u} - u_0 \in R(F'(\hat{x})^*F'(\hat{u})^\nu), 0 < \nu \leq 1$, general source condition $\hat{u} - u_0 \in R(\phi((F'(\hat{x})^*F'(\hat{x})))$, with index functions ϕ (Argyros et al. (2014); Mahale and Nair (2007); Argyros et al. (2013); Semenova (2010)) and the new variational source conditions (Hofmann et al. (2016)). In our study, we will be using Hölder-type and the general source conditions with respect to the operator $F'(u_0)$.

1.2.2 Choice of regularization parameter

In general, a regularized solution u_α^δ can be written as $u_\alpha^\delta = R_\alpha f^\delta$ where R_α is a regularization function. A choice of $\alpha = \alpha_\delta$ of the regularization parameter may be made in either in a prior or a posterior way (Groetsch (1993)).

In practical applications, it is desirable that α is chosen independent of the source function φ, but may depend on the data (δ, f^δ) and therefore on the regularized solutions. In a posteriori methods the parameter α is determined during the course of computation of u_α^δ. For linear type ill-posed problems there exists many posteriori parameter choice strategies. For example, Groetsch and Guacaneme

(1987) considered the discrepancy principle

$$\|F(u_\alpha^\delta) - f^\delta\| = \frac{\delta}{\sqrt{\alpha}},$$

for choosing α. Many a posteriori strategies for linear type ill-posed problems are available in the literature (Guacaneme (1990); George and Nair (1993); Engl et al. (1996); Tautenhahn (2002)).

In our studies we have considered the adaptive parameter choice strategy, proposed by Pereverzev and Schock (2005). Pereverzev and Schock (2005), considered an adaptive selection of th parameter which does not involve even the regularization method explicitly. Let us discuss this adaptive method in shortly and more generally by approximating $\hat{u}$ with elements from a set $\{u_\alpha^\delta : \alpha > 0, \delta > 0\}$.

Assume that there exist increasing functions $\varphi(p)$ and $\phi(p)$ for $p > 0$ such that

$$\lim_{p\to 0} \varphi(p) = 0 = \lim_{p\to 0} \phi(p),$$

and

$$\|\hat{u} - u_\alpha^\delta\| \leq \varphi(p) + \frac{\delta}{\phi(p)}$$

for all $\alpha > 0, \delta > 0$. Note that the $\varphi(\alpha) + \frac{\delta}{\phi(\alpha)}$ attains its minimum for the choice $\alpha := \alpha_\delta$ such that $\varphi(\alpha) = \frac{\delta}{\phi(\alpha)}$, that is for

$$\alpha_\delta = (\varphi\phi)^{-1}(\delta)$$

and in that case

$$\|\hat{u} - u_{\alpha_\delta}^\delta\| \leq 2\varphi(\alpha_\delta).$$

In an "aposteriori"choice, one finds a parameter α_δ without making use of the unknown source function φ such that one obtains an error estimates of the form

$$\|\hat{u} - u_{\alpha_\delta}^\delta\| \leq c\varphi(\alpha_\delta),$$

for some $c > 0$ with $\alpha_\delta = (\varphi\phi)^{-1}(\delta)$. The procedure considered by Pereverzev and Schock (2005) starts with a finite number of positive reals, $\alpha_0, \alpha_1, \cdots, \alpha_N$, such that

$$\alpha_0 < \alpha_1 < \cdots < \alpha_N.$$

Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\phi(\alpha_i)}$ and for some $\mu > 1$,

$$\phi(\alpha_i) \leq \mu\phi(\alpha_{i-1}) \quad \forall i \in \{0, 1, 2, \cdots, N\}$$

Let

$$l := max\left\{i : \varphi(\alpha_i) \leq \frac{\delta}{\phi(\alpha_i)}\right\} < N,$$

and

$$k := max\left\{i : \|u_i^\delta - u_j^\delta\| \leq 4\frac{\delta}{\phi(\alpha_j)}, j = 0, 1, 2, \cdots, i-1\right\}.$$

Then, (see George and Nair (2008)) $l \leq k$ and

$$\|\hat{u} - u_{\alpha_\delta}^\delta\| \leq 6\mu\varphi(\alpha_\delta), \qquad \alpha_\delta = (\varphi\phi)^{-1}(\delta).$$

We will be using, the above parameter procedure extensively in our study.

1.3 Iterative methods and Convergence analysis

Obtaining closed form solution for regularization methods are desirable, but by considering most of the practical problems it may not be possible. So, iterative methods are used to obtain an approximation for regularized solution. In the last few years many authors considered iterative regularization methods, for example, Landweber method(Hanke et al. (1995)), Levenberg-Marquardth method(Hanke (1997a)), Gauss-Newton(Bakushinskii (1992)), Conjugate gradient(Hanke (1997b)) and Newton like methods(Hofmann et al. (2016), Hanke (1997a)). Iterative methods have the following form in common:

(1) Begin with an initial value u_0;

(2) Successive approximates $u_i, i = 1, 2...$, to $\hat{u}$ are computed with the aid of an iterative function $G : E \to E$, defined by $G(u_i) = u_{i+1}, i = 1, 2, \ldots$;

(3) If $\hat{u}$ is a fixed point of G, i.e., $G(\hat{u}) = \hat{u}$, all fixed points of G are also zeros of F, and if G is continuous in a neighbourhood of each of its fixed points, then each limit point of the sequence $u_i, i = 1, 2, ...$, is a fixed point of G.

1.3.1 Order of Convergence

Iterative methods can be classified by the rate of convergence. A sequence $\{u_k\}$ in E with $\lim u_k = u^*$ is said to be converging at an order p to u^*, if there exist real positive numbers γ and C such that, for all $k \in \mathbb{N}$,

$$\|u_k - u^*\| \leq Ce^{-\gamma p^k}.$$

Further extensive discussion of convergence rate can be seen in, Ortega and Rheinboldt (2000); Kelley (1995).

1.4 Motivation of Research

As mentioned earlier, iterative methods are used for solving (1.0.1) (see Argyros and George (2015); Ji'an et al. (2008); Liu (2005); Buong and Hung (2005)). In fact, most of the existing methods for solving (1.0.1) in Banach space setting, the error estimate is realized under the source condition (see Ji'an et al. (2008); Buong and Phuong (2012); Liu (2005))

$$u_0 - \hat{u} = F'(\hat{u})v. \tag{1.4.1}$$

One can easily note that, the above source condition is depending on the unknown solution $\hat{u}$. To our knowledge, for ill posed operator equation (1.0.1) in the setting of Banach space, no error estimate is known for $\| u_\alpha^\delta - \hat{u} \|$ under the general Hölder type condition

$$u_0 - \hat{u} = F'(\hat{u})^\nu v \quad 0 < \nu \leq 1. \tag{1.4.2}$$

This motivates us to bridge the above stated gap, by studying a source condition which is depending only on the initial data. We were also interested in obtaining a better order of convergence by modifying some already existed iterative methods.

1.5 Research Objectives

Our central aim in this thesis is to introduce new iterative methods with higher order of convergence to approximate $\hat{u}$. The overall objectives can be summarized as follows:

1. Introduce and study higher order iteration methods for approximating the Lavrentiev solution u_α^δ, in Banach space setting.

2. Obtain error estimates for the proposed methods using a generalised source condition.

3. The parameter choice strategy for the above methods is another problem of interest in our work.

1.6 Outline of the thesis

Throughout the thesis we considered adaptive parameter choice strategy considered by Pereverzev and Schock (2005) and we have used general Hölder type source condition which is discussed in Chapter 2. At the end of every Chapter, a numerical example is given to illustrate all results produced in the respective Chapters. The rest of the thesis is organized as follows.

Chapter 2 deals with nonlinear ill-posed problems associating $m-$accretive operators in Banach spaces. For the implementation of Lavrentiev regularization method, a derivative and inverse free method is used . Results of this chapter were published as a paper in Acta Mathematica Scientia ,Volume 38, Issue 1, January 2018, Pages 303-314.

In Chapter 3, we study an extended Newton-type iterative scheme that converges cubically to the solution which uses assumptions only on the first Fréchet derivative of the operator. The results of this Chapter were published in Journal of Applied Mathematics and Computing.

Chapter 4 is mainly concentrating on Newton-Kantorovich regularization method and were able to obtain a second order convergence for the solution. As in early studies we did not use any scalar sequences in our methods. The results of this Chapter were published in Rendiconti del Circolo Matematico di Palermo Series 2.

In Chapter 5, we discuss secant-type iteration and were able to obtain a convergence order of atleast 2.20557. We have also provided both local and semi-local types of convergence analysis .

Chapter 6 gives the conclusion of the thesis and discusses some further possible extensions, for future research.

Chapter 2

LAVRENTIEV'S REGULARIZATION METHOD FOR NONLINEAR ILL-POSED EQUATIONS IN BANACH SPACES

This Chapter deals with nonlinear ill-posed problems associating $m-$accretive mappings in Banach spaces. We consider a derivative and inverse free method for the implementing the Lavrentiev regularization method. Using general Hölder type source condition we obtain an optimal order error estimate. Also for choosing the regularization parameter we consider the adaptive parameter choice strategy considered by Pereverzev and Schock (2005). A numerical example is given to illustrate the theoretical results.

2.1 Introduction

In this study we consider the problem of approximately solving the nonlinear ill-posed equation

$$F(u) = f. \tag{2.1.1}$$

Here $F : E \to E$ is an m-accretive, Fréchet differentiable and single valued nonlinear mapping. Since F is m-accretive, by Definition 1.1.8

$$F(u) + \alpha(u - u_0) = f^\delta \tag{2.1.2}$$

has a unique solution u_α^δ for $\alpha > 0$. Here and below, we take u_0 as the initial guess of the exact solution $\hat{u}$ (which is assumed to be exist) of (2.1.1). As mentioned in Section 1.4, in earlier studies the optimal order convergence rate for $\|u_\alpha^\delta - \hat{u}\|$ is obtained under the Hölder type assumption (1.4.1). Under general Hölder type assumption (1.4.2), no error estimate is known for $\|u_\alpha^\delta - \hat{u}\|$. So, precisely, we consider the Hölder type source condition

$$u_0 - \hat{u} = F'(u_0)^\nu v \quad 0 < \nu \leq 1 \tag{2.1.3}$$

and attain the optimal order error estimate for $\|u_\alpha^\delta - \hat{u}\|$ in the Banach space setting. Note that (1.4.1) is depending on the unknown solution $\hat{u}$ but (2.1.3) is depending on the known u_0. This is one of the advantage of our approach. Using our idea one can obtain the optimal order error estimate for $\|u_\alpha^\delta - \hat{u}\|$ under the assumption (1.4.2)(see Corollary 2.2.5).

The rest of the Chapter is organized as follows. In Section 2.2, we consider Hölder type source condition for obtaining error estimate for $\|u_\alpha^\delta - \hat{u}\|$. In Section 2.3 we consider an iterative method and its convergence analysis. A priori choice of the Parameter and adaptive choice of the parameter are considered in Section 2.4. The implementation of the adaptive method and the algorithm are provided in Section 2.5. Finally, the Chapter winds up with a numerical example in Section 2.6.

2.2 Error estimates using Hölder type source condition

We briefly introduce some results from (Buong (2004); Ji'an et al. (2008)) to make the study self-contained. Let u_α^δ be the unique solution of (2.1.2) and u_α is the unique solution of

$$F(u) + \alpha(u - u_0) = f. \tag{2.2.1}$$

Then

$$\|u_\alpha^\delta - u_\alpha\| \leq \frac{\delta}{\alpha} \tag{2.2.2}$$

and

$$\|u_\alpha - \hat{u}\| \leq \|\hat{u} - u_0\|. \tag{2.2.3}$$

The following lemma from Ji'an et al. (2008), is used for proving our results in this Chapter.

LEMMA 2.2.1. *(cf.Ji'an et al. (2008)) Let $F : E \to E$ be accretive and Fréchet differentiable on E. Then for any real number $\alpha > 0$ and $u \in E$, $F'(u) + \alpha I$ is invertible,*

$$\|(F'(u) + \alpha I)^{-1}\| \leq \frac{1}{\alpha} \tag{2.2.4}$$

and

$$\|(F'(u) + \alpha I)^{-1} F'(u)\| \leq 2. \tag{2.2.5}$$

Note that by (2.2.4) we have,

$$\|\alpha(F'(u) + \alpha I)^{-1}\| \leq 1.$$

So for $0 < \nu \leq 1$, we have (see Krasnosel'skii et al. (1976)[page 287]),

$$F'(u)^\nu w = \frac{\sin \pi\nu}{\pi\nu} \int_0^\infty t^\nu (F'(u) + tI)^{-2} F'(u) w dt. \tag{2.2.6}$$

One of the crucial result for proving error estimate is the succeeding lemma, proof of this is analogous to the proof of Lemma 14.1 in (Krasnosel'skii et al. (1976)), but to make this chapter self-contained we will be giving the proof.

LEMMA 2.2.2. *Let $F : E \to E$ be a Fréchet differentiable and monotone operator. Then for $u \in E$ and $0 < \nu \leq 1$,*

$$\|\alpha(F' + \alpha I)^{-1} F'(u)^\nu\| \leq C_0 \alpha^\nu,$$

where $C_0 = \max\{4 \frac{\sin(\pi\nu)}{\pi\nu} \left(\frac{\nu}{1-\nu}\right)^\nu, 2\}$.

Proof. Note that for $\nu = 1$, we have by (2.2.5),

$$\|\alpha(F'(u) + \alpha I)^{-1} F'(u)\| \leq 2\alpha.$$

Now let us consider the case $0<\nu<1$. Then by (2.2.6) we have

$$\begin{aligned}(F'+\alpha I)^{-1}F'(u)^{\nu}w &= \frac{\sin\pi\nu}{\pi\nu}\int_0^{\infty} t^{\nu}(F'+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)wdt\\ &= \frac{\sin\pi\nu}{\pi\nu}[\int_0^{\rho} t^{\nu}(F'+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)wdt\\ &\quad+\int_{\rho}^{\infty} t^{\nu}(F'+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)wdt]\\ &= \frac{\sin\pi\nu}{\pi\nu}[H_1+H_2], \qquad (2.2.7)\end{aligned}$$

where $H_1=\int_0^{\rho} t^{\nu}(F'+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)wdt$ and $H_2=\int_{\rho}^{\infty} t^{\nu}(F'+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)wdt$. So, by (2.2.4) and (2.2.5) we have

$$\begin{aligned}\|H_1\| &= \|\int_0^{\rho} t^{\nu}F'(u)(F'(u)+tI)^{-2}(F'+\alpha I)^{-1}wdt\|\\ &\leq 2\int_0^{\rho}\frac{t^{\nu-1}}{\alpha}\|w\|dt\\ &= 2\frac{\rho^{\nu}}{\nu.\alpha}\|w\| \qquad (2.2.8)\end{aligned}$$

and

$$\begin{aligned}\|H_2\| &= \|\int_{\rho}^{\infty} t^{\nu}F'(u)(F'(u)+tI)^{-2}(F'+\alpha I)^{-1}wdt\|\\ &\leq 2\int_{\rho}^{\infty} t^{\nu-2}\|w\|dt\\ &= 2\frac{\rho^{\nu-1}}{1-\nu}\|w\|. \qquad (2.2.9)\end{aligned}$$

Thus by (2.2.7), (2.2.8) and (2.2.9), we have

$$\|(F'+\alpha I)^{-1}F'(u)^{\nu}w\| \leq 2\frac{\sin(\pi\nu)}{\pi\nu}\left[\frac{\rho^{\nu}}{\nu\alpha}+\frac{\rho^{\nu-1}}{1-\nu}\right]\|w\|.$$

Now the result follows by taking minimum of the right side of the expression above (i.e., $\rho=\frac{\nu\alpha}{1-\nu}$).

ASSUMPTION 2.2.3. *(see Argyros and George (2015); Semenova (2010); Pereverzev and Schock (2005)) There exists a constant $k_0\geq 0$ such that for every $u\in B(u_0,r)$ and $v\in E$ there exists an element $\phi(u,u_0,v)\in E$ such that $[F'(u)-F'(u_0)]v=F'(u_0)\phi(u,u_0,v)$, $\|\phi(u,u_0,v)\|\leq k_0\|v\|\|u-u_0\|$.*

THEOREM 2.2.4. *Let Assumption 2.2.3 and (2.1.3) hold. If* $3k_0r < 1$, *then*

$$\|u_\alpha - \hat{u}\| \leq \frac{C_0}{1 - 3k_0r}\alpha^\nu$$

where v *is as in (2.1.3),* $r = \|\hat{u} - u_0\|$ *and* C_0 *is as in Lemma 2.2.2.*

Proof. Note that for $\nu = 1$, the result follows from similar arguments given in Lemma 2.2.2. Now let us consider the case $0 < \nu < 1$. We have

$$F(u_\alpha) - F(\hat{u}) + \alpha(u_\alpha - u_0) = 0.$$

Thus by mean value theorem of integral calculus, we have

$$\begin{aligned}(F'(u_0) + \alpha I)(u_\alpha - \hat{u}) &= \alpha(u_0 - \hat{u}) \\ &\quad - \int_0^1 [F'(\hat{u} + t(u_\alpha - \hat{u})) - F'(u_0)](u_\alpha - \hat{u})dt.\end{aligned}$$

Therefore by (2.1.3), Lemma 2.2.1, Lemma 2.2.2, Assumption 2.2.3 and (2.2.3), we obtain

$$\begin{aligned}\|u_\alpha - \hat{u}\| &\leq \|\alpha(F'(u_0) + \alpha I)^{-1}F'(u_0)^\nu v\| \\ &\quad + \|(F'(u_0) + \alpha I)^{-1}\int_0^1 [F'(\hat{u} + t(u_\alpha - \hat{u})) - F'(u_0)](u_\alpha - \hat{u})dt\| \\ &\leq C_0\alpha^\nu + 2\int_0^1 \|\phi(\hat{u} + t(u_\alpha - \hat{u}), u_0, u_\alpha - \hat{u})dt\| \\ &\leq C_0\alpha^\nu + 2k_0(\|\hat{u} - u_0\| + \frac{1}{2}\|u_\alpha - \hat{u}\|)\|u_\alpha - \hat{u}\| \\ &\leq C_0\alpha^\nu + 2k_0(\|\hat{u} - u_0\| + \frac{1}{2}\|u_0 - \hat{u}\|)\|u_\alpha - \hat{u}\| \\ &\leq C_0\alpha^\nu + 3k_0\|\hat{u} - u_0\|\|u_\alpha - \hat{u}\| \\ &\leq C_0\alpha^\nu + 3k_0r\|u_\alpha - \hat{u}\|.\end{aligned}$$

Proof of the Theorem is completed.

□

COROLLARY 2.2.5. *Let Assumption 2.2.3 and (1.4.2) hold. If* $k_0r < 1$, *then*

$$\|u_\alpha - \hat{u}\| \leq \frac{C_0}{1 - k_0r}\alpha^\nu$$

where v *is as in (1.4.2),* $r = \|\hat{u} - u_\alpha\|$ *and* C_0 *is as in Lemma 2.2.2..*

Proof. Since

$$F(u_\alpha) - F(\hat{u}) + \alpha(u_\alpha - u_0) = 0,$$

we have

$$\begin{aligned}(F'(\hat{u}) + \alpha I)(u_\alpha - \hat{u}) &= \alpha(u_0 - \hat{u}) \\ &\quad - \int_0^1 [F'(\hat{u} + t(u_\alpha - \hat{u})) - F'(\hat{u})](u_\alpha - \hat{u})dt.\end{aligned}$$

Therefore by (1.4.2), Lemma 2.2.1, Lemma 2.2.2, Assumption 2.2.3 and (2.2.3), we obtain

$$\begin{aligned}\|u_\alpha - \hat{u}\| &\leq \|\alpha(F'(\hat{u}) + \alpha I)^{-1} F'(\hat{u})^\nu v\| \\ &\quad + \|(F'(\hat{u}) + \alpha I)^{-1} \int_0^1 [F'(\hat{u} + t(u_\alpha - \hat{u})) - F'(\hat{u})](u_\alpha - \hat{u})dt\| \\ &\leq C_0\alpha^\nu + 2\int_0^1 \|\varphi(\hat{u} + t(u_\alpha - \hat{u}), \hat{u}, u_\alpha - \hat{u})dt\| \\ &\leq C_0\alpha^\nu + 2k_0\frac{1}{2}\|u_\alpha - \hat{u}\|\|u_\alpha - \hat{u}\| \\ &\leq C_0\alpha^\nu + k_0 r\|u_\alpha - \hat{u}\|.\end{aligned}$$

The remains of the proof is alike to the proof of Theorem 2.2.4.

