Journal of Computational and Applied Mathematics 347 (2019) 95–107

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Unified convergence analysis of frozen Newton-like methods under generalized conditions

Ioannis K. Argyros [a], Santhosh George [b,*]

[a] Department of Mathematical Sciences, Cameron University, Lawton, OK, 73505, USA
[b] Department of Mathematical and Computational Sciences, NIT, Karnataka, 575025, India

ARTICLE INFO

Article history:
Received 12 September 2017
Received in revised form 19 June 2018

MSC:
49D15
65G19
47H17
65H10

Keywords:
Banach space
Frozen Newton-like method
Local and semi-local convergence
Generalized Lipschitz conditions

ABSTRACT

The objective in this article is to present a unified convergence analysis of frozen Newton-like methods under generalized Lipschitz-type conditions for Banach space valued operators. We also use our new idea of restricted convergence domains, where we find a more precise location, where the iterates lie leading to at least as tight majorizing functions. Consequently, the new convergence criteria are weaker than in earlier works resulting to the expansion of the applicability of these methods. The conditions do not necessarily imply the differentiability of the operator involved. This way our method is suitable for solving equations and systems of equations. Numerical examples complete the presentation of this article.

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $\mathcal{D}$ be a subset of $\mathcal{X}$. We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. Let $F : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathcal{Y}$ be a continuous nonlinear operator. Consider the equation

$$F(x) = 0. \tag{1.1}$$

The task of obtaining a locally unique solution p of equation $F(x) = 0$ is very important. Indeed, using mathematical modeling [1–3] numerous problems in optimization, control theory, inverse theory, Mathematical physics, Chemistry, Biology, Economics and also in Engineering, can be made to look like equation $F(x) = 0$ defined on suitable abstract spaces. It is desirable to find p in closed form. However, this task can be achieved only in special cases. That explains why most researchers resort to iterative methods which generate a sequence converging to p under certain conditions.

It is well known [2] that under suitable conditions, Newton's method defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{1.2}$$

where $x_0 \in \mathcal{D}$ is an initial point provides a quadratically convergent iteration $\{x_n\}$ for solving Eq. (1.1). But, there is a plethora of problems that for some reasons, Newton's method cannot apply in its original form. A case of interest occurs when the derivative is not continuously invertible, as for instance, when dealing with small divisors [4]. That is why numerous

* Corresponding author.
E-mail addresses: iargyros@cameron.edu (I.K. Argyros), sgeorge@nitk.ac.in (S. George).

https://doi.org/10.1016/j.cam.2018.08.010

authors have proposed variants of Newton's method which converge under Lipschitz-type conditions provided that certain Kantorovich-type criteria are satisfied.

In this article, we study the unifying class of frozen Newton-like methods defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - L(y_{t_n})^{-1}F(x_n), \tag{1.3}$$

where x_0 is an initial point, $L(.) : \mathcal{D} \longrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$, t_n is a nondecreasing sequence of integers satisfying the conditions

$$t_0 = 0, \;\; t_n \le n \text{ for each } n = 0, 1, 2, \ldots \tag{1.4}$$

and y_{t_n} is the highest indexed point $x_0, x_1, \ldots, x_{t_n}$ for which $L(y_{t_n})^{-1}$ exists. Suppose that $L(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. It is well known that from the numerical efficiency point of view, it is not advantageous to change the operator $L(y_{t_n})^{-1}$ at each step of the iterative method. We obtain more efficient iterative methods, if we keep this operator piece wise constant. According to the dimension of the space optimal methods can be obtained [3]. Many popular methods can be obtained from method (1.3) with an appropriate choice of the sequence $\{t_n\}$ for each $n = 0, 1, 2 \ldots$:

SINGLE POINT METHODS:

Newton's method (1.2):

$$t_n = n, \; L(y_{t_n}) = F'(x_n).$$

Modified Newton's method:

$$\begin{aligned} x_{n+1} &= x_n - F'(x_0)^{-1}F(x_n), \\ t_n = 0, \qquad & L(y_{t_n}) = F'(x_0). \end{aligned}$$

Stirling's method for the equation $G(x) = x$, $\mathcal{X} = \mathcal{Y}$ and $F(x) = x - G(x)$:

$$\begin{aligned} x_{n+1} &= x_n - F'(x_n)^{-1}F(x_n), \\ t_n = n, \qquad & L(y_{t_n}) = F'(x_n). \end{aligned}$$

Picard's method [5]:

$$\begin{aligned} x_{n+1} &= x_n - F(x_n), \\ t_n = n, \qquad & L(y_{t_n}) = I, \end{aligned}$$

where I is the identity operator on $\mathcal{X}$ and $\mathcal{X} = \mathcal{Y}$.

Traub method: $t_{km+j} = km$, $j = 0, 1, \ldots, m-1$, $k = 0, 1, 2, \ldots$. Then, method (1.3) reduces to an iterative method studied by Traub [3]. The parameter m is chosen according to the dimension of the space in order to maximize the numerical efficiency of the method [6].

TWO POINT METHODS:

We can write method (1.3) in the form

$$x_{n+1} = x_n - L(y_{s_n}, y_{t_n})^{-1}F(x_n), \tag{1.5}$$

where $L(.) = L(y_{s_n}, .) : \mathcal{D} \longrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$, s_n is a nondecreasing sequence of integers satisfying the conditions

$$s_0 = -1, \; s_n \le t_n \le n \text{ for each } n = 1, 2, \ldots. \tag{1.6}$$

Secant method:

$$\begin{aligned} x_{n+1} &= x_n - [x_{n-1}, x_n; F]^{-1}F(x_n), \\ s_n &= n-1, \; t_n = n, \; [.,.; F] : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y}) \end{aligned}$$

is a consistent approximation of the Fréchet derivative F' of F and $L(y_{s_n}, y_{t_n}) = [x_{n-1}, x_n; F]$.

Modified secant method:

$$\begin{aligned} x_{n+1} &= x_n - [x_{-1}, x_0; F]^{-1}F(x_n), \\ s_n &= -1, \; t_n = 0, \text{ for each } n = 0, 1, 2, \ldots \end{aligned}$$

and $L(y_{s_n}, y_{t_n}) = [x_{-1}, x_0; F]$.

Traub method [3]:

$s_{km+j} = km - 1$, $t_{km+j} = km$, $s_{-1} = s_0 = -1$, $j = 0, 1, 2, \ldots, m-1$, $k = 0, 1, 2, \ldots$. Then, method (1.5) reduces to a procedure considered by Traub for scalar equations [3].

MULTI POINT METHODS

Consider the method

$$x_{n+1} = x_n - L(y_{t_0}, y_{t_1}, \ldots, y_{t_{n-1}}, y_{t_n})^{-1}F(x_n), \tag{1.7}$$

where $L(y_{t_0}, y_{t_1}, \ldots, y_{t_{n-1}}, y_{t_n}) : \underbrace{\mathcal{D}\times\mathcal{D}\ldots\times\mathcal{D}}_{n+1-times} \longrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$.

The local as well as semi-local convergence analysis of the preceding special cases of method (1.3) has been given in the aforementioned references (see also [5,7,8]) under Lipschitz-type conditions implying the Fréchet-differentiability of F although F' may not even be appearing in method (1.3). Therefore, these conditions limit the applicability of method (1.3). Even, if the previously introduced conditions on F' are satisfied, the convergence domain is in general small.

In the present article we address these problems. Hence, the motivation for this article. More precisely, we have the advantages:

(1) Method (1.3) is always defined, since we can choose $y_{t_n} = x_0$ for each $n = 0, 1, 2, \ldots$.
(2) Many methods are special cases of method (1.3) as already indicated. We study the semi-local convergence of method (1.3) assuming generalized Lipschitz conditions and only the continuity of operator F (see, e.g., condition (a3)), whereas the differentiability of F is not necessarily implied. Therefore, method (1.3) is also suitable for solving non-differentiable equations.
(3) Concerning the new convergence criteria, these are at least as weak as the ones appearing in the special methods. This is due to two facts: (i) We utilize the center-Lipschitz condition (see (a2)) to find upper bounds on the norms of the inverses involved instead of the Lipschitz condition used before which leads to less precise upper bounds. (ii) The introduction of the center-Lipschitz condition helps us define a subset of D containing the iterates leading to tighter majorant functions than before. At this generality the convergence of method (1.3) is linear. However, if the majorant functions specialize, the convergence order increases. We refer the reader to Remarks 2.1 and 2.4 for more details about the importance of method (1.3) as well as to the numerical examples. Similar advantages are obtained in the local convergence case. Hence, the applicability of method (1.3) as well as its special cases is extended.

These improvements are obtained under the same computational cost, since in practice the computation of the old majorant functions requires the computation of the new majorant functions as special cases.

The rest of the article is structured as follows: Sections 2 and 3 contain the semi-local and local convergence analyses of method (1.3). Special cases and numerical examples are presented in Section 4 and Section 5 is conclusion.

2. Semi-local convergence analysis

We present the semi-local convergence analysis of method (1.3) using some Lipschitz-type conditions and parameters. Let

$$R = \sup\{t \geq 0 : \bar{U}(x_0, t) \subseteq \mathcal{D}\}.$$

The semi-local convergence of method (1.3) is based on the conditions $(\mathcal{A})$:

(a1) Operator $F : \bar{U}(x_0, R) \longrightarrow \mathcal{Y}$ is continuous. There exist $x_0 \in \mathcal{D}$, such that $L(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and $\eta \geq 0$ such that

$$\|L(x_0)^{-1}F(x_0)\| \leq \eta.$$

(a2) There exists non-decreasing function $\gamma : [0, +\infty) \longrightarrow [0, +\infty)$ such that for each $x \in \bar{U}(x_0, r)$, $0 \leq r \leq R$

$$\|L(x_0)^{-1}(L(x) - L(x_0))\| \leq \gamma(\|x - x_0\|).$$

For $\gamma(R) < 1$, set $\beta = \max\{0 \leq t \leq R : \gamma(t) < 1\}$ and define $U_0 := \bar{U}(x_0, r) \cap U(x_0, \beta)$.

(a3) There exists non-decreasing function $\alpha : [0, \beta) \longrightarrow [0, +\infty)$ such that for each $u, v, z \in U_0$

$$\|L(x_0)^{-1}(F(v) - F(z) - L(u)(v - z))\| \leq \alpha(r)\|v - z\|.$$

(a4) Equation

$$\left(1 + \frac{g(\eta)}{1 - g(t)}\right)\eta - t = 0$$

has at least one positive solution. Denote by r_0 the smallest such solution, where

$$g(t) = \frac{\alpha(\eta)}{1 - \gamma(t)}.$$

(a5) $0 \leq g(r_0) < 1$ and $\gamma(r_0) < 1$.
(a6) $\bar{U}(x_0, r_0) \subseteq \bar{U}(x_0, R)$. Notice that in view of (a4) and (a5) $r_0 > \eta$.
(a7) There exists $r_1 \geq r_0$ such that $\alpha(r_1) + \gamma(r_0) < 1$. Set $U_1 = \bar{U}(x_0, r) \cap \bar{U}(x_0, r_1)$.

Remark 2.1. Let us comment further (see also the introduction) on the motivation for the convergence hypotheses $(\mathcal{A})$ one at a time before we present the proof of the semi-local convergence of method (1.3).

(a1) Researchers usually assume that F is continuously Fréchet differentiable although F' may not even appear in the method. We only assume that F is continuous. Moreover, in the first two examples operator F is continuous but not differentiable. The hypothesis on η is needed to show how close to the solution the initial point should be for convergence and is a standard hypothesis.

(a2) We use the center Lipschitz condition to determine an at least as small function γ as the one used (by others) call it $\tilde{\gamma}$ for the Lipschitz condition on $\bar{U}(x_0, R)$. Then, $\gamma \le \tilde{\gamma}$. This way the upper bound on $\|L(x)^{-1}L(x_0)\|$ is at least as tight as before.

(a3) In (a2) we have established U_0 where the iterates lie and since $U_0 \subseteq \bar{U}(x_0, r)$ our Lipschitz-type condition (where function α depends on γ and U_0) is such that $\alpha \le \tilde{\alpha}$, where function $\tilde{\alpha}$ depends on $\bar{U}(x_0, r)$.

(a4) The equation here is more precise than the usual corresponding one in (a4) using $\tilde{\gamma}$, $\tilde{\alpha}$ and $\tilde{g}$, since also $g \le \tilde{g}$.

(a5)–(a7) These conditions are also more precise than the ones in (a3) using $\tilde{\gamma}, \tilde{\alpha}, \tilde{g}$ and $\tilde{r}_0$.

In view of the above and the estimates given in the proof that follows, we have the additional advantages over using $\tilde{\gamma}, \tilde{\alpha}, \tilde{g}$ and $\tilde{r}_0$:

(i) Larger convergence domain,
(ii) Tighter error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - p\|$. That is fewer iterations are needed to achieve a desired error tolerance $\varepsilon > 0$.
(iii) An at least as precise information on the location of the solution.

Indeed, the old equation [2] is given by

$$\left(1 + \frac{\tilde{g}(\eta)}{1 - \tilde{g}(t)}\right)\eta - t = 0,$$

where

$$\tilde{g}(t) = \frac{\tilde{\alpha}(\eta)}{1 - \tilde{\gamma}(t)}.$$

Then, it follows that if the old equation has a solution, so does the new equation but not necessarily vice versa. Let us provide an academic example by specializing the functions.