□

2.3 Iterative Method and Convergence analysis

In this Section, we assume that E is a real Banach algebra and $F : E \longrightarrow E$ is twice Fréchet differentiable accretive operator. Before moving to the method, it is comfortable to open up a few notations. For $\alpha > 0$, let

$$R_\alpha(u) := F(u) + \alpha(u - u_0) - f^\delta \tag{2.3.1}$$

and let

$$R'_\alpha(.)u := F'(.)u + \alpha u. \tag{2.3.2}$$

We study the iterative sequence defined by

$$u^\delta_{n+1,\alpha} = u^\delta_{n,\alpha} - \frac{2[R_\alpha(u^\delta_{n,\alpha})]^2}{R_\alpha(u^\delta_{n,\alpha} + R_\alpha(u^\delta_{n,\alpha})) - R_\alpha(u^\delta_{n,\alpha} - R_\alpha(u^\delta_{n,\alpha}))}, \tag{2.3.3}$$

where $u^\delta_{0,\alpha} = u_0$ is our initial point. As in previous papers like (Buong (2003)-Buong and Hung (2005); Vasin and George (2014)) etc., we select the parameter $\alpha = \alpha_i$ from some finite set

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\},$$

by the adaptive method considered by Perverzev and Schock (Pereverzev and Schock (2005)). For the sake of comfort, we use the notation

$$e_n = u^\delta_{n,\alpha} - u^\delta_\alpha \text{ for each } n = 0, 1, 2. \cdots, \tag{2.3.4}$$

where u^δ_α is the solution of $R_\alpha(u) = 0$.

Let

$$C_\beta := \min\{\frac{\|F(u_0) - f^\delta\|}{(2 + \beta_1/\alpha_0)(\beta_1 + \alpha_N)}, 2\}, \quad \delta < \frac{C_\beta}{2}\alpha_0 \tag{2.3.5}$$

and

$$\|\hat{u} - u_0\| \leq r \text{ with } r < \min\{\frac{1}{3k_0}, \frac{1}{2}(\frac{C_\beta}{2} - \frac{\delta}{\alpha_0})\}. \tag{2.3.6}$$

Further, we assume that

$$\|F'(.)\| \leq \beta_1 \text{ and } \|F''(.)\| \leq \beta_2.$$

We start in proving few lemmas which will help us to to prove our main result (Theorem 2.3.5).

LEMMA 2.3.1. *Let e_n be as in (2.3.4). Then*

$$\|e_0\| \leq 2r + \frac{\delta}{\alpha_0}.$$

Proof. Note that, by (2.2.2) and (2.2.3) we have

$$\|u^\delta_\alpha - \hat{u}\| \leq \frac{\delta}{\alpha} + \|u_0 - \hat{u}\|. \tag{2.3.7}$$

The result now follows from the following triangle inequality and (2.3.7)

$$\|u^\delta_\alpha - u_0\| \leq \|u^\delta_\alpha - \hat{u}\| + \|\hat{u} - u_0\|.$$

□

Let us first define the operators $M(u), M_1(u)$ and $M_2(u)$;

$$M(u) = \int_0^1 R''_\alpha(u^\delta_\alpha + t(u - u^\delta_\alpha))(1-t)dt \text{ for each } u \in D(F), \tag{2.3.8}$$

$$M_1(u) = \int_0^1 R''_\alpha(u^\delta_\alpha + t(u + R_\alpha(u) - u^\delta_\alpha))(1-t)dt, \text{ for each } u \in D(F) \tag{2.3.9}$$

and

$$M_2(u) = \int_0^1 R''_\alpha(u^\delta_\alpha + t(u - R_\alpha(u) - u^\delta_\alpha))(1-t)dt, \text{ for each } u \in D(F). \tag{2.3.10}$$

Let

$$\Gamma_1 := \frac{[M_1(u^\delta_{n,\alpha}) - M_2(u^\delta_{n,\alpha})][(e_n)^2 + (R_\alpha(u^\delta_{n,\alpha}))^2]}{2R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}, \tag{2.3.11}$$

and

$$\Gamma_2 := \frac{[M_1(u^\delta_{n,\alpha}) + M_2(u^\delta_{n,\alpha})]e_n R_\alpha(u^\delta_{n,\alpha})}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}. \tag{2.3.12}$$

LEMMA 2.3.2. *Let R'_α be as in (2.3.2), Γ_1 and Γ_2 be as above. Then*

$$R_\alpha(u^\delta_{n,\alpha} + R_\alpha(u^\delta_{n,\alpha})) - R_\alpha(u^\delta_{n,\alpha} - R_\alpha(u^\delta_{n,\alpha})) = 2R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})[1 + \Gamma_1 + \Gamma_2].$$

Proof. Applying the Taylor expansion to the operator $R_\alpha(u)$ around the solution u^δ_α of $R_\alpha(u) = 0$, we get

$$R_\alpha(u^\delta_{n,\alpha}) = R'_\alpha(u^\delta_\alpha)(u^\delta_{n,\alpha} - u^\delta_\alpha) + M(u^\delta_{n,\alpha})(u^\delta_{n,\alpha} - u^\delta_\alpha)^2. \tag{2.3.13}$$

Similarly the Taylor expansion of $R_\alpha(u^\delta_{n,\alpha} + R_\alpha(u^\delta_\alpha))$ and $R_\alpha(u^\delta_{n,\alpha} - R_\alpha(u^\delta_\alpha))$ around the solution u^δ_α of $R_\alpha(u) = 0$, we get

$$\begin{aligned} R_\alpha(u^\delta_{n,\alpha} + R_\alpha(u^\delta_{n,\alpha})) &= R'_\alpha(u^\delta_\alpha)(u^\delta_{n,\alpha} - u^\delta_\alpha + R_\alpha(u^\delta_{n,\alpha})) \\ &\quad + M_1(u^\delta_{n,\alpha})(u^\delta_{n,\alpha} - u^\delta_\alpha + R_\alpha(u^\delta_{n,\alpha}))^2 \\ &= R'_\alpha(u^\delta_\alpha)[(u^\delta_{n,\alpha} - u^\delta_\alpha) + R_\alpha(u^\delta_{n,\alpha})] + M_1(u^\delta_{n,\alpha})[(u^\delta_{n,\alpha} - u^\delta_\alpha)^2 \\ &\quad + (R_\alpha(u^\delta_{n,\alpha}))^2 + 2(u^\delta_{n,\alpha} - u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})] \\ &= R'_\alpha(u^\delta_\alpha)[e_n + R_\alpha(u^\delta_{n,\alpha})] + M_1(u^\delta_{n,\alpha})[(e_n)^2 \qquad (2.3.14) \\ &\quad + (R_\alpha(u^\delta_{n,\alpha}))^2 + 2e_n R_\alpha(u^\delta_{n,\alpha})] \end{aligned}$$

and

$$\begin{aligned}
R_\alpha(u^\delta_{n,\alpha} - R_\alpha(u^\delta_{n,\alpha})) &= R'_\alpha(u^\delta_\alpha)(u^\delta_{n,\alpha} - u^\delta_\alpha - R_\alpha(u^\delta_{n,\alpha})) \\
&\quad + M_2(u^\delta_{n,\alpha})(u^\delta_{n,\alpha} - u^\delta_\alpha - R_\alpha(u^\delta_{n,\alpha}))^2 \\
&= R'_\alpha(u^\delta_\alpha)[(u^\delta_{n,\alpha} - u^\delta_\alpha) - R_\alpha(u^\delta_{n,\alpha})] + M_2(u^\delta_{n,\alpha})[(u^\delta_{n,\alpha} - u^\delta_\alpha)^2 \\
&\quad + (R_\alpha(u^\delta_{n,\alpha}))^2 - 2(u^\delta_{n,\alpha} - u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})] \\
&= R'_\alpha(u^\delta_\alpha)[e_n - R_\alpha(u^\delta_{n,\alpha})] + M_2(u^\delta_{n,\alpha})[(e_n)^2 \qquad (2.3.15) \\
&\quad + (R_\alpha(u^\delta_{n,\alpha}))^2 - 2e_n R_\alpha(u^\delta_{n,\alpha})].
\end{aligned}$$

From (2.3.14) and (2.3.15) we have

$$\begin{aligned}
& R_\alpha(u^\delta_{n,\alpha} + R_\alpha(u^\delta_{n,\alpha})) - R_\alpha(u^\delta_{n,\alpha} - R_\alpha(u^\delta_{n,\alpha})) \\
= \; & 2R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha}) \\
& + [M_1(u^\delta_{n,\alpha}) - M_2(u^\delta_{n,\alpha})]((e_n)^2 + (R_\alpha(u^\delta_{n,\alpha}))^2) \\
& + 2[M_1(u^\delta_{n,\alpha}) + M_2(u^\delta_{n,\alpha})]e_n R_\alpha(u^\delta_{n,\alpha}) \\
= \; & 2R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})[1 + \Gamma_1 + \Gamma_2]. \qquad (2.3.16)
\end{aligned}$$

□

LEMMA 2.3.3. *Let R_α, R'_α, Γ_1 and Γ_2 be as in (2.3.1), (2.3.2), (2.3.11) and (2.3.12) respectively. Then*

(a)

$$\|R_\alpha(u^\delta_{n,\alpha})\| \leq (\beta_1 + \alpha)\|e_n\| + \frac{\beta_2 + \alpha}{2}\|e_n\|^2$$

(b)

$$\|(R_\alpha(u^\delta_{n,\alpha}))^2(\Gamma_1 + \Gamma_2)\| = O(\|e_n\|^3).$$

Proof. Note that (a) follows from (2.3.13) and for all $u \in E$ the inequalities

$$\|R'_\alpha(u)\| \leq \beta_1 + \alpha \;\text{ and }\; \|M(u)\| \leq \frac{\beta_2 + \alpha}{2}. \qquad (2.3.17)$$

For proving (b), we observe that

$$\begin{aligned}
\|R_\alpha(u^\delta_{n,\alpha})\| &= \|R'_\alpha(u^\delta_\alpha)^{-1}R'_\alpha(u^\delta_\alpha)(R_\alpha(u^\delta_{n,\alpha}))\| \\
&\leq \frac{1}{\alpha}\|R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})\| \qquad (2.3.18)
\end{aligned}$$

and hence

$$\begin{aligned}\|(R_\alpha(u^\delta_{n,\alpha}))^2(\Gamma_1+\Gamma_2)\| &\leq \|\frac{1}{\alpha}R_\alpha(u^\delta_{n,\alpha})([M_1(u^\delta_{n,\alpha})-M_2(u^\delta_{n,\alpha})][(e_n)^2+(R_\alpha(u^\delta_{n,\alpha}))^2]\\ &\quad +[M_1(u^\delta_{n,\alpha})+M_2(u^\delta_{n,\alpha})]e_nR_\alpha(u^\delta_{n,\alpha}))\|\\ &= O(\|e_n\|^3).\end{aligned}$$

The last step follows from (a), (2.3.17) and the inequality $\|M_i(u)\| \leq \frac{\beta_2+\alpha}{2}$, for $i=1,2$.

□

LEMMA 2.3.4. *Let R_α and R'_α be as in (2.3.1)and (2.3.2) respectively. Suppose $\|u^\delta_{n,\alpha}-u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$ for each $n=1,2,\cdots$. Then*

$$\frac{1}{\|R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})\|} \leq \frac{1}{\alpha(\|F(u_0)-f^\delta\|-(\beta_1+\alpha)\|u^\delta_{n,\alpha}-u_0\|)} \text{ for each } n=1,2,\cdots.$$

Proof. Observe that

$$\begin{aligned}R_\alpha(u^\delta_{n,\alpha}) &= F(u^\delta_{n,\alpha})-f^\delta+\alpha(u^\delta_{n,\alpha}-u_0)\\ &= F(u_0)-f^\delta+F(u^\delta_{n,\alpha})-F(u_0)+\alpha(u^\delta_{n,\alpha}-u_0)\\ &= F(u_0)-f^\delta+[\int_0^1 F'(u_0+t(u^\delta_{n,\alpha}-u_0))dt+\alpha I](u^\delta_{n,\alpha}-u_0).\end{aligned}$$

So

$$\begin{aligned}\|R_\alpha(u^\delta_{n,\alpha})\| &\geq \|F(u_0)-f^\delta\|-\|[\int_0^1 F'(u_0+t(u^\delta_{n,\alpha}-u_0))dt+\alpha I](u^\delta_{n,\alpha}-u_0)\|\\ &\geq \|F(u_0)-f^\delta\|-(\beta_1+\alpha)\|u^\delta_{n,\alpha}-u_0\| \qquad (2.3.19)\end{aligned}$$

for each $n=1,2,\cdots$. The result now follows from (2.3.18) and (2.3.19).

□

THEOREM 2.3.5. *Let R_α be as in (2.3.1) and u^δ_α be the solution of $R_\alpha(u)=0$. Moreover the first and second Fréchet derivative of F exists for all $u \in D(F)$. Then the sequence defined in (2.3.3) converges quadratically to u^δ_α. Furthermore*

$$\|u^\delta_{n+1,\alpha}-u^\delta_\alpha\| = \frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0)-f^\delta\|-(\beta_1+\alpha)2\|e_0\|)}\|e_n\|^2+O(\|e_n\|^3).$$

Proof. Let $\Theta = \Gamma_1 + \Gamma_2$. Then by (2.3.3), (2.3.16) and (2.3.18), we have

$$\begin{aligned} e_{n+1} &= e_n - \frac{(R_\alpha(u^\delta_{n,\alpha}))^2}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})(1+\Theta)} \\ &= e_n - \frac{(R_\alpha(u^\delta_{n,\alpha}))^2}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}[I - \Theta + \Theta^2 - \cdots] \\ &= e_n - \frac{(R_\alpha(u^\delta_{n,\alpha}))^2}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}(I-\Theta) \\ &\quad - \frac{(R_\alpha(u^\delta_{n,\alpha}))^2}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})} \times \text{higher order terms in } \Theta \\ &= \frac{1}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}[R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})e_n - R_\alpha(u^\delta_{n,\alpha})^2(I-\Theta) \\ &\quad -(R_\alpha(u^\delta_{n,\alpha}))^2 \times \text{higher order terms in } \Theta]. \end{aligned} \tag{2.3.20}$$

Therefore, we have

$$\begin{aligned} \|e_{n+1}\| &\leq \|\frac{1}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}\|[\|R'_\alpha(u^\delta_\alpha)\|\|R_\alpha(u^\delta_{n,\alpha})\|\|e_n\| \\ &\quad + \|(R_\alpha(u^\delta_{n,\alpha}))^2\| + \|(R_\alpha(u^\delta_{n,\alpha}))^2\|\|\Theta\| \\ &\quad + \text{higher order terms in } \|\Theta\|]. \end{aligned}$$

If $\|u^\delta_{n,\alpha} - u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$, then using Lemma 2.3.1-2.3.4, one can prove that

$$\|e_{n+1}\| \leq \frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0)-f^\delta\| - (\beta_1+\alpha)\|u^\delta_{n,\alpha} - u_0\|)}\|e_n\|^2 + O(\|e_n\|^3). \tag{2.3.21}$$

Now it remains to show that $\|u^\delta_{n,\alpha} - u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$. This can be shown as follows; since $\frac{2(\beta_1/\alpha+1)(\beta_1+\alpha)}{\|F(u_0)-f^\delta\|}\|e_0\| \leq \frac{2(\beta_1/\alpha_0+1)(\beta_1+\alpha_N)}{\|F(u_0)-f^\delta\|}\|e_0\| \leq 1$ by (2.3.5) and (2.3.6),

$$\begin{aligned} \|u^\delta_{1,\alpha} - u_0\| &\leq \|u^\delta_{1,\alpha} - u^\delta_\alpha\| + \|u^\delta_\alpha - u_0\| \\ &\leq \frac{2(\beta_1/\alpha+1)(\beta_1+\alpha)}{\|F(u_0)-f^\delta\|}\|e_0\|^2 + O(\|e_0\|^3) + \|u^\delta_\alpha - u_0\| \\ &\leq 2\|e_0\| \leq C_\beta < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha} \end{aligned}$$

(by ignoring higher order terms in $\|e_0\|$). Again by (2.3.21) and (2.3.6) we have,

$$\begin{aligned}\|u^\delta_{2,\alpha} - u_0\| &\leq \|u^\delta_{2,\alpha} - u^\delta_\alpha\| + \|u^\delta_\alpha - u_0\| \\ &\leq \frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0) - f^\delta\| - (\beta_1+\alpha)\|u^\delta_{1,\alpha} - u_0\|)}\|e_1\|^2 \\ &\quad + O(\|e_1\|^3) + \|u^\delta_\alpha - u_0\| \\ &\leq 2\|u^\delta_\alpha - u_0\| = 2\|e_0\| \leq C_\beta < \frac{\|F(u_0) - f^\delta\|}{\beta_1+\alpha}.\end{aligned}$$

(by ignoring higher order terms in $\|e_0\|$ and observing that (by (2.3.6)),

$$\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0) - f^\delta\| - (\beta_1+\alpha)\|u^\delta_{1,\alpha} - u_0\|)}\|e_0\| < 1$$

) which shows $\|u^\delta_{n,\alpha} - u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$ for $n = 2$. By simply replacing $u^\delta_{2,\alpha}$ by $u^\delta_{k+1,\alpha}$ in the preceding estimates we arrive at $\|u^\delta_{k+1,\alpha} - u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$. Thus by induction $\|u^\delta_{n,\alpha} - u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$ for $n > 0$. From the above relation it follows that

$$\begin{aligned}\|u^\delta_{n+1,\alpha} - u^\delta_\alpha\| &\leq \frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0) - f^\delta\| - (\beta_1+\alpha)2\|e_0\|)}\|e_n\|^2 \\ &\quad + O(\|e_n\|^3). \qquad (2.3.22)\end{aligned}$$

The proof of the Theorem is completed. □

REMARK 2.3.6. *Note that, by repeated application of (2.3.22) we have the following estimate*

$$\|e_{n+1}\| \leq \left(\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0) - f^\delta\| - 2(\beta_1+\alpha)\|e_0\|)}\right)^{2^{n+1}-1}\|e_0\|^{2^{n+1}} + O(\|e_n\|^{2^n+3}).$$

Since $\|e_0\| < 1$, we neglect the terms with order $\|e_0\|^{2^{n+1}+3}$ and will get

$$\|e_{n+1}\| \leq C_\alpha e^{-\gamma 2^{n+1}}, \qquad (2.3.23)$$

where $C_\alpha := \left(\frac{2(\beta_1+\alpha_N)^2}{\alpha_0(\|F(u_0)-f^\delta\| - 2(\beta_1+\alpha)\|e_0\|)}\right)^{2^{n+1}-1}$, $\gamma = -\log(\|e_0\|)$. *Note that*

$$C_\alpha e^{-\gamma 2^{n+1}} = [C_\alpha e^{-\gamma 2^n}]e^{-\gamma 2^n},$$

and for large n, $C_\alpha e^{-\gamma 2^n} \leq C$ *for all* $C > 0$. *So for a bigger* n, *from (2.3.23), (2.2.2) and Theorem 2.2.4, we have*

$$\|u^\delta_{n+1,\alpha} - \hat{u}\| \leq Ce^{-\gamma 2^n} + \frac{\delta}{\alpha} + \frac{C_0}{1-3k_0 r}\alpha^\nu.$$

Let

$$n_\delta := min\{n : e^{-\gamma 2^n} \leq \frac{\delta}{\alpha} \ \& \ C_\alpha e^{-\gamma 2^n} \leq C\} \tag{2.3.24}$$

for some constant C. In light of above discussed Remark, we state the succeeding Theorem.

THEOREM 2.3.7. *Let* $u^\delta_{n_\delta+1,\alpha}$ *be as in (2.3.3) and let the assumptions in Theorem 2.2.4 and Theorem 2.3.5 be satisfied, where* n_δ *be as in (2.3.24). Then we have the following;*

$$\|u^\delta_{n_\delta+1,\alpha} - \hat{u}\| \leq \bar{C}(\alpha^\nu + \frac{\delta}{\alpha}) \tag{2.3.25}$$

where $\bar{C} = \max\{C+1, \frac{C_0}{1-3k_0 r}\}$.

2.4 A Priori Choice of the Parameter

Observe that the error $\alpha^\nu + \frac{\delta}{\alpha}$ in (2.3.25) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies, $\alpha_\delta^{1+\nu} = \delta$. That is $\alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Hence by (2.3.25) we come to the following Theorem.

THEOREM 2.4.1. *Let the assumptions in Theorem 2.3.7 holds. For* $\delta > 0$, *let* $\alpha := \alpha_\delta = \delta^{\frac{1}{1+\nu}}$. *Let* n_δ *be as in (2.3.24). Then*

$$\|u^\delta_{n_\delta,\alpha} - \hat{u}\| = O(\delta^{\frac{\nu}{1+\nu}}).$$

2.4.1 Adaptive Scheme and Stopping Rule

We use the adaptive selection of the parameter strategy considered by Pereverzev and Schock (Pereverzev and Schock (2005)), adjusted appropriately for the situation for choosing the parameter α. For easiness, take $u^\delta_i := u^\delta_{n_i,\alpha_i}$. Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$l := max\left\{i : \alpha_i^{\nu} \le \frac{\delta}{\alpha_i}\right\} < N \text{ and} \tag{2.4.1}$$

$$k := max\left\{i : \|u_i^{\delta} - u_j^{\delta}\| \le 4\bar{C}\frac{\delta}{\alpha_j}, j = 0, 1, 2, \cdots, i-1\right\} \tag{2.4.2}$$

where $\bar{C}$ is as in Theorem 2.3.7. Now we state the succeeding Theorem.

THEOREM 2.4.2. *Assume that there exists $i \in \{0, 1, \cdots, N\}$ such that $\alpha_i^{\nu} \le \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 2.3.7 be fulfilled, and let k and l be as in (2.4.2) and (2.4.1) respectively. Then*

$$l \le k;$$

and

$$\|\hat{u} - u_k^{\delta}\| \le 6\bar{C}\mu\delta^{\frac{\nu}{1+\nu}}.$$

Proof. For proving $k \ge l$, it is sufficient to prove that, for all $i \in \{1, 2, \ldots N\}$, $\alpha_i^{\nu} \le \frac{\delta}{\alpha_i} \Longrightarrow \|u_i^{\delta} - u_j^{\delta}\| \le 4\bar{C}\frac{\delta}{\alpha_j}$, $\forall j = 0, 1, 2, \ldots i-1$. For $j < i$, we have

$$\begin{aligned} \| u_i^{\delta} - u_j^{\delta} \| &\le \| u_i^{\delta} - \hat{u} \| + \| \hat{u} - u_j^{\delta} \| \\ &\le \bar{C}(\alpha_i^{v} + \frac{\delta}{\alpha_i}) + \bar{C}(\alpha_j^{v} + \frac{\delta}{\alpha_j}) \\ &\le 2\bar{C}\frac{\delta}{\alpha_i} + 2\bar{C}\frac{\delta}{\alpha_j} \\ &\le 4\bar{C}\frac{\delta}{\alpha_j}. \end{aligned}$$

Thus we have proved the relation $k \ge l$. Notice that

$$\| \hat{u} - u_k^{\delta} \| \le \| \hat{u} - u_l^{\delta} \| + \| u_k^{\delta} - u_l^{\delta} \|$$

where

$$\| \hat{u} - u_l^{\delta} \| \le \bar{C}(\alpha_l^{v} + \frac{\delta}{\alpha_l}) \le 2\bar{C}\frac{\delta}{\alpha_l}.$$

Now since $l \le k$, we have

$$\| u_k^{\delta} - u_l^{\delta} \| \le 4\bar{C}\frac{\delta}{\alpha_l}.$$

Hence

$$\| \hat{u} - u_k^\delta \| \leq 6\bar{C}\frac{\delta}{\alpha_l}$$

Now,since $\alpha_\delta = \delta^{\frac{1}{1+\nu}} \leq \alpha_{l+1} \leq \mu\alpha_l$, it follows that

$$\frac{\delta}{\alpha_l} \leq \frac{\mu\delta}{\alpha_\delta} = \mu\delta^{\frac{\nu}{1+\nu}}.$$

This completes the proof.

2.5 Adaptive choice rule implementation

Conclusively the balancing algorithm linked with the parameter choice detailed in Theorem 2.4.2 contains the sequential steps:

- Select $\alpha_0 > 0$ so that $\delta < \alpha_0$ and $\mu > 1$.
- Set $\alpha_i := \mu^i\alpha_0, i = 0, 1, 2 \cdots, N$.

2.5.1 Algorithm

a. Choose $i = 0$.

b. Set $n_i := \min\left\{n : e^{-\gamma 2^n} \leq \frac{\delta}{\alpha_i} \ \& \ C_\alpha e^{-\gamma 2^n} \leq C\right\}$.

c. Solve $u_i := u_{n_i,\alpha_i}^\delta$ by using the iteration (2.3.3).

d. If $\|u_i - u_j\| > 4\bar{C}\frac{\delta}{\alpha_j}$, $j < i$, then take $k = i - 1$ and return u_k.

e. Else set $i = i + 1$ and go to b.

2.6 Numerical Example

EXAMPLE 2.6.1. *Let $F : D(F) \subseteq C[0,1] \longrightarrow C[0,1]$ defined by*

$$F(u) := \int_0^1 k(t,s)u^3(s)ds, \tag{2.6.1}$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1 \\ (1-s)t, & 0 \le t \le s \le 1 \end{cases} .$$

Using Definition 1.1.7 and 1.1.8, one can easily verify that F *is m-accretive.*

$$F'(u)w = 3\int_0^1 k(t,s)u^2(s)w(s)ds. \tag{2.6.2}$$

In our computation, we take $f(t) = \frac{6sin(\pi t)+sin^3(\pi t)}{9\pi^2}$ *and* $f^\delta = f + \delta$. *Then the exact solution is* $\hat{u}(t) = sin(\pi t)$. *We use* $u_0(t) = sin(\pi t) + \frac{3[t\pi^2 - t^2\pi^2 + sin^2(\pi t)]}{4\pi^2}$, *as our initial guess, so that the function* $u_0 - \hat{u}$ *satisfies the source condition*

$$u_0 - \hat{u} = F'(u_0)\left(\frac{\hat{u}^2}{4u_0^2}\right).$$

We choose $\alpha_0 = \mu\delta$ *and* $\mu = 1.01$. *We use the Gauss-Legendre quadrature formula:*

$$\int_0^1 f(t)dt \approx \sum_{j=1}^{n} w_j f(t_j),$$

where the abscissa t_j *and the weight* w_j *for* $n = 25$ *are given Table 2.1, to discretize equation (2.6.1).*

The discretized form of (2.3.3) is as follows:

$$u^\delta_{k+1,\alpha}(t_i) = u^\delta_{k,\alpha}(t_i) - \frac{2[R_\alpha(u^\delta_{k,\alpha})(t_i)]^2}{R_\alpha(u^\delta_{k,\alpha} + R_\alpha(u^\delta_{k,\alpha})(t_i)) - R_\alpha(u^\delta_{k,\alpha} - R_\alpha(u^\delta_{k,\alpha})(t_i))} \tag{2.6.3}$$

where $R_\alpha(u(t_i)) = F(u(t_i)) + \alpha(u(t_i) - u_0(t_0)) - (f(t_i) + \delta)$ *and* $F(u(t_i)) = \sum_{j=1}^{25} a_{ij}u(t_j)^3$ *with* $a_{ij} = \begin{cases} w_j t_j(1-t_i), & \text{if } j \le i \\ w_j t_i(1-t_j), & \text{if } i < j. \end{cases}$

The relative error $\frac{||u_k - \hat{u}||}{||\hat{u}||}$ *and the residual error* $\frac{||F(u_k) - f^\delta||}{||f^\delta||}$ *are given in Table 2.2.*

Table 2.1: Abscissa and weights of Gauss-Legendre quadrature formula

i	t_i	w_i
1	0.0022215151047509	0.0056968992505131
2	0.0116680392702412	0.0131774933075160
3	0.0285127143855128	0.0204695783506531
4	0.0525040010608623	0.0274523479879176
5	0.0832786856195830	0.0340191669061784
6	0.1203703684813212	0.0400703501675005
7	0.1632168157632658	0.0455141309914818
8	0.2111685348793885	0.0502679745335253
9	0.2634986342771425	0.0542598122371318
10	0.3194138470953061	0.0574291295728558
11	0.3780665581395058	0.0597278817678923
12	0.4385676536946448	0.0611212214951550
13	0.5000000000000000	0.0615880268633577
14	0.5614323463053552	0.0611212214951550
15	0.6219334418604942	0.0597278817678923
16	0.6805861529046939	0.0574291295728558
17	0.7365013657228575	0.0542598122371318
18	0.7888314651206115	0.0502679745335253
19	0.8367831842367342	0.0455141309914818
20	0.8796296315186788	0.0400703501675005
21	0.9167213143804170	0.0340191669061784
22	0.9474959989391377	0.0274523479879176
23	0.9714872856144872	0.0204695783506531
24	0.9883319607297588	0.0131774933075160
25	0.9977784848952490	0.0056968992505131

Table 2.2: The relative error and residual error

δ	α	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|F(u_k)-f^\delta\|\|}{\|\|f^\delta\|\|}$
0.01	0.010406040100000	0.163037448543162	0.994143636281628
0.001	0.001051010050100	0.025530357947278	0.993949284345280
0.005	0.005203020050000	0.043443121882413	0.993312588078136

Chapter 3

EXTENDED NEWTON-TYPE ITERATION FOR NONLINEAR ILL-POSED EQUATIONS IN BANACH SPACE

We produce an extended Newton-type iterative scheme that converges cubically to the solution u_α^δ of the equation (2.1.2) using assumptions only on the first Fréchet derivative of the operator F. As illustrated in Chapter 2 we use general Hölder type source condition and obtain an error estimate. For choosing the regularization parameter accordingly we use the adaptive parameter choice strategy considered by Pereverzev and Schock (2005). A numerical example is given to illustrate the theoretical results.

3.1 Introduction

One of our primary research objective of our study (as mentioned in Section 1.5) is to obtain a higher order convergence for approximating u_α^δ the solution of (2.1.2), in a Banach space setting. We consider at the iterative method considered by Xiao and Yin (2016) for approximating solution x^* of the equation $G(x) = 0$, where $G : \mathbb{R}^n \to \mathbb{R}^n$, is properly modified to approximate u_α^δ. Xiao and Yin (2017) considered the method defined iteratively for $k = 0, 1, 2, \ldots$ by

$$\begin{aligned} v_k &= u_k - aG'(u_k)^{-1}G(u_k) \\ w_k &= u_k - \frac{1}{2}\left\{(\frac{1}{a}G'(v_k) + (1-\frac{1}{a})G'(u_k))^{-1} + G'(u_k)^{-1}\right\}G(u_k), \\ u_{k+1} &= w_k - \left\{\frac{1}{a}G'(v_k) + (1-\frac{1}{a})G'(u_k)\right\}^{-1}(w_k). \end{aligned}$$

Xiao and Yin (2017) proved that the method defined above is well defined and converges cubically to $\hat{u}$.

We modified the above method of Xiao and Yin (2017) to solve the ill-posed equation (2.1.1). Precisely, we consider the iteration defined for each $k = 0, 1, 2, \ldots$ by

$$v_k = u_k - aR'_\alpha(u_k)^{-1}R_\alpha(u_k), \tag{3.1.1}$$

$$w_k = u_k - \frac{1}{2}\left\{(\frac{1}{a}R'_\alpha(v_k) + (1-\frac{1}{a})R'_\alpha(u_k))^{-1} + R'_\alpha(u_k)^{-1}\right\}R_\alpha(u_k), \tag{3.1.2}$$

$$u_{k+1} = w_k - \left\{\frac{1}{a}R'_\alpha(v_k) + (1-\frac{1}{a})R'_\alpha(u_k)\right\}^{-1}R_\alpha(w_k), \tag{3.1.3}$$

where,

$$R_\alpha(u) := F(u) + \alpha(u - u_0) - f^\delta, \tag{3.1.4}$$

$$R'_\alpha(.)h := F'(.)h + \alpha h, \tag{3.1.5}$$

where $\alpha > 0$ is the regularization parameter and the scalar parameter a will be defined later.

In this study we use assumptions only on the first Fréchet derivative of F to obtain the error estimate for $\|u_k - \hat{u}\|$ under the general source condition (2.1.3) for $0 < \nu \leq 1$.

The rest of the Chapter is organized as follows. The convergence analysis of the iterative scheme is given in Section 3.2. Error estimate using Hölder-type source condition is given in Section 3.3. Parameter choice strategy is given in Section 3.4. The Chapter ends with a numerical example given in Section 3.5.

3.2 Iterative Method with Convergence analysis

To present the convergence analysis, it is helpful in introducing some notations. Let,

$$e_k = u_k - u_\alpha^\delta, \tag{3.2.1}$$

$$\hat{e_k} = v_k - u_\alpha^\delta, \tag{3.2.2}$$

$$\bar{e_k} = w_k - u_\alpha^\delta. \tag{3.2.3}$$

Let $r \geq \|\hat{u} - u_0\|$ and $r_0 \leq 2r + 1$. Next, we define some scalar parameters: For $0 < k_0 < \frac{\sqrt{17}-3}{4}$, let

$$\hat{R} = \frac{1}{1 - k_0 r_0}, \quad C^{k_0,a} = |1-a| + ak_0 + ak_0(1+k_0)\hat{R},$$
$$C = \frac{k_0[C_{k_0,a} + |1-a|]}{a}, \quad \bar{R} = \frac{1}{1 - Cr_0},$$
$$\tilde{C} = k_0 + (1+k_0)(\frac{\bar{R}C}{2} + \frac{\hat{R}k_0}{2}) \text{ and } \quad \Lambda = \tilde{C}C\bar{R}(1 + k_0\tilde{C}) + k_0\tilde{C}^2.$$

The preceding constants depend on k_0, r_0 and a. We shall replace them with constants depending on k_0 and a which constitute part of the initial data. Choose $r_0 \in (0, \frac{1}{2k_0})$. Then, $\hat{R} \leq \hat{R}_1 := 2$. Define

$$C_1^{k_0,a} = |1-a| + ak_0 + 2ak_0(1+k_0),$$
$$C_1 = \frac{k_0[C_1^{k_0,a} + |1-a|]}{a}$$

and

$$\tilde{R}_1 = \frac{1}{1 - C_1 r_0}.$$

Then, we have

$C^{k_0,a} \leq C_1^{k_0,a}$ and $C \leq C_1$. Choose $r_0 \in (0, \min\{\frac{1}{2k_0}, \frac{1}{2C_1}\})$. Then, we have

$$\bar{R} \leq \tilde{R}_1 \leq \hat{R}_1 = 2.$$

Moreover, define $\tilde{C}_1 = k_0 + (1+k_0)(C_1 + k_0)$ and $\Lambda_1 = 2\tilde{C}_1 C_1(1 + k_0\tilde{C}_1) + k_0\tilde{C}_1^2$. Then, we have

$$\tilde{C} \leq \tilde{C}_1$$

and

$$\Lambda \leq \Lambda_1.$$

Hereafter, we assume that

$$\delta \in \left(\min\{\alpha, \frac{\alpha}{k_0}, \frac{\alpha}{C_1}, \frac{\alpha}{\tilde{C}_1}, \frac{\alpha}{\Lambda_1}\}, \alpha_0 \right), \tag{3.2.4}$$

for some $\alpha_0 > \min\left\{\alpha, \frac{\alpha}{k_0}, \frac{\alpha}{C_1}, \frac{\alpha}{\tilde{C}_1}, \frac{\alpha}{\Lambda_1}\right\}$. Moreover, we assume that

$$0 < a < \frac{2}{2k_0^2 + 3k_0 + 1}. \tag{3.2.5}$$

Furthermore, we assume that

$$r < r_1 := \frac{1}{2} \min \left\{ 1 - \frac{\delta}{\alpha}, \frac{1}{k_0} - \frac{\delta}{\alpha}, \frac{1}{C_1} - \frac{\delta}{\alpha}, \frac{1}{\tilde{C}_1} - \frac{\delta}{\alpha}, \frac{1}{\Lambda_1} - \frac{\delta}{\alpha} \right\}, \tag{3.2.6}$$

where δ is as in (3.2.4). Notice that r_1 depends only on the initial data α, a, k_0.

REMARK 3.2.1. *Note that by (3.2.5) and (3.2.6) we have*

$$r_0 < \bar{r}_0 := \min\left\{1, \frac{1}{k_0}, \frac{1}{C_1}, \frac{1}{\tilde{C}_1}, \frac{1}{\Lambda_1}\right\} \text{ and } C_1^{k_0, a} < 1. \tag{3.2.7}$$

We shall assume that

$$0 < r_0 < \min\{2r_1 + 1, \bar{r}_0, \frac{1}{2k_0}, \frac{1}{2C_1}\}. \tag{3.2.8}$$

Notice that r_0 depends on α, a and k_0. Next, we see that the Lipschitz-type constant k_0 depends on $D(F)$ which is part of the initial data.

The following assumption is used to prove the results in this Chapter.

ASSUMPTION 3.2.2. *(c.f Argyros and George (2015); Shubha et al. (2015); Vasin and George (2014); Semenova (2010); George and Nair (2008)) There exist two constants $0 \leq l_0$ and $l_1 < \frac{\sqrt{17}-3}{4}$ such that for every $u_1, u_2 \in D(F)$ and $v \in E$ there exists an element $\phi(u_2, u_1, v) \in E$ such that $[F'(u_2) - F'(u_1)]v = F'(u_1)\phi(u_2, u_1, v)$, $\|\phi(u_2, u_1, v)\| \leq l_0\|v\|\|u_2 - u_1\|$, $\|\frac{d}{dv}\phi(u_2 + tv, u_2, v)\| \leq l_1\|v\|$ for $t \in [0, 1]$ and $B(u_\alpha^\delta, r_0) \subseteq D(F)$.*

Let $k_0 = \max\{l_0, 2l_1\}$. Notice that $k_0 = k_0(D(F))$,i.e., k_0 depends on the initial data. Then, knowing the rest of the initial data a and α we can compute all the preceding introduced parameters.

Let,

$$R_\alpha(u_k) = F(u_k) + \alpha(u_k - u_0) - f^\delta \tag{3.2.9}$$

and $\Gamma = F'(u_\alpha^\delta) + \alpha I$. Then since $R_\alpha(u_\alpha^\delta) = F(u_\alpha^\delta) + \alpha(u_\alpha^\delta - u_0) - f^\delta = 0$, we have by Assumption 3.2.2,

$$\begin{aligned} R_\alpha(u_k) &= F(u_k) - F(u_\alpha^\delta) + \alpha(u_k - u_\alpha^\delta) \\ &= \int_0^1 F'(u_\alpha^\delta + te_k)e_k dt + \alpha e_k \\ &= [F'(u_\alpha^\delta) + \alpha I]e_k + \int_0^1 [F'(u_\alpha^\delta + te_k) - F'(u_\alpha^\delta)]e_k dt \\ &= \Gamma\{e_k + \Gamma^{-1}\int_0^1 [F'(u_\alpha^\delta + te_k) - F'(u_\alpha^\delta)]e_k dt\} \\ &= \Gamma\{e_k + \int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt\}. \end{aligned} \tag{3.2.10}$$

Differentiating (3.2.10) with respect to e_k we obtain,

$$R'_\alpha(u_k)(h) = \Gamma\left\{I + \frac{d}{de_k}\{\int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt\}\right\}(h). \tag{3.2.11}$$

Let $M_k(e_k) = \int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt$ and $\bar{M}_k = \frac{d}{de_k}M_k(e_k)$, then

$$R'_\alpha(u_k)(h) = \Gamma\{I + \bar{M}_k\}(h). \tag{3.2.12}$$

Suppose that $u_k \in B(u_\alpha^\delta, r_0)$. Then, we have

$$\begin{aligned} \|\bar{M}_k\| &= \|\int_0^1 \frac{d}{de_k}\{\Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt\}\| \\ &\leq \int_0^1 \|\Gamma^{-1}F'(u_\alpha^\delta)\|\|\frac{d}{de_k}\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)\|dt \\ &\leq 2l_1\|e_k\| \leq k_0\|e_k\| \\ &\leq k_0 r_0 < 1. \end{aligned}$$

The last inequality follows from (3.2.7) and Assumption 3.2.2. Therefore $(I+\bar{M}_k)$ is invertible and its inverse is given by

$$(I+\bar{M}_k)^{-1} = I - \bar{M}_k + \bar{M}_k^2 \cdots . \tag{3.2.13}$$

So by (3.2.12), we have

$$R'_\alpha(u_k)^{-1} = (I - \bar{M}_k + \bar{M}_k^2 \cdots)\Gamma^{-1}. \tag{3.2.14}$$

Now by replacing e_k by $\hat{e_k}$ and u_k by v_k in (3.2.11) we get

$$R'_\alpha(v_k)(h) = \Gamma \left\{ I + \frac{d}{d\hat{e_k}} \{ \int_0^1 \Gamma^{-1} F'(u_\alpha^\delta)\phi(u_\alpha^\delta + t\hat{e_k}, u_\alpha^\delta, \hat{e_k})dt \} \right\} (h). \tag{3.2.15}$$

We obtain again by (3.1.1),

$$\begin{aligned}
\hat{e_k} &= e_k - aR'_\alpha(u_k)^{-1}R_\alpha(u_k) \\
&= e_k - a\left\{ \{I - \bar{M}_k + M_k^2 \cdots\}\Gamma^{-1}\Gamma\{e_k + \int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt\} \right\} \\
&= (1-a)e_k - a\int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt + a\bar{M}_k(I - \bar{M}_k + \bar{M}_k^2 \cdots)e_k \\
&\quad + a\bar{M}_k(I - \bar{M}_k + \bar{M}_k^2 \cdots)\int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|\hat{e_k}\| &= \|(1-a)e_k - a\int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt \\
&\quad + a\bar{M}_k(I - \bar{M}_k + \bar{M}_k^2 \cdots)e_k \\
&\quad + a\bar{M}_k(I - \bar{M}_k + \bar{M}_k^2 \cdots)\int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt\| \\
&\le |1-a|\|e_k\| + ak_0\|e_k\|^2 + a\|e_k\|\frac{\|\bar{M}_k\|}{1-\|\bar{M}_k\|} + ak_o\|e_k\|^2\frac{\|\bar{M}_k\|}{1-\|\bar{M}_k\|} \\
&\le |1-a|\|e_k\| + ak_0\|e_k\|^2 + a\|e_k\|^2k_0\hat{R} + ak_o\|e_k\|^3k_0\hat{R} \\
&\le \|e_k\|\left\{ |1-a| + ak_0 + ak_0(1+k_0)\hat{R} \right\} \\
&= \|e_k\|C_1^{k_0,a}.
\end{aligned} \tag{3.2.16}$$

In the last, but one step we use the fact that $\|e_k\| \le r_0 < 1$. Therefore by (3.2.16) and (3.2.7) we get $v_k \in B(u_\alpha^\delta, r_0)$.