Example 2.2. Choose $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\mathcal{D} = U(x_0, 1 - \xi)$, $x_0 = 1$, $\xi \in (0, \frac{1}{2})$ and $R = 1 - \xi$. Define function F on $\mathcal{D}$ by

$$F(x) = x^3 - \xi.$$

Consider Newton's method. Then, we have $\gamma(t) = (3-\xi)t$, $\alpha(t) = 2\left(1 + \frac{1}{3-\xi}\right)t$, $\beta = \frac{1}{3-\xi}$, $\tilde{\gamma}(t) = (2-\xi)t$, $\tilde{\alpha}(t) = 2(2-\xi)t$ and $\eta = \frac{1}{3}(1-\xi)$. Notice that for each $t \in (0, \frac{1}{2})$

$$\gamma(t) < \tilde{\gamma}(t),$$

$$\alpha(t) < \tilde{\alpha}(t)$$

and

$$g(t) < \tilde{g}(t).$$

Choose $\xi = 0.49$. Then $r_0 = 0.3056$, but the old equation with tilde function has no real solution. Hence, there is no assurance that Newton's method converges to $p = \sqrt[3]{3}$ under the old approach. However, Newton's method converges under our approach.

These advantages are obtained under the same computational cost as if, we were to use $\tilde{\gamma}, \tilde{\alpha}, \tilde{g}$ and $\tilde{r}_0$, since in practice the new functions γ, α, g and parameter r_0 are special cases of the aforementioned ones. As an example, we have in the proof of the theorem that follows

$$\|x_{n+1} - x_n\| \le \frac{\alpha(\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|}{1 - \gamma(\|y_n - x_0\|)}$$

which is tighter than

$$\|x_{n+1} - x_n\| \le \frac{\tilde{\alpha}(\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|}{1 - \tilde{\gamma}(\|y_n - x_0\|)}$$

used before, if $\alpha < \tilde{\alpha}$ or $\gamma < \tilde{\gamma}$ (see also Example 4.1).

Next, we present the semi-local convergence analysis of method (1.3) under the conditions $(\mathcal{A})$. The technique of proof can be reduced to the one given in the works by Ezquerro, Hernandex et al. (see, e.g. [2]) in the special case, when the majorant conditions are the same.

Theorem 2.3. *Assume that the conditions $(\mathcal{A})$ hold. Then, sequence $\{x_n\}$ generated by method (1.3) is well defined in $U(x_0, r_0)$, remains in $U(x_0, r_0)$ for each $n = 0, 1, 2, \ldots$ and converges to a solution p of equation $F(x) = 0$ which is the only solution of this equation in U_1. Moreover, the following error bounds hold*

$$\|x_1 - x_0\| \le \eta, \tag{2.1}$$

$$\|x_2 - x_1\| \le g(\eta)\|x_1 - x_0\| \le g(\eta)\eta, \tag{2.2}$$

$$\|x_n - x_0\| \le \frac{1 - g(r_0)^{n-1}}{1 - g(r_0)}\|x_2 - x_1\| + \eta \le r_0 \text{ for each } n = 2, 3, \ldots, \tag{2.3}$$

$$\|x_0 - p\| \le (\frac{g(\eta)}{1 - g(r_0)} + 1)\eta, \tag{2.4}$$

$$\|x_1 - p\| \le \frac{g(\eta)\eta}{1 - g(r_0)}, \tag{2.5}$$

$$\|x_{n+1} - x_n\| \le g(r_0)^{n-1}\|x_2 - x_1\| \text{ for each } n = 2, 3, \ldots. \tag{2.6}$$

and

$$\|x_n - p\| \le \frac{g(r_0)^{n-1}}{1 - g(r_0)}\|x_2 - x_1\| \text{ for each } n = 2, 3, \ldots. \tag{2.7}$$

Furthermore, p is the only solution of equation $F(x) = 0$ in U_1.

Proof. By conditions (a1) and (a4) we have $x_1 \in U(x_0, r_0)$. Let $x \in U(x_0, r_0)$. Then, using (a1) and (a2)

$$\|L(x_0)^{-1}(L(x) - L(x_0))\| \le \gamma(\|x - x_0\|) \le \gamma(r_0) \le \gamma(R) < 1, \tag{2.8}$$

by the definition of β. If follows from (2.8) and the Banach lemma on invertible operators [1] that $L(x)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\|L(x)^{-1}L(x_0)\| \le \frac{1}{1 - \gamma(\|x - x_0\|)}. \tag{2.9}$$

In particular, we have for $x = x_1$ that since $x_1 \in U(x_0, r_0)$, $L(x_1)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and by (a1) and (4.9)

$$\|L(x_1)^{-1}L(x_0)\| \le \frac{1}{1 - \gamma(\eta)}. \tag{2.10}$$

Iterate x_2 is well defined since x_1 is well defined and $y_1 = x_0$ or $y_1 = x_1$. Notice that (2.1) holds by method (1.3) for $n = 0$ and (a1). Suppose that x_{m+1} and y_m exist for all $m = 0, 1, 2 \ldots, n$. We can write by method (1.3)

$$F(x_{m+1}) = F(x_{m+1}) - F(x_m) - L(y_m)(x_{m+1} - x_m). \tag{2.11}$$

In particular for $m = 0$, (a3), (2.10), (a3) and (a4), we get in turn

$$\begin{aligned} \|x_2 - x_1\| &\le \|L(y_1)^{-1}L(x_0)\|\|L(x_0)^{-1}(F(x_1) - F(x_0)) - L(y_0)(x_1 - x_0)\| \\ &\le \frac{\alpha(\|x_1 - x_0\|)\|x_1 - x_0\|}{1 - \gamma(\|x_1 - x_0\|)} \\ &\le \frac{\alpha(\eta)}{1 - \gamma(\eta)}\|x_1 - x_0\| \\ &\le \frac{\alpha(\eta)\eta}{1 - \gamma(\eta)} = g(\eta)\eta, \end{aligned} \tag{2.12}$$

which shows (2.2) ($\|x_2 - x_1\| \le \|x_1 - x_0\|$ by (a5)) and

$$\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| \le g(\eta)\eta + \eta < r_0, \tag{2.13}$$

so $x_2 \in U(x_0, r_0)$ and (2.4) holds for $n = 2$. Similarly by (2.11) for $m = 1$, $y_2 = x_0$ or x_1 or x_2 (so $y_2 \in U(x_0, r_0)$ and (4.9) holds for $x = y_2$), (a3)-(a5) we have in turn that

$$\begin{aligned} \|x_3 - x_2\| &\le \|L(y_2)^{-1}L(x_0)\|\|L(x_0)^{-1}(F(x_2) - F(x_1)) - L(y_1)(x_2 - x_1)\| \\ &\le \frac{\alpha(\|x_2 - x_1\|)\|x_2 - x_1\|}{1 - \gamma(\|y_2 - x_0\|)} \\ &\le \frac{\alpha(\eta)}{1 - \gamma(r_0)}\|x_2 - x_1\| \\ &= g(r_0)\|x_2 - x_1\| < \|x_2 - x_1\|, \end{aligned} \tag{2.14}$$