Let $N_k(\hat{e_k}) = \int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + t(\hat{e_k}), u_\alpha^\delta, \hat{e_k})dt$ and $\bar{N}_k = \frac{d}{d\hat{e_k}} N_k(\hat{e_k})$. Then,

$$R'_\alpha(v_k)(h) = \Gamma\{I + \bar{N}_k\}(h). \qquad (3.2.17)$$

We also have,

$$\begin{aligned}
\|\bar{N}_k\| &= \| \int_0^1 \frac{d}{d\hat{e_k}}\{\Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + t\hat{e_k}, u_\alpha^\delta, \hat{e_k})dt\}\| \\
&\leq \int_0^1 \|\Gamma^{-1}F'(u_\alpha^\delta)\|\|\frac{d}{d\hat{e_k}}\{\phi(u_\alpha^\delta + t\hat{e_k}, u_\alpha^\delta, \hat{e_k})\|dt\} \\
&\leq 2l_1\|\hat{e_k}\| \leq k_0\|\hat{e_k}\|.
\end{aligned}$$

Let $H_k = \frac{1}{a}R'_\alpha(v_k) + (1 - \frac{1}{a})R'_\alpha(u_k)$.
Then,

$$\begin{aligned}
H_k &= \Gamma\left\{\frac{1}{a}\{I + \bar{N}_k\} + (1 - \frac{1}{a})\{I + \bar{M}_k\}\right\} \qquad (3.2.18) \\
&= \Gamma\left\{I + \frac{1}{a}\bar{N}_k + (1 - \frac{1}{a})\bar{M}_k\right\} \\
&= \Gamma\left\{I + P_k\right\}
\end{aligned}$$

where $P_k = \frac{1}{a}\bar{N}_k + (1 - \frac{1}{a})\bar{M}_k$. Now,

$$\begin{aligned}
\|P_k\| &= \|\frac{1}{a}\bar{N}_k + (1 - \frac{1}{a})\bar{M}_k\| \qquad (3.2.19) \\
&\leq \frac{\|\hat{e_k}\|k_0}{a} + \frac{|a-1|}{a}\|e_k\|k_0 \\
&\leq \|e_k\|\left\{\frac{k_0C_1^{k_0,a} + |a-1|k_0}{a}\right\} \\
&< r_0\frac{k_0[C_1^{k_0,a} + |1-a|]}{a} = r_0C_1 < 1.
\end{aligned}$$

The last inequality follows from (3.2.7). This implies H_k is invertible and its inverse is given by:

$$H_k^{-1} = \{I - P_k + P_k^2 \cdots\}\Gamma^{-1}. \qquad (3.2.20)$$

From (3.1.2) we have

$$
\begin{aligned}
\bar{e_k} &= e_k - \frac{1}{2}\left\{H_k^{-1} + R_\alpha(u_k)^{-1}\right\} R_\alpha(u_k) \\
&= e_k - \frac{1}{2}\left\{\{(I - P_k + P_k^2 \cdots) + (I - \bar{M}_k + \bar{M}_k^2 \cdots)\}\Gamma^{-1}\Gamma\right. \\
&\quad \left.\times\{e_k + \int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt\}\right\} \\
&= -\int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt + \frac{1}{2}P_k(I - P_k + P_k^2 \cdots)e_k \\
&\quad + \frac{1}{2}\bar{M}_k(I - \bar{M}_k + \bar{M}_k^2 \cdots)e_k \\
&\quad + \frac{1}{2}P_k(I - P_k + P_k^2 \cdots)\int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt \\
&\quad + \frac{1}{2}\bar{M}_k(I - \bar{M}_k + \bar{M}_k^2 \cdots)\int_0^1 \Gamma^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\bar{e_k}\| &\leq \int_0^1 \|\Gamma^{-1}F'(u_\alpha^\delta)\|\|\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)\|dt + \frac{1}{2}\|e_k\|\frac{\|P_k\|}{1 - \|P_k\|} \\
&\quad + \frac{1}{2}\|e_k\|\frac{\|\bar{M}_k\|}{1 - \|\bar{M}_k\|} \\
&\quad + \frac{1}{2}\frac{\|P_k\|}{1 - \|P_k\|}\int_0^1 \|\Gamma^{-1}F'(u_\alpha^\delta)\|\|\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)\|dt \\
&\quad + \frac{1}{2}\frac{\|\bar{M}_k\|}{1 - \|\bar{M}_k\|}\int_0^1 \|\Gamma^{-1}F'(u_\alpha^\delta)\|\|\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)\|dt \\
&\leq \|e_k\|^2\left\{k_0 + \frac{1}{2}C_1\tilde{R}_1 + \frac{1}{2}k_0\hat{R}_1 + k_0\frac{1}{2}C_1\tilde{R}_1 + \frac{1}{2}k_0^2\hat{R}_1\right\} \\
&= \tilde{C}_1\|e_k\|^2.
\end{aligned} \tag{3.2.21}
$$

Therefore, by (3.2.21) and (3.2.7) we get $w_k \in B(u_\alpha^\delta, r_0)$.

Next, using the preceding notation we will prove our central result of this section.

THEOREM 3.2.3. *Let R_α be as in (3.1.4) and suppose that u_k, v_k* and *$w_k \in B(u_\alpha^\delta, r_0)$. Further let the first derivative of F exists in $B(u_\alpha^\delta, r_0)$. Then $u_{k+1} \in B(u_\alpha^\delta, r_0)$ and the iteration defined in (3.1.1)– (3.1.3) converges cubically to u_α^δ.*

Moreover

$$\|u^{\delta}_{k+1,\alpha} - u^{\delta}_{\alpha}\| = O(e^{-\gamma 3^k}),$$

where $\gamma = -ln(\|e_0\|)$.

Proof. Since, $u_0 \in B(u^{\delta}_{\alpha}, r_0)$, by (3.2.16), (3.2.21) and Remark 3.2.1, we have $v_0, w_0 \in B(u^{\delta}_{\alpha}, r_0)$. Suppose $u_k \in B(u^{\delta}_{\alpha}, r_0)$. Then by (3.2.16), (3.2.21) and Remark 3.2.1, we have $v_k, w_k \in B(u^{\delta}_{\alpha}, r_0)$. Then from (3.1.1)-(3.1.3), we have

$$\begin{aligned}
e_{k+1} &= \bar{e_k} - \{H_k\}^{-1} R_{\alpha}(w_k) \\
&= \bar{e_k} - \{I - P_k + P_k^2 \cdots\}\Gamma^{-1}\Gamma\{\bar{e_k} + \int_0^1 \Gamma^{-1} F'(u^{\delta}_{\alpha})\phi(u^{\delta}_{\alpha} + t|\bar{e_k}|, u^{\delta}_{\alpha}, \bar{e_k})dt\} \\
&= -\int_0^1 \Gamma^{-1} F'(u^{\delta}_{\alpha})\phi(u^{\delta}_{\alpha} + t|\bar{e_k}|, u^{\delta}_{\alpha}, \bar{e_k})dt + P_k(I - P_k + P_k^2 \cdots)\bar{e_k} \\
&\quad + P_k(I - P_k + P_k^2 \cdots)\int_0^1 \Gamma^{-1} F'(u^{\delta}_{\alpha})\phi(u^{\delta}_{\alpha} + t|\bar{e_k}|, u^{\delta}_{\alpha}, \bar{e_k})dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|e_{k+1}\| &\leq k_0\|\bar{e_k}\|^2 + \|\bar{e_k}\|\frac{\|P_k\|}{1 - \|P_k\|} + k_0\|\bar{e_k}\|^2\frac{\|P_k\|}{1 - \|P_k\|} \\
&\leq k_0\tilde{C}_1^2\|e_k\|^4 + \|e_k\|^3\tilde{C}_1 C_1 \bar{R} + k_0\|e_k\|^5\tilde{C}_1^2 C_1 \tilde{R}_1 \\
&\leq \|e_k\|^3\left\{C_1\tilde{C}_1\tilde{R}_1(1 + k_0\tilde{C}_1) + k_0\tilde{C}_1^2\right\} \\
&= \Lambda_1\|e_k\|^3.
\end{aligned} \tag{3.2.22}$$

Therefore by (3.2.22) and (3.2.7) we get $u_{k+1} \in B(u^{\delta}_{\alpha}, r_0)$.
Repeated application of (3.2.22) above leads to

$$\|e_{k+1}\| \leq \Lambda_1^{\frac{3^k-1}{2}}\|e_0\|^{3^k} \quad = \quad \Lambda_1^{\frac{3^k-1}{2}} e^{-\gamma 3^k}, \tag{3.2.23}$$

where $\gamma = -log\|e_0\|$.

□

3.3 Error estimates using Hölder type source condition

From Lemma 2.2.2, analogous to the proof of Theorem 2.2.4, one can prove the following theorem.

THEOREM 3.3.1. *Let Assumption 3.2.2 and (2.1.3) hold. If* $3k_0r < 1$, *then*

$$\|u_\alpha - \hat{u}\| \leq \hat{C}\alpha^\nu, \qquad \text{for } 0 < \nu \leq 1$$

where $\hat{C} = \frac{C_0}{1-3k_0r} \quad \leq \hat{C}_1 := 2C_0$

Combining Theorem 3.2.3 and Theorem 3.3.1, we have the following:

THEOREM 3.3.2. *Let* u_k *be as in (3.1.3) and let the assumptions in Theorem 3.2.3 and Theorem 3.3.1 be satisfied. Let*

$$k_\delta := \min\{k : e^{-\gamma 3^k} \leq \frac{\delta}{\alpha}\}. \tag{3.3.1}$$

Then we have the following;

$$\|u_k - \hat{u}\| \leq \bar{C}_1(\alpha^\nu + \frac{\delta}{\alpha}), \tag{3.3.2}$$

where $\bar{C}_1 = \max\{\Lambda_1^{\frac{3^k-1}{2}} + 1, \hat{C}_1\}$.

As already seen in Chapter 2, the error $\alpha^\nu + \frac{\delta}{\alpha}$ in (3.3.2) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies, $\alpha_\delta^{1+\nu} = \delta$. That is $\alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Hence by (3.3.2) we have the following Theorem.

THEOREM 3.3.3. *Let the assumptions in Theorem 3.3.2 holds. For* $\delta > 0$, *let* $\alpha := \alpha_\delta = \delta^{\frac{1}{1+\nu}}$. *Let* k_δ *be as in (3.3.1). Then*

$$\|u_k - \hat{u}\| = O(\delta^{\frac{\nu}{1+\nu}}).$$

In order to obtain the above order, without knowing ν, we use the adaptive selection of the parameter strategy considered by Pereverzev and Schock (2005), modified appropriately for the scenario in choosing the parameter α. For convenience, take $u_i := u_{k_i}$. Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$l := \max\left\{i : \alpha_i^\nu \leq \frac{\delta}{\alpha_i}\right\} < N \quad \text{and} \tag{3.3.3}$$

$$k := \max\left\{i : \|u_i - u_j\| \leq 4\bar{C}_1 \frac{\delta}{\alpha_j}, j = 0, 1, 2, \cdots, i-1\right\} \tag{3.3.4}$$

where $\bar{C}_1$ is as in Theorem 3.3.2.

THEOREM 3.3.4. *(cf. George and Nair (2008)) Assume that there exists $i \in \{0, 1, \cdots, N\}$ such that $\alpha_i^\nu \leq \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 3.3.2 be fulfilled, and let l and k be as in (3.3.3) and (3.3.4) respectively. Then $l \leq k$; and*

$$\|\hat{u} - u_k\| \leq 6\bar{C}_1 \mu \delta^{\frac{\nu}{1+\nu}}.$$

Proof. For proving $k \geq l$, it is sufficient to prove that, for all $i \in \{1, 2, \ldots N\}$, $\alpha_i^\nu \leq \frac{\delta}{\alpha_i} \implies \|u_i - u_j\| \leq 4\bar{C}\frac{\delta}{\alpha_j}$, $\forall j = 0, 1, 2, \ldots i-1$. For $j < i$, we have

$$\begin{aligned}
\| u_i - u_j \| &\leq \| u_i - \hat{u} \| + \| \hat{u} - u_j \| \\
&\leq \bar{C}_1(\alpha_i^v + \frac{\delta}{\alpha_i}) + \bar{C}_1(\alpha_j^v + \frac{\delta}{\alpha_j}) \\
&\leq 2\bar{C}_1\frac{\delta}{\alpha_i} + 2\bar{C}_1\frac{\delta}{\alpha_j} \\
&\leq 4\bar{C}_1\frac{\delta}{\alpha_j}.
\end{aligned}$$

Thus we have proved the relation $k \geq l$. Notice that

$$\| \hat{u} - u_k \| \leq \| \hat{u} - u_l \| + \| u_k - u_l \|,$$

where

$$\| \hat{u} - u_l^\delta \| \leq \bar{C}_1(\alpha_l^v + \frac{\delta}{\alpha_l}) \leq 2\bar{C}_1\frac{\delta}{\alpha_l}.$$

Now since $l \leq k$, we have

$$\| u_k - u_l \| \leq 4\bar{C}_1\frac{\delta}{\alpha_l}.$$

Hence,

$$\| \hat{u} - u_k \| \leq 6\bar{C}_1 \frac{\delta}{\alpha_l}.$$

It follows as in Theorem 2.4.2 that

$$\frac{\delta}{\alpha_l} \leq \frac{\mu\delta}{\alpha_\delta} = \mu\delta^{\frac{\nu}{1+\nu}}.$$

□

3.4 Adaptive choice rule implementation

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 3.3.4 involves the following steps:

- Select $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Set $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \cdots, N$.

3.4.1 Algorithm

a. Choose $i = 0$.

b. Set $k_i := \min \left\{ k : e^{-\gamma 3^k} \leq \frac{\delta}{\alpha_i} \right\}$.

c. Solve $u_i := u_{k_i}$ by using the iteration (3.1.1).

d. If $\|u_i - u_j\| > 4\bar{C}_1 \frac{\delta}{\alpha_j}, j < i$, then take $k = i - 1$ and return u_k.

e. Else set $i = i + 1$ and go to b.

3.5 Numerical Example

We apply the algorithm by choosing a sequence of finite dimensional subspace (V_N) of $L^2(0,1)$ with $dim\ V_N = N + 1$. Precisely, we choose V_N as the linear span of $\{v_1, v_2, v_3, ..., v_{N+1}\}$, where $v_i, i = 1, 2, ..., N + 1$ are linear splines in a uniform grid of $N + 1$ points in $[0, 1]$.

EXAMPLE 3.5.1. *(see Semenova (2010), section 4.3) Let* $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ *defined by*

$$F(u) := \int_0^1 k(t,s)u^3(s)ds,$$

where

$$k(t,s) = \begin{cases} (1-s)t, & 0 \le t \le s \le 1 \\ (1-t)s, & 0 \le s \le t \le 1 \end{cases}.$$

Then for all $u(t), v(t) : u(t) > v(t)$:

$$\langle F(u) - F(v), u - v\rangle = \int_0^1 \left[\int_0^1 k(t,s)(u^3 - v^3)(s)ds\right]$$

$$\times (u-v)(t)dt \ge 0.$$

So the operator F *is monotone. The Fréchet derivative of* F *is given by*

$$F'(u)w = 3\int_0^1 k(t,s)u^2(s)w(s)ds. \tag{3.5.1}$$

Note that for $u, v > 0$,

$$\begin{aligned}[F'(v) - F'(u)]w = 3\int_0^1 k(t,s)u^2(s)\frac{[v^2(s) - u^2(s)]w(s)ds}{u^2(s)} \\ := F'(u)\Phi(v,u,w)\end{aligned}$$

where $\Phi(v,u,w) = \frac{[v^2-u^2]w}{u^2}$. *Note that*

$$\Phi(v,u,w) = \frac{[v^2 - u^2]w}{u^2} = \frac{[u+v][v-u]w}{u^2}$$

and

$$\begin{aligned}\|\frac{d}{dw}\Phi(u+tw,u,w)\| &= \left\|\frac{d}{dw}\frac{[2tuw + t^2w^2]w}{u^2}\right\| \\ &= \left\|\frac{4tuw + 3t^2w^2}{u^2}\right\| \\ &\le \left\|\frac{4tu + 3t^2w}{u^2}\right\| \|w\|.\end{aligned}$$

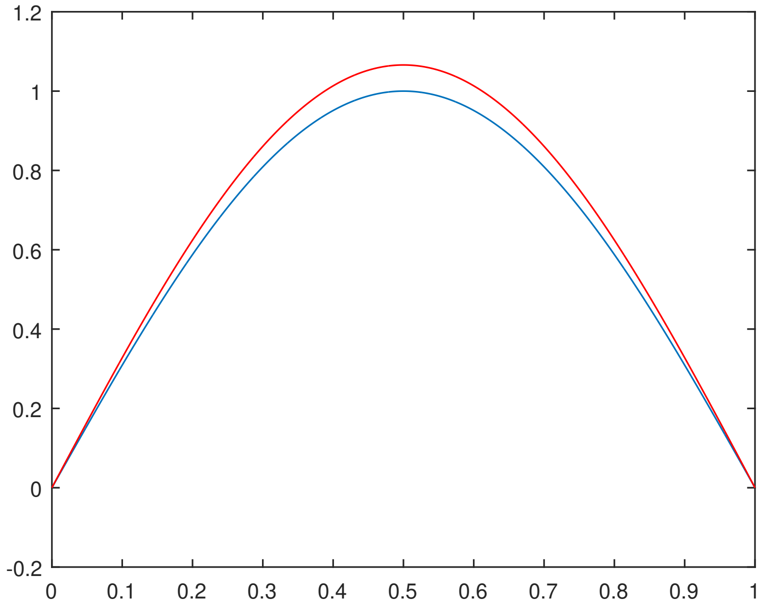


Figure 3.1: Curves of the approximate(red) and exact(blue) solutions with N=1024 and $\delta = 0.1$

So Assumption 3.2.2 satisfies with $k_0 \geq \max\{2\left\|\frac{4tu+3t^2w}{u^2}\right\|, \left\|\frac{u+v}{u^2}\right\|\}$. *In our computation, we take* $f(t) = \frac{6sin(\pi t)+sin^3(\pi t)}{9\pi^2}$ *and* $f^\delta = f + \delta$. *Then, the exact solution is* $\hat{u}(t) = sin(\pi t)$. *We use* $u_0(t) = sin(\pi t) + \frac{3[t\pi^2 - t^2\pi^2 + sin^2(\pi t)]}{4\pi^2}$ *as our initial guess, so that the function* $u_0 - \hat{u}$ *satisfies the source condition*

$$u_0 - \hat{u} = F'(\hat{u_0})\left(\frac{\hat{u}^2}{4u_0^2}\right).$$

Thus we look forward to obtain the rate of convergence $O(\delta^{\frac{1}{2}})$.