which shows (2.3) for $n = 2$ and

$$\begin{aligned}\|x_3 - x_0\| &\le \|x_3 - x_2\| + \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\le g(r_0)\|x_2 - x_1\| + \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\le \frac{1 - g(r_0)^2}{1 - g(r_0)}\|x_2 - x_1\| + \|x_1 - x_0\| \\ &< \left(\frac{g(\eta)}{1 - g(r_0)} + 1\right)\eta = r_0,\end{aligned}$$

so $x_3 \in U(x_0, r_0)$ and (2.4) holds for $n = 3$. Similarly, we obtain

$$\begin{aligned}\|x_4 - x_3\| &\le \frac{\alpha(\|x_3 - x_2\|)\|x_3 - x_2\|}{1 - \gamma(\|y_3 - y_0\|)} \qquad (2.15)\\ &\le \frac{\alpha(\eta)}{1 - \gamma(r_0)}\|x_3 - x_2\| = g(r_0)\|x_3 - x_2\| \\ &< \|x_3 - x_2\| < \|x_2 - x_1\|, \\ \|x_4 - x_3\| &\le g(r_0)^2\|x_2 - x_1\| \\ &\vdots \\ \|x_{m+1} - x_m\| &\le g(r_0)\|x_m - x_{m-1}\| < \|x_m - x_{m-1}\| \\ &< \|x_2 - x_1\| \qquad (2.16)\\ \|x_{m+1} - x_m\| &\le g(r_0)^{m-1}\|x_2 - x_1\| \qquad (2.17)\end{aligned}$$

and

$$\begin{aligned}\|x_{m+1} - x_0\| &\le \|x_{m+1} - x_m\| + \cdots + \|x_2 - x_1\| + \|x_1 - x_0\| \qquad (2.18)\\ &\le \frac{1 - g(r_0)^m}{1 - g(r_0)}\|x_2 - x_1\| + \eta \\ &\le \left(\frac{g(\eta)}{1 - g(r_0)} + 1\right)\eta = r_0, \qquad (2.19)\end{aligned}$$

which complete the induction for (2.3) and (2.4). We can also have

$$\begin{aligned}\|x_{m+i} - x_m\| &\le \|x_{m+i} - x_{m+i-1}\| + \cdots + \|x_{m+1} - x_m\| \\ &\le (g(r_0)^{m+i-2} + \cdots + g(r_0)^{m-1})\|x_2 - x_1\| \\ &= g(r_0)^{m-1}\frac{1 - g(r_0)^i}{1 - g(r_0)}\|x_2 - x_1\|. \qquad (2.20)\end{aligned}$$

It follows from (2.20) that sequence $\{x_m\}$ is complete in a Banach space $\mathcal{X}$ and as such it converges to some $p \in \bar{U}(x_0, r_0)$ (since $\bar{U}(x_0, r_0)$ is a closed set). By letting $i \longrightarrow +\infty$ in (2.20), we obtain (2.7) for $m = n$. We also get

$$\begin{aligned}\|p - x_0\| &\le \|p - x_2\| + \|x_2 - x_0\| \qquad (2.21)\\ &\le \frac{g(r_0)}{1 - g(r_0)}\|x_2 - x_1\| + (g(\eta) + 1)\eta \\ &\le \left(\frac{g(r_0)}{1 - g(r_0)}g(\eta) + g(\eta) + 1\right)\eta \\ &\le \left(\frac{g(\eta)}{1 - g(r_0)} + 1\right)\eta = r_0\end{aligned}$$

and

$$\begin{aligned}\|p - x_1\| &\le \|p - x_2\| + \|x_2 - x_1\| \qquad (2.22)\\ &\le \frac{g(r_0)}{1 - g(r_0)}g(\eta)\eta + g(\eta)\eta \\ &\le \left(\frac{g(r_0)}{1 - g(r_0)} + 1\right)g(\eta)\eta = \frac{g(\eta)\eta}{1 - g(r_0)}.\end{aligned}$$

which show (2.4) and (2.6), respectively.

Next from (2.11), (a5) the estimate

$$\|L(x_0)^{-1}F(x_{m+1})\| \leq \alpha(\eta)\|x_{m+1}-x_m\| \tag{2.23}$$

and by letting $m \longrightarrow \infty$ in (2.23), we get $F(p) = 0$. To show the uniqueness part, let $p_* \in U_1$ such that $F(p_*) = 0$. Then, we have from (4.9) for $x = y_m$, (a4), (a5) and (a7) that

$$\begin{aligned}\|x_{m+1}-p_*\| &= \|L(y_m)^{-1}[F(x_m)-F(p_*)-L(y_m)(x_m-p_*)]\| \\ &\leq \frac{\alpha(r_1)\|x_m-p_*\|}{1-\gamma(\|y_m-x_0\|)} \\ &\leq \frac{\alpha(r_1)}{1-\gamma(r_0)}\|x_m-p_*\| \\ &= c\|x_m-p_*\| \leq \cdots \leq c^{m+1}\|x_0-p_*\| \leq c^{m+1}r_1,\end{aligned} \tag{2.24}$$

where $c = \frac{\alpha(r_1)}{1-\gamma(r_0)} \in [0, 1)$. By letting $m \longrightarrow +\infty$ in (2.23) we get $\lim_{m\longrightarrow+\infty}x_m = p_*$ and since we showed $\lim_{m\longrightarrow+\infty}x_m = p$, we conclude that $p_* = p$. □

Remark 2.4.

(a) Researchers prefer to leave equations like the one in (a4) as uncluttered as possible [2,6,7]. This equation specifies the smallness of η (or the accuracy of the initial point x_0 to assure convergence of the method). The same equation provides the radius of convergence r_0. Such conditions on η and r_0 are not immediate by just looking at the equation. We can use some stronger conditions that imply the solvability of the equation. As an example, suppose that

$$0 \leq 2g(\eta) < 1. \tag{2.25}$$

Then, by (2.25) there exist $r_0 \geq \eta$ and $\lambda \in (0, 1)$ such that

$$\frac{g(\eta)}{1-g(r)} \leq \lambda < 1. \tag{2.26}$$

Therefore, by the equation in (a4) and (2.26), we must have that

$$(1+\lambda)\eta \leq r_0$$

or

$$\eta \leq \frac{1}{1+\lambda}r_0. \tag{2.27}$$

In practice, we choose $\lambda \in (0, 1)$ and solve equation

$$\frac{g(\eta)}{1-g(t)} = \lambda. \tag{2.28}$$

By (2.26) and the intermediate value theorem equation $\frac{g(\eta)}{1-g(t)} - \lambda = 0$ has positive solutions. Denote by r_0 the smallest such solution. Then, the sufficient convergence criteria replacing (a4) and (a5) are given by (2.25) and (2.27).