We choose $a = 1.5$, $\alpha_0 = \mu\delta$ *and* $\mu = 1.01$. *The results of the calculation are given in Table (3.1)- Table (3.4). The plots of the exact solution and the approximate solution obtained are given in figures, Figure 3.1 to Figure 3.4.*

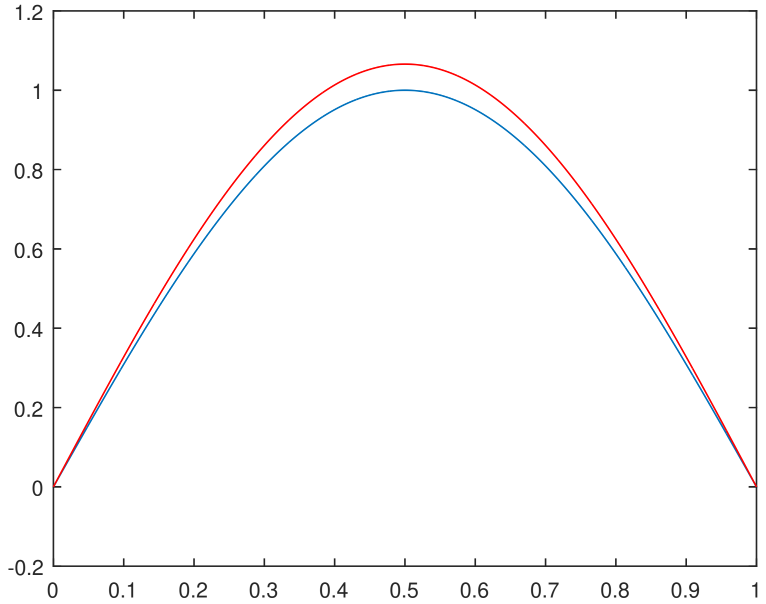


Figure 3.2: Curves of the approximate(red) and exact(blue) solutions with N=1024 and $\delta = 0.01$

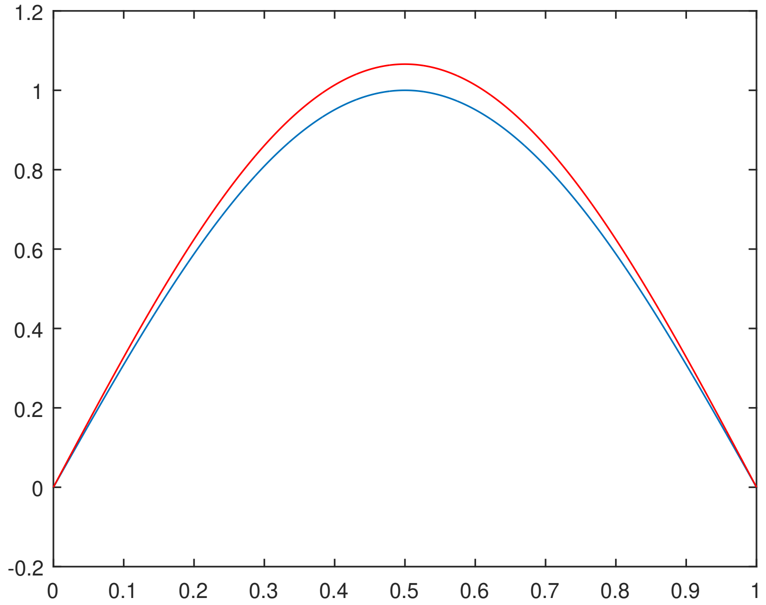


Figure 3.3: Curves of the approximate(red) and exact(blue) solutions with N=1024 and $\delta = 0.004$

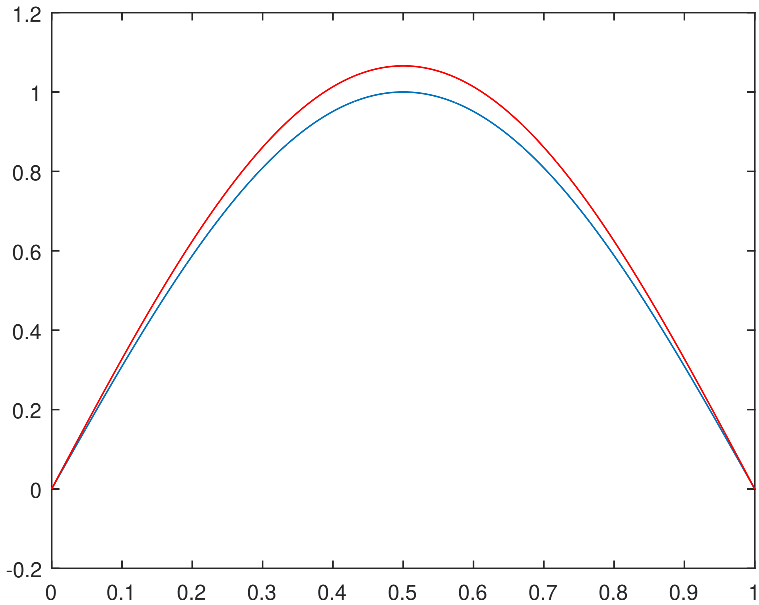


Figure 3.4: Curves of the approximate(red) and exact(blue) solutions with N=1024 and $\delta = 0.002$

Table 3.1: Iterations and corresponding error estimates for $\delta = .1$

N	k	α_k	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|u_k-\hat{u}\|\|}{\delta^{\frac{1}{2}}}$
8	30	0.1363454479	0.0503927259	0.1125936299
16	30	0.1136185917300361	0.1061109950797	0.113661932322
32	30	0.113614603466	0.1063664784762	0.1142351835898
64	30	0.11361360640	0.1064305298623	0.114378926380
128	30	0.11361335714	0.1064465606055	0.114414903744
256	30	0.113613294817	0.1064505677544	0.114423896988
512	30	0.113613279238	0.1064515697139	0.114426145690
1024	30	0.113613275343	0.1064518202046	0.114426707868

Table 3.2: Iterations and corresponding error estimates for $\delta = .01$

N	k	α_k	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|u_k-\hat{u}\|\|}{\delta^{\frac{1}{2}}}$
8	30	0.01382598145	0.05039707647	0.3536092805
16	30	0.013666450897	0.06110993160	0.43127096521
32	30	0.013626568258	0.063664794104	0.44995842369
64	30	0.0136165976	0.064305284639	0.45465153223
128	30	0.01361410493	0.064465606142	0.45582676204
256	30	0.01361348177	0.064505677538	0.45612054021
512	30	0.013613325975	0.064515697139	0.45619399928
1024	30	0.013613287027	0.064518202046	0.45621236423

Table 3.3: Iterations and corresponding error estimates for $\delta = .004$

N	k	α_k	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|u_k-\hat{u}\|\|}{\delta^{\frac{1}{2}}}$
8	30	0.00565801702480	0.050397933465	0.552773098978
16	30	0.0054984864697	0.061109929996	0.67991794194
32	30	0.0054586038309	0.063664794728	0.71092674246
64	30	0.0054486331712	0.064305283708	0.71873562117
128	30	0.0054461405062	0.06446560615	0.72069240541
256	30	0.005445517340	0.064505677538	0.72118164423
512	30	0.0054453615485	0.064515697139	0.72130398266
1024	30	0.0054453226006	0.064518202046	0.72133456791

Table 3.4: Iterations and corresponding error estimates for $\delta = .002$

N	k	α_k	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|u_k-\hat{u}\|\|}{\delta^{\frac{1}{2}}}$
8	30	0.0029353622159	0.0503989450005	0.76746201827
16	30	0.0027758316607	0.06110992901	0.95693295815
32	30	0.0027359490219	0.063664794936	1.0041801675
64	30	0.0027259783622	0.064305283398	1.0161357947
128	30	0.0027234856972	0.064465606149	1.0191352350
256	30	0.0027228625311	0.064505677538	1.0198854104
512	30	0.0027227067395	0.06451569714	1.0200730108
1024	30	0.0027226677916	0.064518202047	1.0201199129

Chapter 4

NEWTON-KANTOROVICH REGULARIZATION METHOD FOR NONLINEAR ILL-POSED EQUATIONS IN BANACH SPACES

We consider Newton-Kantorovich regularization method for implementing of the Lavrentiev regularization method. Optimal order error estimate is given. No scalar sequence (like $\{\alpha_n\}$ in Buong and Hung (2005)) is used in our study. A second order convergence is obtained through our method. Numerical example confirming the theoretical result is also given at the end of the Chapter.

4.1 Introduction

In this Chapter, we assume our space E is a real, reflexive and strictly convex Banach Space with a uniformly Gâteaux differentiable norm (Buong and Phuong (2013)).

Buong and Hung (2005), considered the modified Newton-Kantorovich iterative regularization method, defined iteratively for $n = 0, 1, 2, \ldots$ by

$$F(u_n) + (F'(u_n) + \alpha_n I)(u_{n+1} - u_n) = f, \;\; u_0 \in E,$$

with $\{\alpha_n\}$, $\alpha_n > 0$ is a sequence such that

$$\lim_{n \longrightarrow \infty} \alpha_n = 0, \;\; 1 \leq \frac{\alpha_{n-1}}{\alpha_n} \leq \rho, \;\; n = 1, 2, 3, \ldots,$$

for some constant ρ. The convergence analysis in Buong and Hung (2005), is based on assumptions on the second Gâteaux derivative of F. Buong and Hung (2005), obtained the error estimate $\|u_n - \hat{u}\| = O(\sqrt{\alpha_n})$ under the assumption $-\hat{u} \in R(F'(\hat{u}))$. In this study, we consider the Newton-Kantorovich iterative regularization method defined iteratively for $k = 0, 1, 2, \ldots$ by

$$u_{k+1} = u_k - (F'(u_k) + \alpha I)^{-1}[F(u_k) + \alpha(u_k - u_0) - f^\delta], \; u_0 \in E. \tag{4.1.1}$$

First, we prove that the sequence $\{u_k\}$ defined in (4.1.1) is quadratically convergent to u_α^δ, the unique solution of (2.1.2) (see Theorem 4.2.2).

4.1.1 Advantages

Our approach in this chapter has the following advantages ($\mathcal{A}$):

($\mathcal{A}_1$) We use assumptions only on the first Fréchet derivative of F.

($\mathcal{A}_2$) No scalar sequence (like $\{\alpha_n\}$ in Buong and Hung (2005)) is used in our study.

($\mathcal{A}_3$) We obtained the error estimate for $\|u_k - \hat{u}\|$ under the general source condition (2.1.3) for all $0 < \nu \leq 1$.

The rest of the Chapter is organized as follows. The convergence analysis of method (4.1.1) is given in Section 4.2. Error estimate using Hölder-type source condition is given in Section 4.3. Parameter choice strategy is given in Section 4.4. Implementation of the method is given in Section 4.5. The Chapter ends with some numerical examples given in Section 4.6.

4.2 Convergence analysis

For us to prove our results, it is helpful in introducing some notations and functions. Let $u_0 \in E$ and u_k be as in (4.1.1). Let

$$\sigma_k = \| u_{k+1} - u_k \|,$$

$$\eta := k_0 r_0^2 + 2r_0 + 1,$$

where

$$0 < k_0 < \frac{\sqrt{5}-1}{2} \quad \text{and} \quad r_0 \in (0, \frac{\sqrt{2(1+\sqrt{5}-2k_0)}-2}{2k_0}).$$

Define

$$h(t) := k_0^2 t^2 + k_0 t, \quad \forall t \in [0, \frac{\sqrt{5}-1}{2k_0}).$$

The following properties of h can be verified easily,

($\mathcal{P}_1$) $h(t) < 1$, for all $t \in [0, \frac{\sqrt{5}-1}{2k_0})$.

($\mathcal{P}_2$) For $s \in (0,1)$, $h(st) \leq sh(t)$.

($\mathcal{P}_3$) $s < t \Longrightarrow h(s) < h(t)$.

The following assumption is used extensively to prove our results.

ASSUMPTION 4.2.1. *(see Argyros and George (2015); Shubha et al. (2015); Vasin and George (2014); Semenova (2010); George and Nair (2008)) There exists a constant* $k_0 \geq 0$ *such that for every* $u, u_1 \in B(u_0, r_0)$ *and* $v \in E$ *there exists an element* $\phi(u, u_1, v) \in E$ *such that* $[F'(u) - F'(u_1)]v = F'(u_1)\phi(u, u_1, v)$, $\|\phi(u, u_1, v)\| \leq k_0 \|v\| \|u - u_1\|$.

Next, we shall prove that the sequence $\{u_k\}$ is well defined and remains in $B(u_0, \frac{\eta}{1-h(\eta)})$ and converges to u_α^δ as $n \to \infty$.

THEOREM 4.2.2. *Let* $\alpha \in (\delta, \delta_0]$, *for some* $\delta_0 > 0$, *Assumption 4.2.1 holds and let* $0 < k_0 < \frac{\sqrt{5}-1}{2}$ *and* $r_0 \in (0, \frac{\sqrt{2(1+\sqrt{5}-2k_0)}-2}{2k_0})$. *Then, the sequence* $\{u_k\}$ *defined by (4.1.1) is well defined and remains in* $B(u_0, \frac{\eta}{1-h(\eta)})$,

$$0 < \sigma_0 < \frac{\sqrt{5}-1}{2k_0},$$

$$\sigma_k \leq h(\sigma_{k-1})\sigma_{k-1}$$

and

$$\sigma_k \leq h(\eta)^{2^k-1}\eta.$$

Moreover, u_k converges to u_α^δ as $k \to \infty$ and

$$\|u_k - u_\alpha^\delta\| \leq \beta e^{-\gamma 2^k}, \tag{4.2.1}$$

where $\beta = \frac{\eta}{h(\eta)(1-h(\eta))}$ and $\gamma = -\log(h(\eta))$.

Proof. Note that $u_0 \in B(u_0, \frac{\eta}{1-h(\eta)})$. We have in turn that

$$\begin{aligned}
u_0 - u_1 &= (F'(u_0) + \alpha I)^{-1}[F(u_0) - f^\delta] \\
&= (F'(u_0) + \alpha I)^{-1}[F(u_0) - F(\hat{u}) - F'(u_0)(u_0 - \hat{u}) \\
&\quad + F'(u_0)(u_0 - \hat{u}) + F(\hat{u}) - f^\delta] \\
&= (F'(u_0) + \alpha I)^{-1}[\int_0^1 (F'(\hat{u} + t(u_0 - \hat{u})) - F'(u_0))(u_0 - \hat{u})dt \\
&\quad + F'(u_0)(u_0 - \hat{u}) + f - f^\delta] \\
&= (F'(u_0) + \alpha I)^{-1} \int_0^1 (F'(\hat{u} + t(u_0 - \hat{u})) - F'(u_0))(u_0 - \hat{u})dt \\
&\quad + (F'(u_0) + \alpha I)^{-1} F'(u_0)(u_0 - \hat{u}) + (F'(u_0) + \alpha I)^{-1}(f - f^\delta) \\
&= (F'(u_0) + \alpha I)^{-1} F'(u_0) \int_0^1 \phi(\hat{u} + t(u_0 - \hat{u}), u_0, (u_0 - \hat{u}))dt \\
&\quad + (F'(u_0) + \alpha I)^{-1} F'(u_0)(u_0 - \hat{u}) + (F'(u_0) + \alpha I)^{-1}(f - f^\delta).
\end{aligned}$$

Thus from (2.2.4), (2.2.5) and Assumption 4.2.1, we have

$$\begin{aligned}
\sigma_0 &= \|u_1 - u_0\| \\
&= \|(F'(u_0) + \alpha I)^{-1} F'(u_0) \int_0^1 \phi(\hat{u} + t(u_0 - \hat{u}), u_0, (u_0 - \hat{u}))dt \\
&\quad + (F'(u_0) + \alpha I)^{-1} F'(u_0)(u_0 - \hat{u}) + (F'(u_0) + \alpha I)^{-1}(f - f^\delta)\| \\
&\leq k_0 \| u_0 - \hat{u} \|^2 + 2 \| u_0 - \hat{u} \| + \frac{\delta}{\alpha} \\
&\leq k_0 r_0^2 + 2r_0 + 1 \\
&< \frac{\sqrt{5} - 1}{2k_0}.
\end{aligned}$$

That is $h(\sigma_0)$ is well defined. Suppose $\sigma_{k-1} < \frac{\sqrt{5}-1}{2k_0}$. Then, we get in turn that

$$\begin{aligned}
& u_{k+1} - u_k \\
= \; & -(F'(u_k) + \alpha I)^{-1}[F(u_k) + \alpha(u_k - u_0) - f^\delta] \\
= \; & -(F'(u_k) + \alpha I)^{-1}[F(u_k) - F(u_{k-1}) - F'(u_{k-1})(u_k - u_{k-1})] \\
= \; & -(F'(u_k) + \alpha I)^{-1} \int_0^1 (F'(u_{k-1} + t(u_k - u_{k-1})) - F'(u_{k-1}))(u_k - u_{k-1})dt \\
= \; & -(F'(u_k) + \alpha I)^{-1} F'(u_{k-1}) \int_0^1 \phi(u_{k-1} + t(u_k - u_{k-1}), u_{k-1}, u_k - u_{k-1})dt \\
= \; & -(F'(u_k) + \alpha I)^{-1} F'(u_k) \int_0^1 \phi(u_{k-1} + t(u_k - u_{k-1}), u_{k-1}, u_k - u_{k-1})dt \\
& -(F'(u_k) + \alpha I)^{-1}(F'(u_{k-1}) - F'(u_k)) \\
& \times \int_0^1 \phi(u_{k-1} + t(u_k - u_{k-1}), u_{k-1}, u_k - u_{k-1})dt \\
= \; & -(F'(u_k) + \alpha I)^{-1} F'(u_k) \int_0^1 \phi(u_{k-1} + t(u_k - u_{k-1}), u_{k-1}, u_k - u_{k-1})dt \\
& -(F'(u_k) + \alpha I)^{-1} F'(u_k) \\
& \times \int_0^1 \phi(u_{k-1}, u_k, \phi(u_{k-1} + t(u_k - u_{k-1}), u_{k-1}, u_k - u_{k-1}))dt,
\end{aligned}$$

so by (2.2.4), (2.2.5) and Assumption 4.2.1, we have

$$\begin{aligned}
\sigma_k & \leq k_0 \parallel u_k - u_{k-1} \parallel^2 + k_0^2 \parallel u_k - u_{k-1} \parallel^3 \\
& \leq h(\sigma_{k-1})\sigma_{k-1} \leq \sigma_{k-1} \qquad (4.2.2) \\
& < \frac{\sqrt{5}-1}{2k_0},
\end{aligned}$$

which shows $h(\sigma_k)$ is well defined. Therefore, by (4.2.2) and $(\mathcal{P}_2)$, we have

$$\begin{aligned}
\sigma_k & \leq h(\sigma_{k-1})\sigma_{k-1} \\
& \leq h(h(\sigma_{k-2})\sigma_{k-2})h(\sigma_{k-2})\sigma_{k-2} \\
& \leq h(\sigma_{k-2})^2 h(\sigma_{k-2})\sigma_{k-2} \\
& \vdots \\
& \leq h(\sigma_0)^{2^k-1}\sigma_0 \\
& \leq h(\eta)^{2^k-1}\eta.
\end{aligned}$$

Next, we shall prove that $\{u_k\}$ is a Cauchy sequence in $B(u_0, \frac{\eta}{1-h(\eta)})$. Observe, that

$$
\begin{aligned}
\| u_{k+m} - u_k \| &\leq \| u_{k+m} - u_{k+m-1} \| + \| u_{k+m-1} - u_{k+m-2} \| + ... + \| u_{k+1} - u_k \| \\
&\leq \sigma_{k+m-1} + \sigma_{k+m-2} + ... + \sigma_k \\
&\leq h(\eta)^{2^{k+m-1}-1}\eta + h(\eta)^{2^{k+m-2}-1}\eta + ... + h(\eta)^{2^k-1}\eta \\
&\leq h(\eta)^{2^k-1}\eta[1 + h(\eta) + h(\eta)^2 + \ldots + h(\eta)^{2^{m-1}-1}] \\
&\leq h(\eta)^{2^k-1}\frac{1-h(\eta)^{2^{m-1}}}{1-h(\eta)}\eta, \qquad (4.2.3)
\end{aligned}
$$

i.e., $\{u_k\}$ is a Cauchy sequence. Further, note that

$$
\begin{aligned}
\| u_k - u_0 \| &\leq \| u_k - u_{k-1} \| + \| u_{k-1} - u_{k-2} \| + ... + \| u_1 - u_0 \| \\
&\leq \sigma_{k-1} + \sigma_{k-2} + ... + \sigma_0 \\
&\leq h(\eta)^{2^{k-1}-1}\eta + h(\eta)^{2^{k-2}-1}\eta + ... + \eta \\
&\leq \frac{1-h(\eta)^{2^{k-1}}}{1-h(\eta)}\eta \\
&\leq \frac{1}{1-h(\eta)}\eta,
\end{aligned}
$$

so $u_k \in B(u_0, \frac{\eta}{1-h(\eta)})$, for all $k = 0, 1, 2, \ldots$ and hence $\{u_k\}$ converges. By letting $k \to \infty$ in (4.1.1), we conclude that u_k converges to u_α^δ. The estimate (4.2.1) now follows from (4.2.3). This completes the proof.

□

4.3 Error estimates using Hölder type source condition

Combining Theorem 4.2.2 and Theorem 2.2.4, we have the following:

THEOREM 4.3.1. *Let u_k be as in (4.1.1) and let the assumptions in Theorem 2.2.4 and Theorem 4.2.2 be satisfied. Let*

$$k_\delta := \min\{k : e^{-\gamma 2^k} \leq \frac{\delta}{\alpha}\}. \qquad (4.3.1)$$

Then we have the following;

$$\|u_{k_\delta} - \hat{u}\| \le \bar{C}(\alpha^\nu + \frac{\delta}{\alpha}), \tag{4.3.2}$$

where $\bar{C} = \max\{\beta + 1, C_0\}$.

4.4 Choice of the Parameter

As seen in previous Chapters, the error is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies $\alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Hence by (4.3.2) we have the following Theorem.

THEOREM 4.4.1. *Let the assumptions in Theorem 4.3.1 hold. For* $\delta > 0$, *let* $\alpha := \alpha_\delta = \delta^{\frac{1}{1+\nu}}$. *Let* n_δ *be as in (4.3.1). Then*

$$\|u_{n_\delta} - \hat{u}\| = O(\delta^{\frac{\nu}{1+\nu}}).$$

4.4.1 Adaptive Scheme and Stopping Rule

Let $u_i := u_{k_i}$,

$$l := \max\left\{i : \alpha_i^\nu \le \frac{\delta}{\alpha_i}\right\} < N \quad \text{and} \tag{4.4.1}$$

$$k := \max\left\{i : \|u_i - u_j\| \le 4\bar{C}\frac{\delta}{\alpha_j}, j = 0, 1, 2, \cdots, i-1\right\} \tag{4.4.2}$$

where $\bar{C}$ is as in Theorem 4.3.1. Now we have the succeeding Theorem.