(b) We can do even better using the set $U_0^1 := \bar{U}(x_0, r) \cap U(x_1, \beta - \|x_1 - x_0\|)$ provided that $\|x_1 - x_0\| < \beta$. Notice that $U_0^1 \subseteq U_0$, since for $x \in U(x_1, \beta - \|x_1 - x_0\|)$, we have

$$\|x-x_1\| < \beta - \|x_1-x_0\| \Longrightarrow \|x-x_1\| + \|x_1-x_0\| < \beta$$

$$\Longrightarrow \|x-x_0\| < \beta \Longrightarrow x \in U_0^1.$$

Moreover, if $\|x_1-x_0\| < \frac{\beta}{2}$, then $x_0 \in U(x_1, \beta-\|x_1-x_0\|)$, so $x_0 \in U_0^1$, since $x_0 \in \bar{U}(x_0, r)$. Clearly, U_0^1 can then replace U_0 in Theorem 2.3 leading to a tighter function $\tilde{\alpha}$ than α which in turn results an even finer convergence analysis.

(c) The results of Theorem 2.3 extend in the more general setting of inexact frozen Newton-like methods defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - L(y_{t_n})^{-1}F_1(x_n), \tag{2.29}$$

where $F_1(x) = F(x) + q(x)$ and $q : \mathcal{D} \longrightarrow \mathcal{X}$ is the residual operator and $q_n = q(x_n)$ is a null residual sequence. If we simply replace F by F_1 in the hypotheses of Theorem 2.3, then the conclusions of this theorem hold for method (2.29) with the exception of the uniqueness part. The uniqueness part also holds, if we further assume that $q(p) = 0$ (see (2.24)). If $q(x) = 0$ method (2.29) reduces to method (1.3). It is well known that method (2.29) contains the so called multi-step methods [6,7].

3. Local convergence analysis

Let $p \in \mathcal{D}$ be a solution of equation $F(x) = 0$. Define $\bar{R} = \sup\{t \geq 0 : \bar{U}(p,t) \subseteq \mathcal{D}\}$. We base the local convergence of method (1.3) on the conditions $(\mathcal{B})$:

(b1) Operator $F : \bar{U}(p,\bar{R}) \longrightarrow \mathcal{Y}$ is continuous and $L(p)^{-1} \in \mathcal{L}(\mathcal{Y},\mathcal{X})$.

(b2) There exists non-decreasing function $\bar{\gamma} : [0,+\infty) \longrightarrow [0,+\infty)$ such that for each $x \in \bar{U}(p,r),\ 0 \leq r \leq \bar{R}$

$$\|L(p)^{-1}(L(x) - L(p))\| \leq \bar{\gamma}(\|x-p\|).$$

For $\bar{\gamma}(\bar{R}) < 1$, set $\bar{\beta} = \max\{0 \leq t \leq \bar{R} : \bar{\gamma}(t) < 1\}$ and define $V_0 = \bar{U}(p,r) \cap U(p,\bar{\beta})$.

(b3) There exists non-decreasing function $\bar{\alpha} : [0,\bar{\beta}) \longrightarrow [0,+\infty)$ such that for each $u, v \in V_0$

$$\|L(p)^{-1}(F(v) - F(p) - L(u)(v-p))\| \leq \bar{\alpha}(r)\|v-p\|.$$

(b4) Equation

$$\bar{\alpha}(t) + \bar{\gamma}(t) = 1 \tag{3.1}$$

has positive solutions. Denote by $\bar{r}_0$ the smallest such solution.

(b5) $\bar{U}(p,\bar{r}_0) \subseteq \bar{U}(p,\bar{R})$ and $\bar{\gamma}(\bar{r}_0)\bar{r}_0 < 1$.

(b6) There exists $\bar{r}_1 \geq \bar{r}_0$ such that $\bar{\alpha}(\bar{r}_1) + \bar{\gamma}(\bar{r}_0)\bar{r}_0 < 1$. Set $V_1 = \bar{U}(p,r) \cap \bar{U}(p,\bar{r})$.

Remark 3.1. Comments similar to the ones given in Remark 2.1 for the semi-local case can be given for the local case. Let us present an example in the case of Newton's method. Let us present an example in the case of Newton's method.

Example 3.2. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^3$, $\mathcal{D} = \bar{U}(0,1)$, $p = (0,0,0)^T$. Define function F on $\mathcal{D}$ for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have $\bar{\gamma}(t) = (e-1)t$, $\bar{\alpha}(t) = \frac{1}{2}e^{\frac{1}{e-1}}t$, $\bar{\beta} = \frac{1}{e-1}$ and $\bar{r}_0 = 0.38269191223238574472$. The functions used by Traub [3] in (b2) and (b3) are $\bar{\bar{\gamma}}(t) = et$, $\bar{\bar{\alpha}}(t) = \frac{1}{2}et$, whereas the corresponding to (3.1) equation gives $\bar{\bar{r}}_0 = 0.24525296078096148200$1. Notice that

$$\bar{\gamma}(t) < \bar{\bar{\gamma}}(t)$$

and

$$\bar{\alpha}(t) < \bar{\bar{\alpha}}(t),$$

so the new error bounds are also tighter. Hence, the applicability of Newton's method is extended.

Next, we present the local convergence analysis of method (1.3) under the $(\mathcal{B})$ conditions.

Theorem 3.3. *Assume that the conditions $(\mathcal{B})$ hold. Then, sequence $\{x_n\}$ generated for $x_0 \in U(p,r_0) - \{p\}$ by method (1.3) is well defined in $U(p,r_0)$, remains in $\bar{U}(p,\bar{r}_0)$ for each $n = 0,1,2,\ldots$ and converges to p, so that*

$$\|x_{n+1} - p\| \leq c_n\|x_n - p\|,$$

where

$$c_n = \frac{\bar{\alpha}(\bar{r}_1)}{1 - \bar{\gamma}(\|x_n - p\|)} \leq \bar{c} := \frac{\bar{\alpha}(\bar{r}_1)}{1 - \bar{\gamma}(\bar{r}_0)} \in [0,1).$$

Moreover, the point p is the only solution of equation $F(x) = 0$ in V_1.

Proof. Simply follow the proof of (2.24) with $p, \bar{\alpha}, \bar{\gamma}, \bar{r}_0, \bar{r}_1$ replacing $x_0, \alpha, \gamma, r_0, r_1$, respectively. □

Remark 3.4.

(a) As in Remark 2.4 assume that $\|x_0 - p\| \leq \bar{\beta}$. Define the set $V_0^1 = \bar{U}(p,r) \cap U(x_0, \bar{\beta} - \|x_0 - p\|)$. Then, again we have $V_0^1 \subseteq V_0$ and V_0^1 can replace V_0 in Theorem 3.3.

(b) The local results extend in the case of method (2.29), if again we replace F by F_1 in the conditions $(\mathcal{B})$ (see Remark 2.4(c)).

4. Special cases and numerical examples

We first present two numerical examples involving systems defined on $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. In the first example, we show, how to compute the majorant functions appearing in Theorems 2.3 and 3.3. Moreover, we show that the new majorant functions α, γ are tighter than $\bar{\alpha}$ and $\bar{\gamma}$ which are tighter than $\bar{\alpha}_{old}$, $\bar{\gamma}_{old}$ used in [2]. Notice that the results in [2] improved the results in [9,10]. The emphasis in the second example is to show convergence of method (1.3), present some error bounds and the solution p without necessarily verifying the convergence conditions introduced in the previous sections. Furthermore, we provide an example involving a boundary value problem also reduced to solving a system of equations. Finally, in the fourth example, we solve a nonlinear integral equation of Hammerstein type in cases not covered before.

The most widely used conditions for Newton-like methods are [8]:

$$\|L(x_0)^{-1}(F'(x) - F'(y))\| \le \psi(r)\|x - y\| \tag{4.1}$$

$$\|L(x_0)^{-1}(L(x) - L(x_0))\| \le \psi_0(r) \tag{4.2}$$

$$\|L(x_0)^{-1}(F'(x) - L(x))\| \le \psi_1(\|x - x_0\|) + \delta \tag{4.3}$$

for each $x, y \in U(x_0, r)$, where $\psi, \psi_0, \psi_1 : [0, +\infty) \longrightarrow [0, +\infty)$ are continuous and nondecreasing functions and $\delta \ge 0$. Let us connect the conditions of Theorem 2.3 with the conditions (4.1)–(4.3) in the special case, where $t_n = n$ for each $n = 0, 1, 2 \ldots$. Using again (2.11), we get

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - L(x_n)(x_{n+1} - x_n) \\ &= \int_0^1 [F'(x_n + \theta(x_{n+1} - x_n)) - L(x_n)](x_{n+1} - x_n)d\theta \\ &= \int_0^1 [F'(x_n + \theta(x_{n+1} - x_n)) - F'(x_n)](x_{n+1} - x_n)d\theta \\ &\quad +(F'(x_n) - L(x_n))(x_{n+1} - x_n). \end{aligned} \tag{4.4}$$

Then, using (4.1)–(4.4), we get

$$\|L(x_0)^{-1}F(x_{n+1})\| \le \frac{1}{2}\psi(r)\|x_{n+1} - x_n\|^2 + (\psi_1(r) + \delta)\|x_{n+1} - x_n\|. \tag{4.5}$$

Therefore, we must choose

$$\alpha^1(t) = \frac{1}{2}\psi(t)t + \psi_1(t) + \delta \tag{4.6}$$

and

$$\gamma^1(t) = \psi_0(t). \tag{4.7}$$

Notice that

$$\alpha(t) \le \alpha^1(t) \text{ and } \gamma(t) \le \gamma^1(t), \tag{4.8}$$

since $U_0 \subseteq \bar{U}(x_0, r)$. However, if we use our technique of restricted convergence domains and suppose together with (4.2) that

$$\|L(x_0)^{-1}(F'(x) - F'(y))\| \le \bar{\psi}(r)\|x - y\| \tag*{(4.1)$'$}$$

and

$$\|L(x_0)^{-1}(F'(x) - L(x))\| \le \bar{\psi}_1(\|x - x_0\|) + \delta \tag*{(4.3)$'$}$$

for each $x, y \in U_0$ and $\bar{\psi}, \bar{\psi}_1 : [0, \beta) \longrightarrow [0, +\infty)$ continuous, nondecreasing and $\bar{\delta} \ge 0$, then we can choose

$$\alpha(t) = \alpha^0(t) = \frac{1}{2}\bar{\psi}(t)t + \bar{\psi}_1(t) + \bar{\delta}$$

and

$$\gamma(t) = \gamma^0(t) = \gamma^1(t).$$

Then, again we have that (4.8) holds, since

$$\bar{\psi}(t) \le \psi(t), \;\; \bar{\psi}_1(t) \le \psi_1(t) \text{ and } \bar{\delta} \le \delta. \tag{4.9}$$

Therefore, the new results extend the old ones in view of (4.8) and (4.12).

Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^k$ in the rest of the section, where k is a positive integer. In what follows we shall use the definition of the standard divided difference $[a, b; F] := ([a, b; F]_{ij})_{i,j=1}^k \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$ defined for $F = (F_1, F_2, \ldots, F_k)^T$, $a = (a_1, a_2, \ldots, a_k)^T$, $b = (b_1, b_2, \ldots, b_k)^T$, $1 \le i, j \le k$ by

$$[a, b; F]_{i,j} = \frac{F_i(a_1, \ldots, a_j, b_{j+1}, \ldots, b_k) - F_i(a_1, \ldots, a_{j-1}, b_j, b_{j+1}, \ldots, b_k)}{a_j - b_j}. \tag{4.10}$$

The formula defines a bounded linear operator which satisfies $[b, a; F](b - a) = F(b) - F(a)$ for each $a \neq b,\ a, b \in \mathbb{R}^k$.

Example 4.1 ([2,9,10]). Let $k = 3$, $D = U(0, 1)$ and for $h = (h_1, h_2, h_3)^T$ define mapping F on D by

$$F(h) = (h_1 + 0.0125|h_1|, h_2^2 + h_2 + 0.0125|h_2|, e^{h_3} - 1)^T. \tag{4.11}$$

Clearly, a solution of equation $F(h) = 0$ is given by $p = (0, 0, 0)^T$.

Let $F : \mathcal{D} \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, we use the divided differences of first order given by $[u, v; F] = ([u, v; F])_{i,j=1}^4 \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$, where

$$[u, v; F]_{i1} = \begin{cases} \dfrac{F_i(u_1, v_2, v_3) - F_i(v_1, v_2, v_3)}{u_1 - v_1}, & \text{if } u_1 \neq v_1 \\ 0, \ \text{if } u_1 = v_1 \end{cases}$$

$$[u, v; F]_{i2} = \begin{cases} \dfrac{F_i(u_1, u_2, v_3) - F_i(u_1, v_2, v_3)}{u_2 - v_2}, & \text{if } u_2 \neq v_2 \\ 0, \ \text{if } u_2 = v_2 \end{cases}$$

and

$$[u, v; F]_{i3} = \begin{cases} \dfrac{F_i(u_1, u_2, u_3) - F_i(u_1, u_2, v_3)}{u_3 - v_3}, & \text{if } u_3 \neq v_3 \\ 0, \ \text{if } u_3 = v_3. \end{cases}$$

It is easy to see that $[x, y; F] = F(x) - F(y)$ with x and y having some different component. If the three components coincide for two terms $x_n = x_{n+1}$, then no more iterations are needed and $x_n = x_{n+1} = p$.