THEOREM 4.4.2. *(cf. George and Nair (2008)) Assume that there exists* $i \in \{0, 1, \cdots, N\}$ *such that* $\alpha_i^\nu \le \frac{\delta}{\alpha_i}$. *Let assumptions of Theorem 4.3.1 be fulfilled, and let* l *and* k *be as in (4.4.1) and (4.4.2) respectively. Then* $l \le k$; *and*

$$\|\hat{u} - u_k\| \le 6\bar{C}\mu\delta^{\frac{\nu}{1+\nu}}.$$

Proof. For proving $k \ge l$, it is sufficient to prove that, for all $i \in \{1, 2, \ldots N\}$,

$\alpha_i^\nu \leq \frac{\delta}{\alpha_i} \Longrightarrow \|u_i - u_j\| \leq 4\bar{C}\frac{\delta}{\alpha_j}, \ \forall j = 0, 1, 2, \ldots i-1$. For $j < i$, we have

$$\begin{aligned}
\| u_i - u_j \| &\leq \| u_i - \hat{u} \| + \| \hat{u} - u_j \| \\
&\leq \bar{C}(\alpha_i^v + \frac{\delta}{\alpha_i}) + \bar{C}(\alpha_j^v + \frac{\delta}{\alpha_j}) \\
&\leq 2\bar{C}\frac{\delta}{\alpha_i} + 2\bar{C}\frac{\delta}{\alpha_j} \\
&\leq 4\bar{C}\frac{\delta}{\alpha_j}.
\end{aligned}$$

Thus we have proved the relation $k \geq l$. Notice that

$$\| \hat{u} - u_k \| \leq \| \hat{u} - u_l \| + \| u_k - u_l \|,$$

where

$$\| \hat{u} - u_l^\delta \| \leq \bar{C}(\alpha_l^v + \frac{\delta}{\alpha_l}) \leq 2\bar{C}\frac{\delta}{\alpha_l}.$$

Now since $l \leq k$, we have

$$\| u_k - u_l \| \ \leq \ 4\bar{C}\frac{\delta}{\alpha_l}.$$

Hence,

$$\| \hat{u} - u_k \| \leq 6\bar{C}\frac{\delta}{\alpha_l}.$$

It follows as in Theorem 2.4.2 that

$$\frac{\delta}{\alpha_l} \leq \frac{\mu\delta}{\alpha_\delta} = \mu\delta^{\frac{\nu}{1+\nu}}.$$

This completes the proof.

4.5 Adaptive choice rule implementation

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 4.4.2 involves the following steps:

- Select $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Set $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \cdots, N$.

Table 4.1: Iterations and corresponding error estimates

n	k	n_k	δ	α	$\|u_k - \hat{u}\|$	$\frac{\|u_k - \hat{u}\|}{\delta^{1/2}}$
8	2	2	0.0100494106	0.0103539178	0.0356594712	0.3557169858
16	2	2	0.0100123526	0.0103157369	0.043211246	0.4318458202
32	2	2	0.0100030882	0.0103061917	0.0450178076	0.4501085811
64	2	2	0.010000772	0.0103038054	0.0454707028	0.4546894766
128	2	2	0.010000193	0.0103032089	0.0455840673	0.4558362735
256	2	2	0.0100000483	0.0103030597	0.045612402	0.4561229197
512	30	2	0.0100000121	0.0136132905	0.0456194869	0.4561945942
1024	30	2	0.010000003	0.0136132782	0.0456212582	0.456212513

4.5.1 Algorithm

a. Set $i = 0$.

b. Choose $k_i := \min\left\{k : e^{-\gamma 2^k} \leq \frac{\delta}{\alpha_i}\right\}$.

c. Solve $u_i := u_{k_i}$ by using the iteration (4.1.1).

d. If $\|u_i - u_j\| > 4\bar{C}\frac{\delta}{\alpha_j}$, $j < i$, then take $k = i - 1$ and return u_k.

e. Else set $i = i + 1$ and go to b.

4.6 Numerical Example

EXAMPLE 4.6.1. *Returning back to Example (3.5.1), we choose* $\alpha_0 = \mu\delta$ *and* $\mu = 1.01$. *The results of the calculations are given in Table 4.1. The plots of the approximate solution and the exact solution obtained are given in below figures, Figure 4.1 to Figure 4.4.*

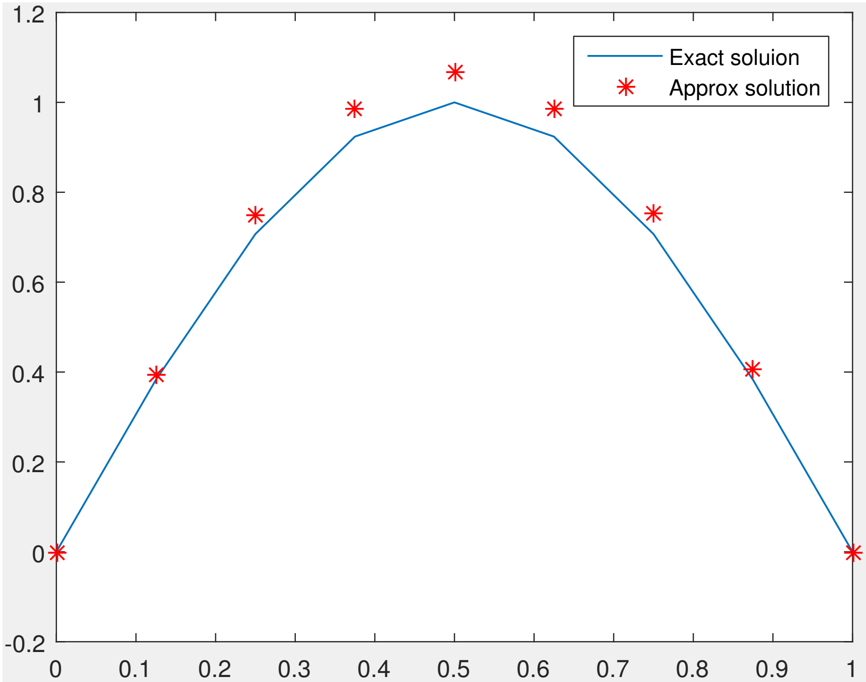


Figure 4.1: Curves of the approximate(red) and exact(blue) solutions with N=1024 and $\delta = 0.1$

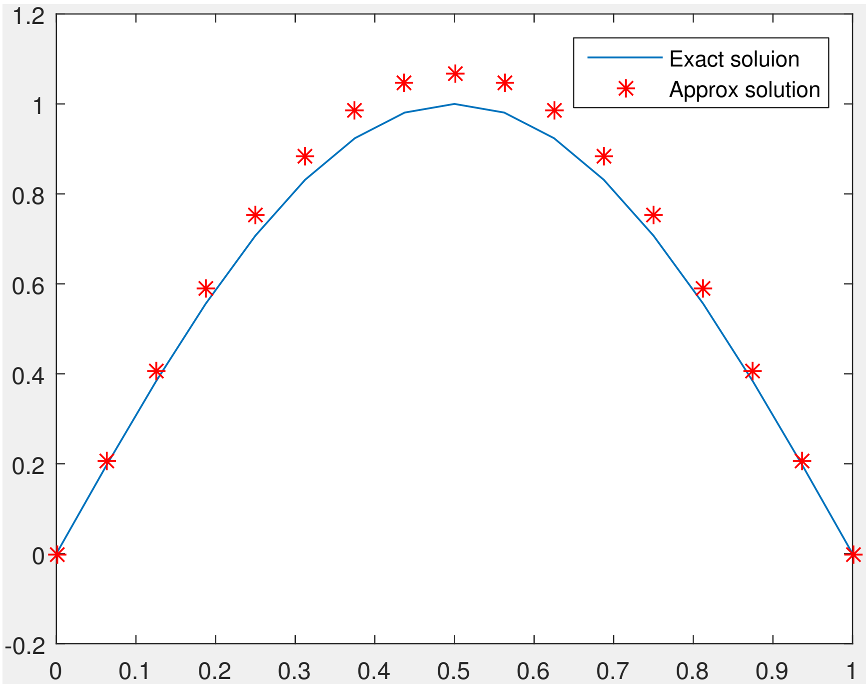


Figure 4.2: Curves of the approximate(red) and exact(blue) solutions with N=1024 and $\delta = 0.01$

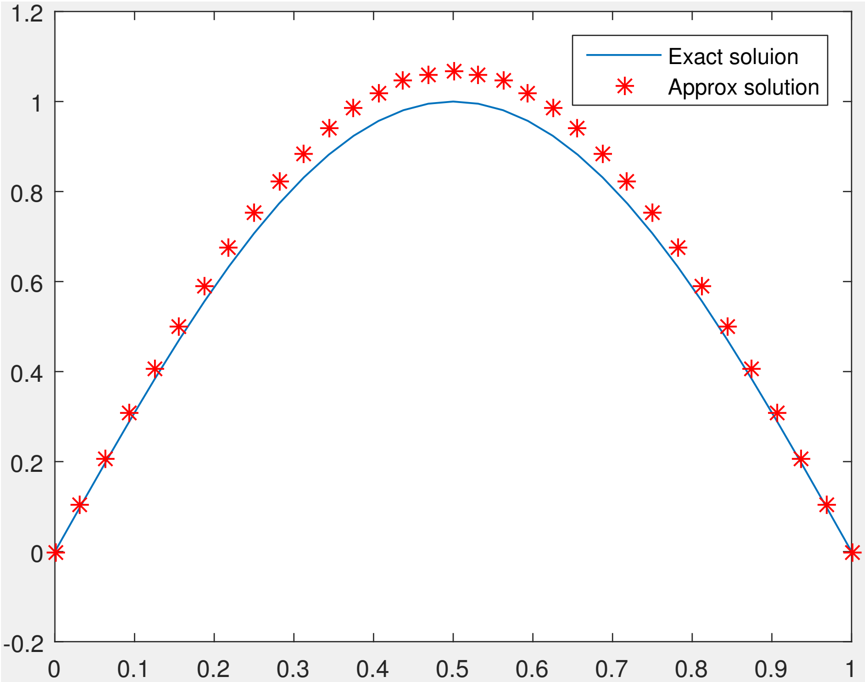


Figure 4.3: Curves of the approximate(red) and exact(blue) solutions with N=1024 and $\delta = 0.004$

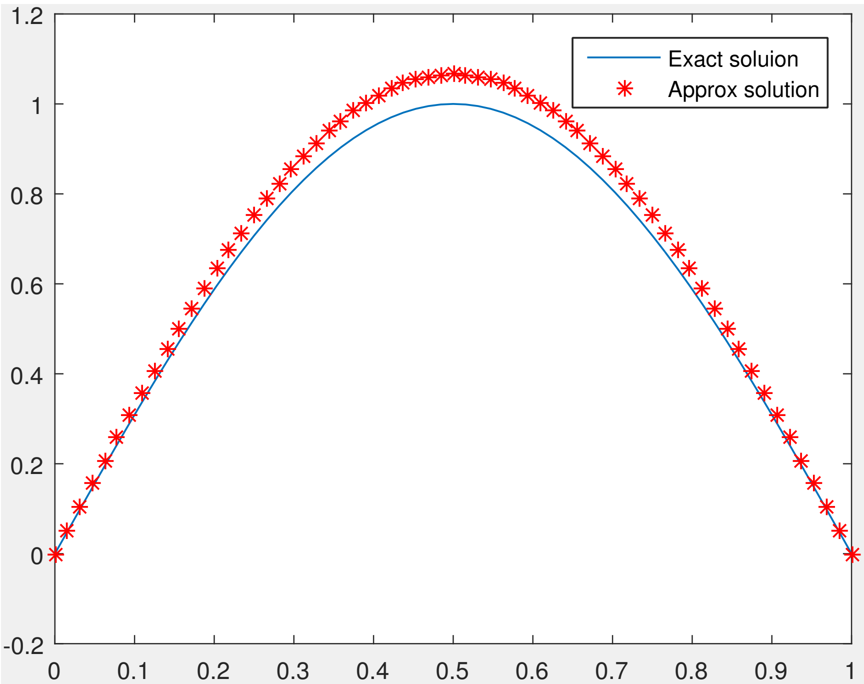


Figure 4.4: Curves of the approximate(red) and exact(blue) solutions with N=1024 and $\delta = 0.002$

Table 4.2: Table showing the number of iterations, alpha and the error for $p = 2.4$

Function ($x \in [0,1]$)	$\delta = 0.0050, \mu = 1.05$				$\delta = 0.0013, \mu = 1.05$			
	k	n_k	$\alpha(k)$	$\frac{\Vert\hat{x}-x^{\delta}_{n,\alpha_k,s}\Vert_p}{\Vert x^{\delta}_{n,\alpha_k,s}\Vert_p}$	k	n_k	$\alpha(k)$	$\frac{\Vert\hat{x}-x^{\delta}_{n,\alpha_k,s}\Vert_p}{\Vert x^{\delta}_{n,\alpha_k,s}\Vert_p}$
$\hat{x} = \min\{x, 1-x\}$	2	85	0.0527	0.2565603753	2	94	0.0204	0.0979871028
$\hat{x} = \max\{x, x-0.5\}$	2	85	0.0527	0.2565459158	2	94	0.0204	0.09799614810
$\hat{x} = x^2$ *if* $0.2 < x < 0.7$, *else* $\hat{x} = x$	2	94	0.0818	0.40659965753	2	99	0.0261	0.13108347524

EXAMPLE 4.6.2. *(see Hofmann et al. (2016)) Consider the parameter identification problem in an elliptic PDE; i.e., to find the source term q in the elliptic boundary value problem*

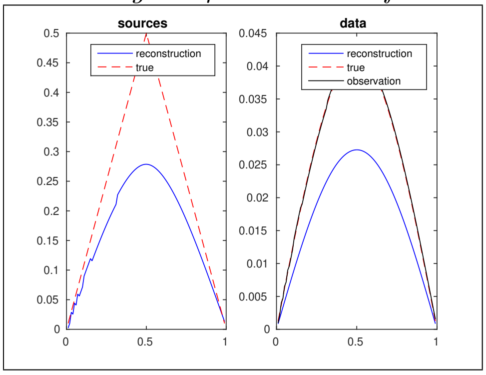
$$\begin{aligned} -\Delta u + \xi(u) &= q \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{4.6.1}$$

from measurement of u in Ω. Here $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz continuously differentiable monotonically increasing function and $\Omega \subseteq \mathbb{R}$ is a smooth domain. The corresponding forward operator in this case is $F : D(F) \subset L^p(\Omega) \longrightarrow L^p(\Omega)$, $p \geq 2$(see Kaltenbacher et al. (2009)) defined by

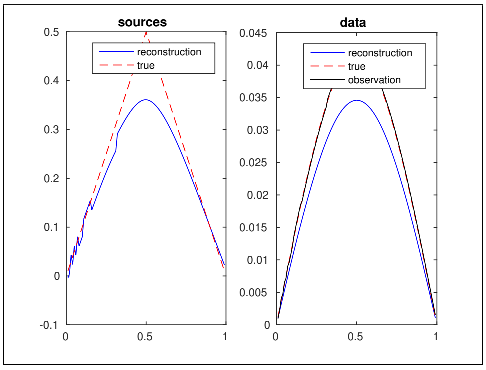
$$F(q) = u \tag{4.6.2}$$

is monotone. Table 4.2 gives the number of iterations, alpha and the relative error. The curves for the exact and approximate solutions are given in Figure 4.5.

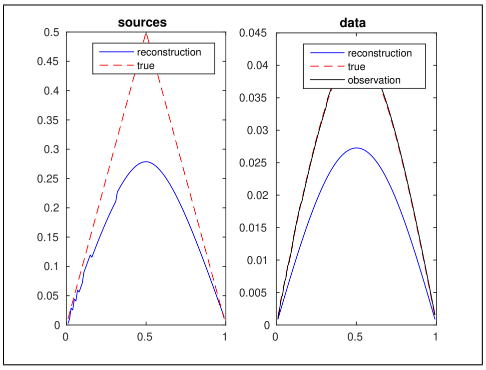
Figure 4.5: Curves of the exact and approximate solutions



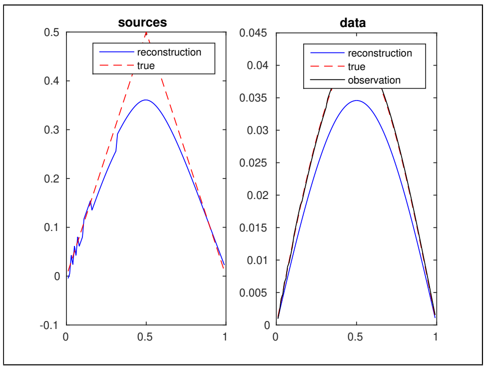
$\delta = 0.0050, \mu = 1.05$



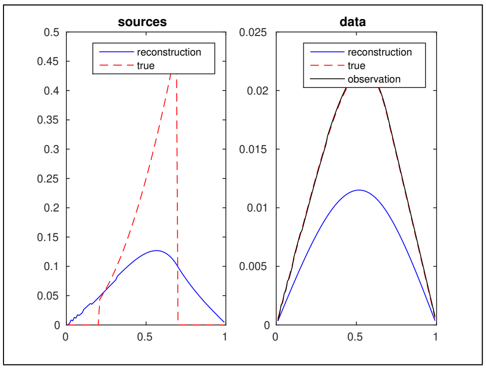
$\delta = 0.0013, \mu = 1.05$



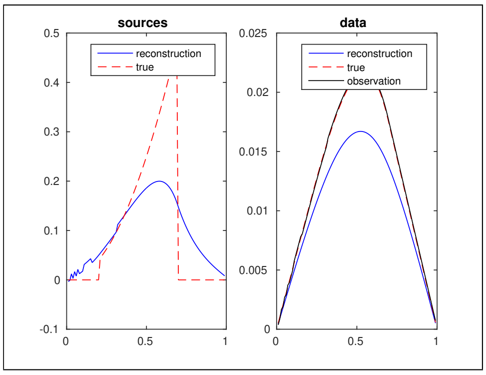
$\delta = 0.0050, \mu = 1.05$



$\delta = 0.0013, \mu = 1.05$



$\delta = 0.0050, \mu = 1.05$



$\delta = 0.0013, \mu = 1.05$

Chapter 5

SECANT-TYPE ITERATION FOR NONLINEAR ILL-POSED EQUATIONS IN BANACH SPACE

We study Secant-type iteration for nonlinear ill-posed equations in Banach spaces. We prove that the proposed iterative scheme has a convergence order at least 2.20557 using assumptions only on first Fréchet derivative of the operator. Both local and semi-local convergence is discussed and a numerical example supporting our theory is given at the end of this Chapter.

5.1 Introduction

Obtaining a closed form solution u_α^δ of (2.1.2) is difficult in general. So, most of the solution methods considered for solving (2.1.2) are iterative. The study of convergence of iterative methods is usually centered into two categories : namely semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point u_0, to obtain conditions ensuring the convergence of the iteration scheme, while the local convergence is based on the information around the solution u_α^δ to find the estimates of the computed radii of the convergence balls. In the local convergence analysis we impose conditions on u_α^δ and in semi-local convergence analysis we impose condition on u_0. In this Chapter we propose a new improved two step Secant-type method (Argyros (2008)) which approximates u_α^δ. The proposed method is defined for each $k = 0, 1, \cdots$, by

$$
\begin{aligned}
u_{k+1} &= u_k - (A_{u_k v_k} + \alpha I)^{-1} R_\alpha(u_k) \\
v_{k+1} &= u_{k+1} - (A_{u_k v_k} + \alpha I)^{-1} R_\alpha(u_{k+1})
\end{aligned} \tag{5.1.1}
$$

where

$$R_\alpha(u) := F(u) + \alpha(u - u_0) - f^\delta$$

and A_{uv} is a divided difference of order one for F or a consistent approximation for F' (Argyros (2008)). Possible choices for A_{uv} involving F' are:

$$
\begin{aligned}
A_{uv} &= \int_0^1 F'(v + t(u - v))dt, \quad \forall u, v \in D(F) \\
&\quad \text{with } u \neq v \text{ and} \\
A_{uv} &= F'(u), \text{ if } u = v.
\end{aligned}
$$

or

$$A_{uv} = \frac{1}{2}(F'(u) + F'(v)), \forall u, v \in D(F).$$

Many other choices not involving F' are also possible (Argyros (2008)). As an example, let $X = \mathbb{R}^i$ and $F = (F_1, F_2, \cdots, F_i)$ are component functions of F. Let $u = (u_1, u_2, \cdots, u_i)$ and $v = (v_1, v_2, \cdots, v_i)$, where $u_j, v_j \in \mathbb{R}$, $j = 1, 2, \cdots i$. Then, we define A_{uv} by

$$A_{uv} = \left(\frac{F_1(u_1) - F_1(v_1)}{u_1 - v_1}, \frac{F_2(u_2) - F_2(v_2)}{u_2 - v_2}, \cdots, \frac{F_i(u_i) - F_i(v_i)}{u_i - v_i} \right). \tag{5.1.2}$$

We shall prove in Section 5.3 that method (5.1.1) is of order at least 2.20557.

REMARK 5.1.1. *Advantages of our approach over other previous studies are:*

(a) A wider choice for the operator A_{uv} in (5.1.1) defined previously.

(b) We provide semi-local and local convergence analysis of the method (5.1.1).

(c) We use assumptions only on the first Fréchet derivative of F to obtain the error estimate for $\|u_k - \hat{u}\|$ under a general source condition (see $(\mathcal{A}_2)$).

The rest of the Chapter is organized as follows. In Section 5.2, we provides basic assumptions and preliminaries. Section 5.3 deals with semi-local convergence analysis of the proposed method while its local convergence analysis is given in Section 5.4. In Section 5.5 we consider adaptive choice of the parameter. In Section 5.6 the implementation of the adaptive method and the algorithm are given, a numerical example illustrating the method is given in Section 5.7.

5.2 Basic assumptions and Preliminaries

The results in this Chapter are based on the following assumptions($\mathcal{A}$):

($\mathcal{A}_0$) $A_{u_1 u_2}(u_1 - u_2) \cong F(u_1) - F(u_2)$.

($\mathcal{A}_1$) There exists $L_0 > 0$, for $x, y, u, v, z \in D = D(F)$ there exists an element $\phi(x, y, u, v, z) \in E$ such that

$$(A_{xy} - A_{uv})(z) = A_{uv}\phi(x, y, u, v, z)$$

with

$$\|\phi(x, y, u, v, z)\| \leq L_0(\|x - u\| + \|y - v\|)\|z\|.$$

($\mathcal{A}_2$) There exists $v \in E$ such that $u_0 - \hat{u} = F'(u_0)^\nu v \quad 0 < \nu \leq 1$.

($\mathcal{A}_3$) There exits a constant $\eta > 0$ such that for each $u, v \in E$ and $\alpha > 0$ the operator $(A_{uv} + \alpha I)$ is invertible,

$$\|(A_{uv} + \alpha I)^{-1} A_{uv}\| \leq \eta$$

and

$$\|(A_{uv} + \alpha I)^{-1}\| \leq \frac{1}{\alpha}.$$

Let $r \geq \|u_0 - \hat{u}\|$ and $\rho = 2r + \frac{\delta}{\alpha_0}$. Let $B(u, \lambda) = \{v \in E : \|u - v\| < \lambda\}$ and $\bar{B}(u, \lambda) = \{v \in E : \|u - v\| \leq \lambda\}$.

REMARK 5.2.1. *From (2.2.2) and (2.2.3) it is clear that* $u_\alpha^\delta \in B(u_0, \rho)$ *for all* $\alpha \geq \alpha_0$.

5.3 Semi-local convergence

We present the semi-local convergence of method (5.1.1) in this section.

Let

$$0 < \Theta < \min\left\{\frac{0.4253}{3\eta L_0}, \frac{\sqrt{1+\eta L_0}-1}{\eta L_0}\right\}, \tag{5.3.1}$$

$\alpha_0 > \frac{\delta}{\Theta}$, $\Delta_1 = \frac{0.5747-0.5747(\eta L_0)^2\Theta^2-2.1494\eta L_0\Theta}{3.1494\eta L_0+1.1494(\eta L_0)^2\Theta}$, $\Delta_2 = \frac{\sqrt{\eta^2+4\eta L_0(\Theta-\frac{\delta}{\alpha_0})}-\eta}{2\eta L_0}$ and $r < \min\{\frac{1-\eta L_0\Theta^2-2\Theta}{2(\eta L_0\Theta+1)}, \Delta_1, \Delta_2\}$.

Using the above notation, we prove the following Lemma which is used to prove the main result of this Section.

LEMMA 5.3.1. *The scalar sequences* $\{g_k\}$ *and* $\{h_k\}$ *defined for each* $k = 0, 1, \ldots$ *by*

$$\begin{aligned} g_0 = 0,\ g_1 &= \Theta,\ h_0 = 2r & (5.3.2)\\ h_{k+1} &= g_{k+1} + \eta L_0(g_{k+1} - g_k + h_k - g_k)(g_{k+1} - g_k) & (5.3.3)\\ g_{k+2} &= g_{k+1} + \frac{\eta L_0(g_{k+1} - g_k + h_k - g_k)(g_{k+1} - g_k)}{1 - \eta L_0(g_{k+1} + h_{k+1} + h_k)} \end{aligned}$$

is well defined, increasing, bounded from above by

$$g = \frac{g_1}{0.4253} \tag{5.3.4}$$

and converges to its unique least upper bound g^* *which satisfies*

$$g_1 \leq g^* \leq g. \tag{5.3.5}$$

Moreover, the following estimates hold

$$h_{k+1} - g_{k+1} \leq 0.5747(g_{k+1} - g_k) \leq 0.5747^{k+1}(g_1 - g_0), \tag{5.3.6}$$

$$g_{k+2} - g_{k+1} \leq 0.5747(g_{k+1} - g_k) \leq 0.5747^{k+1}(g_1 - g_0) \tag{5.3.7}$$

and

$$g_k \leq h_k \quad \forall k = 0, 1, \ldots. \tag{5.3.8}$$

Proof. We shall show (5.3.6)-(5.3.8) using mathematical induction. It follows from the definition of sequences $\{g_k\}$ and $\{h_k\}$ that estimates (5.3.6) and (5.3.7) are true, if

$$0 < \frac{\eta L_0(g_{k+1} - g_k + h_k - g_k)}{1 - \eta L_0(g_{k+1} + h_{k+1} + h_k)} \leq 0.5747 \tag{5.3.9}$$

and

$$0 < \eta L_0(g_{k+1} - g_k + h_k - g_k) \leq 0.5747. \tag{5.3.10}$$

Since by the definition of g_1, we have

$$\eta L_0(g_1 + h_1 + h_0) < 1,$$

so (5.3.9) implies (5.3.10) and (5.3.6)-(5.3.8) hold for $k = 0$. Suppose that (5.3.8)–(5.3.10) are true for all values $0, 1, \ldots, k$. We have by the definition of sequences $\{g_k\}$ and $\{h_k\}$ that,

$$\begin{aligned} h_{k+1} &\leq g_{k+1} + 0.5747(g_{k+1} - g_k) \leq g_{k+1} + 0.5747^{k+1}(g_1 - g_0) \quad (5.3.11)\\ &\leq g_1 + (g_1 - g_0)0.5747 + \cdots + (g_1 - g_0)0.5747^{k+1}\\ &\leq \frac{1 - 0.5747^{k+2}}{1 - 0.5747} g_1 \quad (5.3.12)\\ &\leq \frac{g_1}{1 - 0.5747} = g. \end{aligned}$$

Similarly,

$$\begin{aligned} g_{k+2} &\leq g_{k+1} + 0.5747(g_{k+1} - g_k) \leq g_{k+1} + 0.5747^{k+1}(g_1 - g_0) \quad (5.3.13)\\ &\leq g_1 + (g_1 - g_0)0.5747 + \cdots + (g_1 - g_0)0.5747^{k+1}\\ &\leq \frac{1 - 0.5747^{k+2}}{1 - 0.5747} g_1 \quad (5.3.14)\\ &\leq \frac{g_1}{1 - 0.5747} = g. \end{aligned}$$

Observe that

$$\begin{aligned} \eta L_0(g_{k+1} + h_{k+1} + h_k) &\leq 3\eta L_0 h_{k+1}\\ &\leq 3\eta L_0 \frac{g_1}{1 - 0.5747} < 1 \ \text{(by (5.3.1))}, \end{aligned}$$

so (5.3.9) implies (5.3.10). Therefore, it is enough to prove (5.3.9). Evidently, (5.3.9) is true, if

$$\frac{\eta L_0(g_{k+2} - g_{k+1} + h_{k+1} - g_{k+1})}{1 - \eta L_0(g_{k+2} + h_{k+2} + h_{k+1})} \leq 0.5747. \tag{5.3.15}$$

It follows from (5.3.11)-(5.3.14) and by induction hypothesis, (5.3.15) is true, if

$$\frac{2\eta L_0 0.5747(g_{k+1}-g_k)}{1-\Lambda} \leq 0.5747 \tag{5.3.16}$$

where $\Lambda = \eta L_0 \left(\frac{1-0.5747^{k+2}}{1-0.5747}(g_1-g_0) + \frac{1-0.5747^{k+3}}{1-0.5747}(g_1-g_0) + \frac{1-0.5747^{k+2}}{1-0.5747}(g_1-g_0)\right)$
or

$$2\eta L_0 0.5747^k(g_1-g_0) + \eta L_0\left(\frac{1-0.5747^{k+3}}{1-0.5747} + 2\frac{1-0.5747^{k+2}}{1-0.5747}\right)(g_1-g_0) - 1 \leq 0. \tag{5.3.17}$$

Inequality (5.3.17) inspires us to introduce recurrent polynomials f_k on $(0,1)$ by

$$f_k(t) = 2\eta L_0 t^k(g_1-g_0) + \eta L_0\left(\frac{1-t^{k+3}}{1-t} + 2\frac{1-t^{k+2}}{1-t}\right)(g_1-g_0) - 1. \tag{5.3.18}$$

Then, (5.3.17) is true, if

$$f_k(0.5747) \leq 0 \quad \text{for each k} = 1, 2, \ldots. \tag{5.3.19}$$

Using above definition and with the help of some algebraic manipulations we get a relationship between two consecutive polynomials f_k as

$$f_{k+1}(t) \;=\; f_k(t) + \eta L_0 t^k (g_1-g_0)(t^3+2t^2+2t-2). \tag{5.3.20}$$

We define function f_∞ on $[0,1)$ by

$$f_\infty(t) = \lim_{k\to\infty} f_k(t). \tag{5.3.21}$$

Substituting $t = 0.5747$ in above equation (using (5.3.18)), we have,

$$f_\infty(0.5747) = 3\eta L_0 \frac{(g_1-g_0)}{1-0.5747} - 1, \tag{5.3.22}$$

and by (5.3.20), we have

$$f_\infty(0.5747) = f_{k+1}(0.5747) = f_k(0.5747) \quad \text{for each k}. \tag{5.3.23}$$

So, (5.3.19) is satisfied, if

$$f_\infty(0.5747) \leq 0, \tag{5.3.24}$$

which is true by (5.3.1). Hence, we showed (5.3.9) and (5.3.10) and hence (5.3.6)-(5.3.8) are satisfied. Thus the sequences $\{g_k\}$ and $\{h_k\}$ are increasing, bounded from above by g and as such it converges to its unique least upper bound g^* which satisfies (5.3.5).

THEOREM 5.3.2. *Suppose there exists $u_0, v_0 \in \bar{B}(\hat{u}, r)$ and $(\mathcal{A}_0)$–$(\mathcal{A}_3)$ hold. Moreover hypothesis of Lemma 5.3.1 hold. Then, the sequences defined in (5.1.1) for $\alpha > \alpha_0$ is well defined and remains in $\bar{B}(u_0, g^*)$ and converges to u_α^δ. Moreover, the following estimates hold for each $k = 0, 1, \ldots,$*

$$\|u_k - u_\alpha^\delta\| \leq g^* - g_k. \tag{5.3.25}$$

Proof. We will first prove that $u_k, v_k \in \bar{B}(u_0, g^*)$ by using induction. Clearly $u_0 \in \bar{B}(u_0, g^*)$. Now since,

$$\|v_0 - u_0\| \leq \|v_0 - \hat{u}\| + \|\hat{u} - u_0\| \leq 2r = h_0 \leq g^*, \tag{5.3.26}$$

$v_0 \in \bar{B}(u_0, g^*)$. From the definition of u_k, we have,

$$\begin{aligned}
& u_1 - u_0 \\
= & -(A_{u_0v_0} + \alpha I)^{-1} R_\alpha(u_0) \\
= & -(A_{u_0v_0} + \alpha I)^{-1} \left(F(u_0) - F(\hat{u}) + f - f^\delta\right) \\
= & -(A_{u_0v_0} + \alpha I)^{-1} \left(A_{u_0\hat{u}}(u_0 - \hat{u}) - A_{u_0v_0}(u_0 - \hat{u}) + A_{u_0v_0}(u_0 - \hat{u}) + f - f^\delta\right) \\
= & -(A_{u_0v_0} + \alpha I)^{-1}(A_{u_0\hat{u}} - A_{u_0v_0})(u_0 - \hat{u}) \\
& -(A_{u_0v_0} + \alpha I)^{-1} A_{u_0v_0}(u_0 - \hat{u}) - (A_{u_0v_0} + \alpha I)^{-1}(f - f^\delta),
\end{aligned}$$

so,

$$\begin{aligned}
& \|u_1 - u_0\| \\
\leq & \|(A_{u_0v_0} + \alpha I)^{-1}(A_{u_0\hat{u}} - A_{u_0v_0})(u_0 - \hat{u})\| \\
& + \|(A_{u_0v_0} + \alpha I)^{-1} A_{u_0v_0}(u_0 - \hat{u})\| + \|(A_{u_0v_0} + \alpha I)^{-1}(f - f^\delta)\| \\
\leq & \|(A_{u_0v_0} + \alpha I)^{-1} A_{u_0v_0} \phi(u_0, \hat{u}, u_0, v_0, u_0 - \hat{u})\| \\
& + \eta\|u_0 - \hat{u}\| + \frac{\delta}{\alpha} \\
\leq & \eta L_0 r^2 + \eta r + \frac{\delta}{\alpha_0} \\
\leq & g_1 \text{ (by (5.3.1))} \leq g^*,
\end{aligned} \tag{5.3.27}$$

i.e., $u_1 \in \bar{B}(u_0, g^*)$. Again by the definition of u_k, v_k we have

$$\begin{aligned}
v_1 - u_1 &= -(A_{u_0v_0} + \alpha I)^{-1} R_\alpha(u_1) \\
&= -(A_{u_0v_0} + \alpha I)^{-1} \left(F(u_1) - F(u_0) + \alpha(u_1 - u_0) - (A_{u_0v_0} + \alpha I)(u_1 - u_0)\right) \\
&= -(A_{u_0v_0} + \alpha I)^{-1}(A_{u_1u_0} - A_{u_0v_0})(u_1 - u_0),
\end{aligned}$$

and hence,

$$
\begin{aligned}
\|v_1 - u_1\| &= \|(A_{u_0v_0} + \alpha I)^{-1}(A_{u_1u_0} - A_{u_0v_0})(u_1 - u_0)\| \\
&= \|(A_{u_0v_0} + \alpha I)^{-1}A_{u_0v_0}\phi(u_1, u_0, u_0, v_0, u_1 - u_0)\| \\
&\leq \eta\|\phi(u_1, u_0, u_0, v_0, u_1 - u_0)\| \\
&\leq \eta L_0(\|u_1 - u_0\| + \|u_0 - v_0\|)\|u_1 - u_0\| \\
&\leq \eta L_0(h_0 - g_0 + g_1 - g_0)(g_1 - g_0) \qquad (5.3.28) \\
&= h_1 - g_1.
\end{aligned}
$$

Thus,

$$
\|v_1 - u_0\| \leq \|v_1 - u_1\| + \|u_1 - u_0\| \leq h_1 - g_1 + g_1 - g_0 = h_1 \leq g^*, \qquad (5.3.29)
$$

and hence $v_1 \in \bar{B}(u_0, g^*)$. We have for $\|x\| \leq 1$,

$$
\begin{aligned}
&\|(A_{u_0v_0} + \alpha I)^{-1}((A_{u_1v_1} + \alpha I) - (A_{u_0v_0} + \alpha I))x\| \\
\leq\ &\|(A_{u_0v_0} + \alpha I)^{-1}(A_{u_1v_1} - A_{u_0v_0})x\| \\
\leq\ &\|(A_{u_0v_0} + \alpha I)^{-1}A_{u_0v_0}\phi(u_1, v_1, u_0, v_0, x)\| \\
\leq\ &\|(A_{u_0v_0} + \alpha I)^{-1}A_{u_0v_0}\|\|\phi(u_1, v_1, u_0, v_0, x)\| \\
\leq\ &\eta L_0(\|u_1 - u_0\| + \|v_1 - v_0\|)\|x\| \\
\leq\ &\eta L_0(\|u_1 - u_0\| + \|v_1 - u_0\| + \|u_0 - v_0\|) \\
\leq\ &\eta L_0(g_1 - g_0 + h_1 - g_1 + g_1 - g_0 + h_0) \\
\leq\ &\eta L_0(h_1 + g_1 + h_0) \\
<\ &1 \text{ (by (5.3.1))}.
\end{aligned}
$$

Therefore, by Banach lemma on invertible operators (Argyros (2008)), we have

$$
\|(A_{u_1v_1} + \alpha I)^{-1}(A_{u_0v_0} + \alpha I)\| \leq \frac{1}{1 - \eta L_0(h_1 + g_1 + h_0)}. \qquad (5.3.30)
$$

Also, by using (5.3.28) and (5.3.30), we have,

$$\begin{aligned}\|u_2 - u_1\| &= \|(A_{u_1v_1} + \alpha I)^{-1} R_\alpha(u_1)\| \\ &= \|(A_{u_1v_1} + \alpha I)^{-1}(A_{u_0v_0} + \alpha I)(A_{u_0v_0} + \alpha I)^{-1} R_\alpha(u_1)\| \\ &\leq \|(A_{u_1v_1} + \alpha I)^{-1}(A_{u_0v_0} + \alpha I)\| \|(A_{u_0v_0} + \alpha I)^{-1} R_\alpha(u_1)\| \\ &\leq \frac{1}{1 - \eta L_0(h_1 + g_1 + h_0)} \|v_1 - u_1\| \\ &\leq \frac{\eta L_0(h_0 - g_0 + g_1 - g_0)(g_1 - g_0)}{1 - \eta L_0(h_1 + g_1 + h_0)} = g_2 - g_1 \end{aligned} \tag{5.3.31}$$

and hence,

$$\|u_2 - u_0\| \leq \|u_2 - u_1\| + \|u_1 - u_0\| \leq g_2 - g_1 + g_1 - g_0 = g_2 \leq g^*, \tag{5.3.32}$$

i.e., $u_2 \in \bar{B}(u_0, g^*)$. The induction is completed by simply replacing u_0, v_0, u_1, v_1, u_2 by $u_k, v_k, u_{k+1}, v_{k+1}, u_{k+2}$ in the preceding estimates. Thus by induction, $u_k, v_k \in \bar{B}(u_0, g^*)$, for all $k = 0, 1, \ldots$. The sequence $\{u_k\}$ is a complete sequence in $\bar{B}(u_0, g^*)$ and converges. By letting $k \longrightarrow \infty$ in (5.3.25), we conclude that u_k converges to u_α^δ. The estimate (5.3.25) now follows by using standard majorizing techniques (Argyros (2008); Ortega and Rheinboldt (2000)).

□

5.4 Local convergence

In this Section we present a local convergence of method for (5.1.1). We assume that $\alpha_0 > 3\eta L_0 \delta$. and

$$r < \frac{1}{2}\left(\frac{1}{3\eta L_0} - \frac{\delta}{\alpha_0}\right).$$

Observe that by the above choice, we have

$$3\rho\eta L_0 < 1. \tag{5.4.1}$$

THEOREM 5.4.1. *Let conditions ($\mathcal{A}_0$)–($\mathcal{A}_3$) hold. Let $\bar{B}(u_\alpha^\delta, \rho) \subseteq D$ where ρ is as defined in Section 5.2. Suppose that there exists $u_0, v_0 \in \bar{B}(\hat{u}, r)$. Then, the sequences $\{u_k\}$ defined in (5.1.1) for $\alpha > \alpha_0$ is well defined and remains in*

$\bar{B}(u_\alpha^\delta, \rho)$ *and converges to* u_α^δ *with order of at least* 2.20557. *Moreover,*

$$\|u_{k+1} - u_\alpha^\delta\| = O(e^{-\gamma(2.20557)^k}),$$

where $\gamma = -\frac{ln(\|e_0\|)}{2.20557^3}$.