Local case

$$\begin{aligned} \|[\tilde{z}, p; F]^{-1}([x, y; F] - [u, v; F])\| &\le d[\frac{e}{2}(\|x - u\| + \|y - v\|) + 0.025] \\ &\quad \text{for each } x, y, u, v \in D \\ \|[\tilde{z}, p; F]^{-1}([\tilde{z}, x; F] - [\tilde{z}, p; F])\| &\le d[\|x - \tilde{z}\| + 0.025] \\ &\quad \text{for each } x, y \in D \end{aligned}$$

and

$$\begin{aligned} \|[\tilde{z}, p; F]^{-1}([x, y; F] - [u, v; F])\| &\le d[\frac{e^{\rho_0}}{2}(\|x - u\| + \|y - v\|) + 0.025] \\ &\quad \text{for each } x, y, u, v \in V_0 \end{aligned}$$

where $d = \|[p, \tilde{z}; F]^{-1}\|$. Therefore, we can define , $\bar{R} = 1$,

$$\bar{\alpha}_{old} = d(et + 0.025),$$

$$\bar{\gamma}(t) = \bar{\gamma}_{old} = d(2t + 0.025)$$

and

$$\bar{\alpha}(t) = d(e^{\bar{\rho}} t + 0.025),$$

where $\bar{\rho} = \min\{1, \bar{\beta}\}$. Then, we have that

$$\bar{\alpha}(t) < \bar{\alpha}_{old}(t)$$

and

$$\gamma(t) \le \bar{\alpha}_{old}(t) \tag{4.12}$$

for each $t > 0$. Then, in [2,9,10] using the secant method (1.4) for $\tilde{z} = (0.01, 0.01, 0.01)^T$, we obtain for: $d = 1.7206$ by solving equation $\bar{\alpha}_{old}(t) + \bar{\gamma}_{old}(t) = 1$ that $\bar{r}_{old} = 0.1126$.

In the case of Theorem 3.3 (i.e., using method (1.3)), we have $\bar{\beta} = \frac{1}{d} - 0.025$ so $\bar{\rho} = \rho_0$ and (4.2) holds as a strict inequality. Moreover, we have by solving equation $\bar{\alpha}(t) + \bar{\gamma}(t) = 1$ that

$$\bar{r}_0 = 0.1419. \tag{4.13}$$

It follows from the above that for the secant method, the new results improve the ones in [2,9,10], since

$$\bar{r}_{old} < \bar{r}_0. \tag{4.14}$$

Semi-local case [2,9,10]
We also obtain for $R = 1$,

$$\gamma(t) = \gamma_{old}(t) = d_0(2t + 0.025),$$

$$\alpha_{old}(t) = d_0(et + 0.025)$$

and

$$\alpha(t) = d_0(e^{r^*}t + 0.025) \tag{4.15}$$

where $d_0 = \|[x_{-1}, x_0; F]^{-1}\|$, $\beta = 0.5562$, $r^* = \min\{1, \beta\} = 0.5562$ and $\gamma_{old}, \alpha_{old}$ are as γ, α respectively but defined on D (see also [2,9,10] and (2.20)). Then, again we have that

$$\alpha(t) < \alpha_{old}(t) \tag{4.16}$$

and

$$\gamma(t) < \alpha_{old}(t) \tag{4.17}$$

Choose $x_0 = (0.1, 0.1, 0.01)^T$, and $x_{-1} = (0.11, 0.11, 0.02)^T$. Then, we get that $\eta = \|x_1 - x_0\| = 0.1123$, $d_0 = 0.8180$. Then, by solving equation [2,9,10]

$$\left[1 + \frac{g_{old}(\eta)}{1 - g_{old}(t)}\right]\eta - t = 0,$$

where $g_{old}(t) = \frac{\alpha_{old}(\eta)}{1-\gamma_{old}(t)}$, we get $r_{old} = 0.1747$. If we use the secant method, we must solve the equation [2,9,10] (see (a4))

$$\left[1 + \frac{g(\eta)}{1 - g(t)}\right]\eta - t = 0, \tag{4.18}$$

to obtain $r_0 = 0.1890$, so the uniqueness of the solution is established in a larger ball than in [2,9,10].

Example 4.2. Let us consider the system for $h = (h_1, h_2)^T$

$$f_1(h) = 3h_1^2h_2 + h_2^2 - 1 + |h_1 - 1| = 0 \tag{4.19}$$

$$f_2(h) = h_1^4 + h_1h_2^3 - 1 + |h_2| = 0 \tag{4.20}$$

which can be written as $F(h) = 0$, where $F = (f_1, f_2)^T$. Using the analog to the divided difference given in Example 4.1, $([a, b; F]_{ij})_{i,j=1}^2 \in L(\mathbb{R}^2, \mathbb{R}^2)$, for $x_{-1} = (1, 0)^T$, $x_0 = (5, 5)^T$, we obtain by (1.3).

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	5
1	1	0	5
2	0.909090909090909	0.363636363636364	3.0636E−01
3	0.894886945874111	0.329098638203090	3.453E−02
4	0.894655531991499	0.327827544745569	1.271E−03
5	0.894655373334793	0.327826521746906	1.022E−06
6	0.8946655373334687	0.327826521746298	6.089E−13
7	0.8946655373334687	0.327826421746298	2.710E−20

Hence, the solution p is given by $p = (0.894655373334687, 0.3278626421746298)^T$. Notice that mapping F is not differentiable, so the earlier results mentioned in the introduction of this study cannot be used.

Example 4.3. We consider the boundary value problem appearing in many studies of applied sciences [2] given by

$$\varphi'' + \varphi^{1+\mu} + \varphi^2 = 0, \quad \mu \in [0, 1] \tag{4.21}$$
$$\varphi(0) = \varphi(1) = 0.$$

Let $h = \frac{1}{l}$, where l is a natural integer and set $s_i = ih, i = 1, 2, \ldots, l - 1$. The boundary conditions are then given by $\varphi_0 = \varphi_n = 0$. We shall replace the second derivative φ'' by the popular divided difference

$$\varphi''(t) \approx \frac{[\varphi(t + h) - 2\varphi(t) + \varphi(t - h)]}{h^2} \tag{4.22}$$

$$\varphi''(s_i) = \frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{h^2}, \ i = 1, 2, \ldots, l - 1.$$

Using (4.21) and (4.22), we obtain the system of equations defined by

$$\begin{aligned} 2\varphi_1 - h^2\varphi_1^{1+\mu} - h^2\varphi_1^2 - \varphi_2 &= 0 \\ -\varphi_{i-1} + 2\varphi_i - h^2\varphi_i^{1+\mu} - h^2\varphi_i^2 - \varphi_{i+1} &= 0 \\ -\varphi_{l-2} + 2\varphi_{l-1} - h^2\varphi_{l-1}^{1+\mu} - h^2\varphi_{l-1}^2 &= 0. \end{aligned} \tag{4.23}$$

Define operator $F : \mathbb{R}^{l-1} \longrightarrow \mathbb{R}^{l-1}$ by

$$F(\varphi) = M(x) - h^2 f(\varphi), \tag{4.24}$$

where

$$M = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{bmatrix}$$

and

$$f(\varphi) = [\varphi_1^{1+\mu} + \varphi_1^2, \varphi_2^{1+\mu} + \varphi_2, \dots, \varphi_{l-1}^{1+\mu} + \varphi_{l-1}^2]^T.$$

Then, the Fréchet-derivative F' of operator F is given by

$$\begin{aligned} F'(\varphi) = M - (1+\mu)h^2 &\begin{bmatrix} \varphi_1^{\mu} & 0 & 0 & \dots & 0 \\ 0 & \varphi_2^{\mu} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \varphi_{l-1}^{\mu} \end{bmatrix} \\ -2h^2 &\begin{bmatrix} \varphi_1 & 0 & 0 & \dots & 0 \\ 0 & \varphi_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \varphi_{l-1} \end{bmatrix}. \end{aligned} \tag{4.25}$$

We shall use a special case of method (1.3) given by

$$\begin{aligned} \psi_n^{(1)} &= \psi_n - F'(\psi_n)^{-1}F(\psi_n) \\ \psi_n^{(2)} &= \psi_n^{(1)} - F'(\psi_n)^{-1}F(\psi_n^{(1)}) \\ &\ \ \vdots \\ \psi_n^{(k)} &= \psi_n^{(k-1)} - F'(\psi_n)^{-1}F(\psi_n^{(k-1)}) \\ \psi_{n+1} &= \psi_n^{(k)}. \end{aligned} \tag{4.26}$$

Let $\mu = \frac{1}{2}$, $k = 3$ and $l = 10$. This way we obtain a 9×9 system. A good initial approximation is $10 \sin \pi t$, since a solution of (4.21) vanishes at the end points and is positive at the interior. This approximation gives the vector

$$\xi = \begin{bmatrix} 3.0901699423 \\ 5.877852523 \\ 8.090169944 \\ 9.510565163 \\ 10 \\ 9.510565163 \\ 8.090169944 \\ 5.877852523 \\ 3.090169923 \end{bmatrix}$$

leading using (4.26) to

$$\psi_0 = \begin{bmatrix} 2.396257294 \\ 4.698040582 \\ 6.677432200 \\ 8.038726637 \\ 8.526409945 \\ 8.038726637 \\ 6.6774432200 \\ 4.698040582 \\ 2.396257294 \end{bmatrix}.$$

Using vector ψ_0 as the initial vector in (4.26), we get the solution ψ^* given by

$$\psi^* = \psi_6 = \begin{bmatrix} 2.394640795 \\ 4.694882371 \\ 6.672977547 \\ 8.033409359 \\ 8.520791424 \\ 8.033409359 \\ 6.672977547 \\ 4.694882371 \\ 2.394640795 \end{bmatrix}.$$

Notice that operator F' given in (4.25) is not Lipschitz.

Example 4.4. Let $\mathcal{X} = \mathcal{Y} = C[0, 1]$, be the space of continuous functions on $[0, 1]$ equipped with the max-norm. Let $\mathcal{D} = \{x \in C[0, 1] : \|x\| \leq R\}$, $R > 0$. Define F on D by

$$F(x)(s) = x(s) - f(s) - d\int_0^1 P(s, t)x(t)^3 dt,\ c \in C[0, 1], s \in [0, 1],$$

where $f \in C[0, 1]$ is a given function, d is a real constant and the kernel P is the Green's function. In this case, for each $x \in D$, the Fréchet-derivative F' is a linear operator defined by

$$[F'(x)(v)](s) = v(s) - 3s\int_0^1 P(s, t)x(t)^2 v(t)dt,\ v \in C[0, 1],\ s \in [0, 1].$$

Choose $x_0(s) = f(s) = 1$, to obtain $\|I - F'(x_0)\| \leq \frac{3|d|}{8}$. Hence, $|d| < \frac{8}{3}$, $F'(x_0)^{-1}$ is defined and

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8 - 3|d|},\ \|F(x_0)\| \leq \frac{|d|}{9},\ \eta = \|F'(x_0)^{-1}F(x_0)\| \leq \frac{|d|}{8 - 3|d|}.$$

Consider the case of Newton's method. Choosing $d = 1.0$ and $R = 3$, we have $\eta = 0.2$, $\gamma(t) = 2.6t$, $\alpha(t) = 2.28t$ and $\tilde{\gamma}(t) = 3.8t$, $\tilde{\alpha}(t) = 3.8t$. Then, the old equation given in Remark 2.1 has no positive solution, but using the equation in (a4) of Theorem 2.3, we see that $r_0 = 0.01$. Hence, the conclusions of Theorem 2.3 hold in this case.

5. Conclusion

We presented a local as well as a semi-local convergence analysis of frozen Newton-like methods for generating a sequence approximating Banach space valued equations. This method specializes to many popular methods. If the starting inverse exists, then the method is always well defined. Using our idea of the restricted convergence domain, we provide a more precise domain where the iterates lie. Hence, the majorant functions involved are at least as tight as in previous studies. This way, the convergence criteria are at least as weak; the convergence domain is enlarged; the error bounds on the distances $\|x_n - p\|$, $\|x_{n+1} - x_n\|$ are tighter and the information on the location of the solution is at least as precise. The results reduce to earlier ones, if only one majorant condition is used. Moreover, the differentiability of operator F is not assumed or implied as in previous works, making method (1.3) suitable for solving systems of equations.

References

[1] I.K. Argyros, A.A. Magreñán, Iterative Methods and their Dynamics with Applications, CRC Press, New York, USA, 2017.
[2] J.A. Ezquerro, M.A. Hernández, Newton'S Method: An Updated Approach of Kantorovich'S Theory, Birkhauser, Elsevier, Cham, Switzerland, 2017.
[3] J.F. Traub, Iterative Methods for the Solutions of Equations, Prentice-Hall, Englewood Cliffs, New Jersey,USA, 1964.
[4] C.H. Hald, On a Newton-Moser type method, Numer. Math. 23 (1975) 411–426.
[5] I.K. Argyros, S. Hilout, Weaker conditions for the convergence of Newton's method, J. Complexity 28 (2012) 364–387.
[6] S. Amat, S. Busquier, A. Grau, M. Grau-Sánchez, Maximum efficiency for a family of Newton-like methods with frozen derivatives and some applications, Appl. Math. Comput. 219 (2013) 7954–7963.
[7] S. Amat, I.K. Argyros, S. Busquier, M.A. Hernández, On two high-order families of frozen Newton-type methods, Numer. Linear Algebra Appl. 25 (2018) 1–13 e2126.
[8] I.K. Argyros, A unifying local semi-local convergence analysis and applications for two-point Newton-like methods in Banach space, J. Math. Anal. Appl. 298 (2) (2004) 374–397.
[9] M.A. Hernández, M.J.A. Rubio, A uniparameteric family of iterative processes for solving nondifferentiable equations, J. Math. Anal. Appl. 275 (2002) 821–834.
[10] M.A. Hernández, M.J. Rubio, J.A. Ezquerro, Solving a special case of conservative problems by Secant-like methods, Appl. Math. Comput. 169 (2) (2005) 926–9420.