Proof. For convenience we use the notation

$$e_k = \|u_k - u_\alpha^\delta\| \text{ for each } k = 0, 1, 2\cdots$$

and

$$\hat{e}_k = \|v_k - u_\alpha^\delta\| \text{ for each } k = 0, 1, 2\cdots.$$

By the definition of ρ, (2.2.4) and (2.2.5), it is quite clear that $u_0, v_0 \in \bar{B}(u_\alpha^\delta, \rho)$. Note that,

$$\begin{aligned} u_1 - u_\alpha^\delta &= u_0 - u_\alpha^\delta - (A_{u_0v_0} + \alpha I)^{-1}(R_\alpha(u_0)) \\ &= -(A_{u_0v_0} + \alpha I)^{-1}(A_{u_0u_\alpha^\delta} - A_{u_0v_0})(u_0 - u_\alpha^\delta) \\ &= -(A_{u_0v_0} + \alpha I)^{-1}A_{u_0v_0}\phi(u_0, u_\alpha^\delta, u_0, v_0, u_0 - u_\alpha^\delta), \end{aligned}$$

so,

$$\begin{aligned} e_1 &\leq \eta L_0\|\phi(u_0, u_\alpha^\delta, u_0, v_0, u_0 - u_\alpha^\delta)\| \\ &\leq \eta L_0\|u_0 - u_\alpha^\delta\|\|v_0 - u_\alpha^\delta\| \\ &\leq \eta L_0 e_0\hat{e}_0 \leq \rho^2\eta L_0 < \rho. \end{aligned}$$

The last step follows from (5.4.1). Also, we have,

$$\begin{aligned} v_1 - u_\alpha^\delta &= u_1 - u_\alpha^\delta - (A_{u_0v_0} + \alpha I)^{-1}(R_\alpha(u_1)) \\ &= -(A_{u_0v_0} + \alpha I)^{-1}(A_{u_1u_\alpha^\delta} - A_{u_0v_0})(u_1 - u_\alpha^\delta) \\ &= -(A_{u_0v_0} + \alpha I)^{-1}A_{u_0v_0}\phi(u_1, u_\alpha^\delta, u_0, v_0, u_1 - u_\alpha^\delta) \end{aligned}$$

so,

$$\begin{aligned} \hat{e}_1 &\leq \eta L_0\|\phi(u_1, u_\alpha^\delta, u_0, v_0, u_1 - u_\alpha^\delta)\| \\ &\leq \eta L_0\|u_0 - u_\alpha^\delta\|(\|u_1 - u_0\| + \|v_0 - u_\alpha^\delta\|) \\ &\leq \eta L_0\|u_0 - u_\alpha^\delta\|(\|u_1 - u_\alpha^\delta\| + \|u_\alpha^\delta - u_0\| + \|v_0 - u_\alpha^\delta\|) \\ &\leq 3\rho^2\eta L_0 < \rho. \end{aligned}$$

Here also, the last step follows from (5.4.1). Replacing u_0, v_0, u_1, v_1 by $u_k, v_k, u_{k+1}, v_{k+1}$ in the above steps, we arrive at

$$e_{k+1} \leq \eta L_0 e_k \hat{e}_k < \rho \tag{5.4.2}$$

and

$$\hat{e}_{k+1} \leq \eta L_0 e_{k+1}(\hat{e}_k + e_{k+1} + e_k) < \rho. \tag{5.4.3}$$

Therefore, from (5.4.2) and (5.4.3) and by induction we have $u_k, v_k \in B(u_\alpha^\delta, \rho)$, for all $k = 0, 1, \ldots$. Next, we derive some inequalities which will be useful to show the order of convergence. We have by (5.4.2)

$$\begin{aligned} e_{k+1} &\leq \eta L_0 e_k \hat{e}_k \\ &\leq \eta L_0 \rho e_k \quad (5.4.4) \\ &\leq \rho, \quad (5.4.5) \end{aligned}$$

by (5.4.3),

$$\begin{aligned} \hat{e}_{k+1} &\leq \eta L_0 e_{k+1}(\hat{e}_k + e_k + e_{k+1}) \\ &\leq \eta L_0 \eta L_0 e_k \rho (3\rho) \quad \text{(from (5.4.4))} \\ &\leq 3\eta^2 L_0^2 \rho^2 e_k \quad (5.4.6) \end{aligned}$$

and

$$\begin{aligned} e_{k+2} &\leq \eta L_0 e_{k+1} \hat{e}_{k+1} \\ &\leq \eta L_0 \rho 3\eta^2 L_0^2 \rho^2 e_k \quad \text{(from (5.4.5))} \\ &\leq 3\eta^3 L_0^3 \rho^3 e_k. \quad (5.4.7) \end{aligned}$$

So by (5.4.4), (5.4.6) and (5.4.7), we have

$$\begin{aligned} \hat{e}_{k+2} &\leq \eta L_0 e_{k+2}(\hat{e}_{k+1} + e_{k+1} + e_{k+2}) \\ &\leq \eta L_0 e_{k+2}(\eta L_0 e_k \rho + 3\eta^3 L_0^3 \rho^3 e_k + 3\eta^2 L_0^2 \rho^2 e_k) \\ &\leq e_{k+2} e_k (\eta L_0 \rho + 3\eta^3 L_0^3 \rho^3 + 3\eta^2 L_0^2 \rho^2), \quad (5.4.8) \end{aligned}$$

and hence by (5.4.8), we have

$$\begin{aligned} e_{k+3} &\leq \eta L_0 e_{k+2} \hat{e}_{k+2} \\ &\leq \eta L_0 e_{k+2}^2 e_k (3\eta^3 L_0^2 \rho^2 + 3\eta^4 L_0^4 \rho^3 + \eta^2 L_0^2 \rho) \\ &\leq C_\rho e_{k+2}^2 e_k, \quad (5.4.9) \end{aligned}$$

where $C_\rho = \eta L_0(3\eta^3 L_0^2\rho^2 + 3\eta^4 L_0^4\rho^3 + \eta^2 L_0^2\rho)$. Let $\Gamma_k := \sqrt{C_\rho} e_k$. Then, by (5.4.9), we have

$$\Gamma_{k+3} \leq \Gamma_{k+2}^2 \Gamma_k, \tag{5.4.10}$$

for all $k = 0, 1, 2, \ldots$. Next we shall prove by induction that the following inequality holds

$$\Gamma_k \leq \Gamma_0^{F_k} \tag{5.4.11}$$

for all $k \geq 0$, where F_k is the generalized Fibonacci sequence defined recursively by $F_0 = F_1 = F_2 = 1$ and

$$F_{k+3} = 2F_{k+2} + F_k \quad \forall k \geq 0.$$

It is obvious that (5.4.11) holds for $k = 0, 1, 2$. Assume that (5.4.11) holds for $k = 0, 1, \cdots, n+2$, for some integer n. By induction assumptions and (5.4.10) we have,

$$\Gamma_{n+3} \leq \Gamma_{n+2}^2 \Gamma_n \leq \Gamma_0^{2F_{n+2}} \Gamma_0^{F_n} = \Gamma_0^{F_{n+3}} \tag{5.4.12}$$

which means that the inequality (5.4.11) hold for $k = n+3$. Thus by induction, the inequality (5.4.11) hold for all $k \geq 0$.

Next, we shall show that F_k is bounded below for all $k \geq 0$. First, we shall prove that

$$F_k \geq (2+x)^{k-3} \tag{5.4.13}$$

for some $x > 0$. Evidently for $k = 0, 1, 2$, (5.4.13) is true. Assume that (5.4.13) is true for $k = 0, 1, \cdots, n$. Now consider

$$\begin{aligned} F_{n+1} &= 2F_n + F_{n-2} \\ &= 2(2+x)^{n-3} + (2+x)^{n-5} \\ &\geq (2+x)^{n-5}(2+x)^3 \\ &= (2+x)^{n-2} \end{aligned}$$

provided

$$2(2+x)^2 + 1 \geq (2+x)^3. \tag{5.4.14}$$

Notice that (5.4.14) is true if $s(x) = x^3 + 4x^2 + 4x - 1 = (2+x)^3 - 2(2+x)^2 - 1 \leq 0$. Clearly $s(0.20557) \leq 0$. Hence the inequality (5.4.13) hold with $x = 0.20557$.

Therefore from (5.4.11) and (5.4.13) we have

$$e_k \leq Ce^{-\gamma(2.20557)^k}, \tag{5.4.15}$$

where $C = C_\rho^{\frac{(2.20557)^{k-3}-1}{2}}$. This completes the proof of the Theorem.

□

Combining (2.2.2), Theorem 2.2.4 and Theorem 5.4.1, we have the following:

THEOREM 5.4.2. *Let u_k be as in (5.1.1) and let the assumptions in Theorem 2.2.4 and Theorem 5.4.1 be satisfied. Let*

$$k_\delta := \min\{k : e^{-\gamma(2.20557)^k} \leq \frac{\delta}{\alpha}\}. \tag{5.4.16}$$

Then we have the following;

$$\|u_k - \hat{u}\| \leq \bar{C}_1(\alpha^\nu + \frac{\delta}{\alpha}), \tag{5.4.17}$$

where $\bar{C}_1 = \max\{C+1, C_0\}$.

□

5.5 Adaptive choice of the parameter

As detailed in previous Chapters, the error is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies $\alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Hence by (5.4.17) we have the following Theorem.

THEOREM 5.5.1. *Let the assumptions in Theorem 5.4.2 holds. For $\delta > 0$, let $\alpha := \alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Let k_δ be as in (5.4.16). Then*

$$\|u_k - \hat{u}\| = O(\delta^{\frac{\nu}{1+\nu}}).$$

□

In order to obtain the above order, without knowing ν, we use the adaptive selection of the parameter strategy considered by Pereverzev and Schock (2005) (see also George and Nair (2008); Semenova (2010)), modified appropriately for the situation for choosing the parameter α. For convenience, take $u_i := u_{k_i}$. Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$n_i = \min\{k : e^{-\gamma(2.20557)^k} \leq \frac{\delta}{\alpha_i}\} \tag{5.5.1}$$

$$l := \max\left\{i : \alpha_i^\nu \leq \frac{\delta}{\alpha_i}\right\} < N \text{ and} \tag{5.5.2}$$

$$k := \max\left\{i : \|u_i - u_j\| \leq 4\bar{C}_1 \frac{\delta}{\alpha_j}, j = 0, 1, 2, \cdots, i-1\right\} \tag{5.5.3}$$

where $\bar{C}_1$ is as in Theorem 5.4.2. Now we have the following Theorem.

THEOREM 5.5.2. *(cf. George and Nair (2008)) Assume that there exists* $i \in \{0, 1, \cdots, N\}$ *such that* $\alpha_i^\nu \leq \frac{\delta}{\alpha_i}$. *Let assumptions of Theorem 5.4.2 be fulfilled, and let* l *and* k *be as in (5.5.2) and (5.5.3) respectively. Then* $l \leq k$; *and*

$$\|\hat{u} - u_{n_l}\| \leq 6\bar{C}_1 \mu \delta^{\frac{\nu}{1+\nu}}.$$

The proof of the above theorem is analogous to the proof of Theorem 3.3.4.

5.6 Adaptive choice rule implementation

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 5.5.2 involves the following steps:

- Select $\alpha_0 > 0$ such that $3\eta L_0 \delta < \alpha_0$ and $\mu > 1$.
- Set $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \cdots, N$.

5.6.1 Algorithm

a. Set $i = 0$.

b. Choose $k_i := \min\left\{k : e^{-\gamma(2.20557)^k} \leq \frac{\delta}{\alpha_i}\right\}$.

c. Solve $u_i := u_{k_i}$ by using the iteration (5.1.1).

d. If $\|u_i - u_j\| > 4\bar{C}_1 \frac{\delta}{\alpha_j}, j < i$, then take $k = i - 1$ and return u_k.

e. Else set $i = i + 1$ and go to b.

Table 5.1: The relative error and residual error

δ	α	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|F(u_k)-f^\delta\|\|}{\|\|f^\delta\|\|}$
0.01	*0.010303010000000*	*0.102996274871587*	*0.989101691267989*
0.001	*0.001030301000000*	*0.118619906504308*	*0.987872641334813*
0.005	*0.005151505000000*	*0.089419446715342*	*0.988451475811924*

5.7 Numerical Example

In this section we present the numerical Example 2.6 which was discussed in Chapter 1. We have taken $A_{uv} = \frac{F(u)-F(v)}{u-v}$, $u \neq v$.

EXAMPLE 5.7.1. *We consider the equation (2.6.1) in Example 2.6, for the implementation of method (5.1.1) with* $\alpha_0 = \mu\delta$ *and* $\mu = 1.01$. *We use the Gauss-Legendre quadrature formula:* $\int_0^1 f(t)dt \approx \sum_{j=1}^n w_j f(t_j)$, *with the same abscissa* t_j *and the weight* w_j *for* $n = 25$ *given Table 2.1, to discretize equation (2.6.1). The discretized form of (5.1.1) is as follows:*

$$\begin{aligned} u_{k+1}(t_i) &= u_k(t_i) - \frac{u_k(t_i) - v_k(t_i)}{F(u_k(t_i)) - F(v_k(t_i)) + \alpha(u_k(t_i) - v_k(t_i))} R_\alpha(u_k(t_i)) \\ v_{k+1}(t_i) &= u_{k+1}(t_i) - \frac{u_k(t_i) - v_k(t_i)}{F(u_k(t_i)) - F(v_k(t_i)) + \alpha(u_k(t_i) - v_k(t_i))} R_\alpha(u_{k+1}(t_i)), \end{aligned}$$

where $F(u(t_i)) = \sum_{j=1}^{25} a_{ij} u(t_j)^3$, *and* $R_\alpha(u(t_i)) = F(u(t_i)) + \alpha(u(t_i) - u_0(t_0)) - (f(t_i) + \delta)$ *with* $a_{ij} = \begin{cases} w_j t_j(1-t_i), & \text{if } j \leq i \\ w_j t_i(1-t_j), & \text{if } i < j. \end{cases}$

We use,

$$u_0(t) = sin(\pi t) + \frac{3[t\pi^2 - t^2\pi^2 + sin^2(\pi t)]}{4\pi^2}, \quad v_0(t) = u_0(t) + 0.5$$

as our initial guess, so that the function $u_0 - \hat{u}$ *satisfies the source condition. The relative error* $\frac{||u_k-\hat{u}||}{||\hat{u}||}$ *and the residual error* $\frac{||F(u_k)-f^\delta||}{||f^\delta||}$ *are given in Table 5.1.*

Chapter 6

CONCLUSIONS AND FUTURE SCOPES

6.1 CONCLUDING REMARKS

We have mainly concentrated our work on solving nonlinear ill-posed problems involving m-accretive operators in a Banach space setting. We have tried using various iterative schemes. Throughout the work we considered a general Hölder type source condition and for the adaptive parameter choice strategy we considered Pereverzev and Schock (2005) for choosing the regularization parameter.

In Chapter 2, we considered a derivative and inverse free iterative method for the implementation of regularization solution. We were able to obtain a second order convergence and we have illustrated our results with a numerical example at the end of the Chapter.

We studied an iterative scheme that converges cubically to our solution in Chapter 3. The method is also a derivative and inverse free and we have given a numerical example for validation of our results.

Newton-Kantorovich regularization method is investigated in Chapter 4. We obtained a second order convergence without using any scalar sequences. To illustrate our results we have provided a numerical example at the end of the Chapter.

We analyzed Secant-type iteration in Chapter 5 and proved that the method has a convergence order at least 2.20557 using assumptions only on first Fréchet derivative of the operator. We have provided both local and semi-local convergence for the method. Some numerical results were also included at the end of Chapter.

6.2 FUTURE SCOPE OF THE RESEARCH

Various iteration techniques have been highlighted as a part of this thesis. There are many other iterative techniques available for approximately solving nonlinear equation

$$F(x) = 0,$$

in Euclidean as well as Banach space setting. Modifying these methods for ill-posed problem is a challenging task. In future, we intend to study these existing methods, modified suitably for solving ill-posed problem (2.1.1). Further we intend to propose and study new methods for solving ill-posed equations.

Study of ill-posed problems in Banach scale is another area of interest. It is also one of our future goal.

Investigating the parameter choice strategy can further be improved which is one of the most crucial requirements in most of the real-time applications.

References

Alber, Y. and Ryazantseva, I. (2006). *Nonlinear ill-posed problems of monotone type*. Springer, Dordrecht.

Argyros, I. K. (2008). *"Convergence and applications of Newton-type iterations"*. Springer, New York.

Argyros, I. K., Cho, Y. J., and George, S. (2013). Expanding the applicability of lavrentiev regularization methods for ill-posed problems. *Boundary Value Problems*, 2013(1):114.

Argyros, I. K. and George, S. (2015). Iterative regularization methods for nonlinear ill-posed operator equations with m-accretive mappings in banach spaces. *Acta Mathematica Scientia*, 35(6):1318 – 1324.

Argyros, I. K., George, S., and Jidesh, P. (2014). "Inverse free iterative methods for nonlinear ill-posed operator equations". *Int. J. Math. Math. Sci.*, pages Art. ID 754154, 8.

Bakushinskii, A. B. (1992). The problem of the convergence of the iteratively regularized gauss-newton method. *Comput. Math. Math. Phys.*, 32(9):1353–1359.

Buong, N. (2003). Convergence rates in regularization under arbitrarily perturbative operators. *Zh. Vychisl. Mat. Mat. Fiz.*, 43(3):323–327.

Buong, N. (2004). Convergence rates in regularization for nonlinear ill-posed equations under accretive perturbations. *Zh. Vychisl. Mat. Mat. Fiz.*, 44(3):397–402.

Buong, N. and Hung, V. Q. (2005). Newton-Kantorovich iterative regularization for nonlinear ill-posed equations involving accretive operators. *Ukraïn. Mat. Zh.*, 57(2):271–276.

Buong, N. and Phuong, N. T. H. (2012). Convergence rates in regularization for nonlinear ill-posed equations involving m-accretive mappings in Banach spaces. *Appl. Math. Sci. (Ruse)*, 6(61-64):3109–3117.

Buong, N. and Phuong, N. T. H. (2013). Regularization methods for nonlinear ill-posed equations involving m-accretive mappings in Banach spaces. *Russian Math. (Iz. VUZ)*, 57(2):58–64.

Engl, H. W., Hanke, M., and Neubauer, A. (1996). *"Regularization of inverse problems"*, volume 375 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht.

George, S. and Nair, M. T. (1993). An a posteriori parameter choice for simplified regularization of ill-posed problems. *Integral Equations and Operator Theory*, 16(3):392–399.

George, S. and Nair, M. T. (2008). A modified newtonlavrentiev regularization for nonlinear ill-posed hammerstein-type operator equations. *Journal of Complexity*, 24(2):228 – 240.

Groetsch, C. W. (1984). *The theory of Tikhonov regularization for Fredholm equations of the first kind*, volume 105 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA.

Groetsch, C. W. (1993). *Inverse problems in the mathematical sciences*. Vieweg Mathematics for Scientists and Engineers. Friedr. Vieweg & Sohn, Braunschweig.

Groetsch, C. W. and Guacaneme, J. (1987). Arcangeli's method for fredholm equations of the first kind. *Proceedings of the American Mathematical Society*, 99(2):256–260.

Guacaneme, J. E. (1990). A parameter choice for simplified regularization. *Rostock. Math. Kolloq.*, (42):59–68.

Hadamard, J. (1953). *"Lectures on Cauchy's problem in linear partial differential equations"*. Dover Publications, New York.

Hanke, M. (1997a). A regularizing levenberg - marquardt scheme, with applications to inverse groundwater filtration problems. *Inverse Problems*, 13(1):79.

Hanke, M. (1997b). “Regularizing properties of a truncated Newton-CG algorithm for nonlinear inverse problems”. *Numer. Funct. Anal. Optim.*, 18(9-10):971–993.

Hanke, M., Neubauer, A., and Scherzer, O. (1995). A convergence analysis of the landweber iteration fornonlinear ill-posed problems. *Numerische Mathematik*, 72(1):21–37.

Hofmann, B., Kaltenbacher, B., and Resmerita, E. (2016). “Lavrentiev's regularization method in Hilbert spaces revisited. *Inverse Probl. Imaging*, 10(3):741–764.

Ji'an, W., Jing, L., and Zhenhai, L. (2008). Regularization methods for nonlinear ill-posed problems with accretive operators. *Acta Mathematica Scientia*, 28(1):141 – 150.

Kaltenbacher, B., Schöpfer, F., and Schuster, T. (2009). Iterative methods for nonlinear ill-posed problems in Banach spaces: convergence and applications to parameter identification problems. *Inverse Problems*, 25(6):065003, 19.

Keller, J. B. (1976). Inverse problems. *The American Mathematical Monthly*, 83(2):107–118.

Kelley, C. T. (1995). *Iterative methods for linear and nonlinear equations*, volume 16 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. With separately available software.

Krasnosel'skii, M. A., Zabreyko, P. P., Pustylnik, E. I., and Sobolevski, P. E. (1976). *“Integral operators in spaces of summable functions”*. Noordhoff International Publishing, Leiden. Translated from the Russian by T. Ando, Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis.

Liu, Z. (2005). Browdertikhonov regularization of non-coercive evolution hemivariational inequalities. *Inverse Problems*, 21(1):13–20.

Mahale, P. and Nair, M. T. (2007). General source conditions for nonlinear ill-posed equations. 28:111–126.

Nair, M. T. (2009). *Linear operator equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ. Approximation and regularization.

Nashed, M. Z. and Rall, L. B. (1976). Annotated bibliography on generalized inverses and applications. pages 771–1041. Univ. Wisconsin Math. Res. Center Publ., No. 32.

Ortega, J. M. and Rheinboldt, W. C. (2000). *Iterative solution of nonlinear equations in several variables*, volume 30 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. Reprint of the 1970 original.

Pereverzev, S. and Schock, E. (2005). "On the adaptive selection of the parameter in regularization of ill-posed problems". *SIAM J. Numer. Anal.*, 43(5):2060–2076.

Semenova, E. V. (2010). Lavrentiev regularization and balancing principle for solving ill-posed problems with monotone operators. *Comput. Methods Appl. Math.*, 10(4):444–454.

Shubha, V. S., George, S., and Jidesh, P. (2015). A derivative free iterative method for the implementation of lavrentiev regularization method for ill-posed equations. *Numerical Algorithms*, 68(2):289–304.

Tautenhahn, U. (1996). "Error estimates for regularization methods in Hilbert scales". *SIAM J. Numer. Anal.*, 33(6):2120–2130.

Tautenhahn, U. (2002). "On the method of Lavrentiev regularization for nonlinear ill-posed problems". *Inverse Problems*, 18(1):191–207.

Tautenhahn, U. (2004). "Lavrentiev regularization for nonlinear ill-posed problems". *Vietnam J. Math.*, 32:29–41.

Tautenhahn, U. and Jin, Q.-n. (2003). "Tikhonov regularization and a posteriori rules for solving nonlinear ill posed problems". *Inverse Problems*, 19(1):1–21.

Vasin, V. and George, S. (2014). An analysis of lavrentiev regularization method and newton type process for nonlinear ill-posed problems. *Applied Mathematics and Computation*, 230:406 – 413.

Xiao, X. and Yin, H. (2016). Increasing the order of convergence for iterative methods to solve nonlinear systems. *Calcolo*, 53(3):285–300.

Xiao, X. and Yin, H. (2017). Achieving higher order of convergence for solving systems of nonlinear equations. *Applied Mathematics and Computation*, 311(C):251–261.

PUBLICATIONS

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2. C D Sreedeep, Santhosh George & Ioannis . K Argyros. Extended Newton-type iteration for nonlinear ill-posed equations in Banach space. Journal of Applied Mathematics and Computing, DOI= 10.1007/s12190-018-01221-2.

3. C D Sreedeep, Santhosh George & Ioannis . K Argyros. Newton-Kantorovich regularization method for nonlinear ill-posed equations involving $m-$accretive operators in Banach spaces. Rendiconti del Circolo Matematico di Palermo Series 2, DOI=10.1007/s12215-019-00413-4.

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