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Product and factorization of hypo-EP operators

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Abstract: In this article, we derive some necessary and sufficient conditions for the product of hypo-*EP* operators to be hypo-*EP* and we characterize hypo-*EP* operators through factorizations.

Keywords: Hypo-*EP* operator, *EP* operator

MSC: Primary 47A05, 47B20.

1 Introduction

A square matrix A over the complex field is said to be an *EP* matrix if the column spaces of A and A^* are equal. The notion of *EP* matrix was introduced in 1950 by Schwerdtfeger [13]. A few years later, in 1966, Pearl [11] gave a characterization of *EP* matrix through the Moore-Penrose inverse: A square matrix A is an *EP* matrix if and only if A commutes with its Moore-Penrose inverse $A^\dagger$. Using the Pearl's characterization, Campbell and Meyer [3] extended the notion of *EP* matrix to bounded operator with a closed range defined on a Hilbert space. A bounded operator A having a closed range is said to be an *EP* operator if the ranges of A and A^* are equal [3]. Itoh [7] introduced hypo-*EP* operator by weakening the Pearl's characterization: $A^\dagger A - AA^\dagger$ is a positive operator. Hypo-*EP* operator is our focus of attention in this paper and it has been studied in [7, 9, 14].

Throughout this paper, given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denotes the set of all operators, i.e., bounded and linear maps, from $\mathcal{H}$ to $\mathcal{K}$, and we write $\mathcal{B}(\mathcal{H}, \mathcal{H}) = \mathcal{B}(\mathcal{H})$. The class $\mathcal{B}_c(\mathcal{H})$ denotes the set of all operators in $\mathcal{B}(\mathcal{H})$ having closed ranges. For any operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and kernel of A respectively. Given $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is the adjoint operator of A if $\langle Ax, y\rangle = \langle x, By\rangle$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$; in this case the operator B is denoted by A^*. An operator A in $\mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. For any nonempty set $\mathcal{M}$ in $\mathcal{H}$, $\mathcal{M}^\perp$ denotes the orthogonal complement of $\mathcal{M}$. Note that if $A \in \mathcal{B}_c(\mathcal{H})$, then $A^* \in \mathcal{B}_c(\mathcal{H})$, $\mathcal{N}(A)^\perp = \mathcal{R}(A^*)$, $\mathcal{N}(A^*)^\perp = \mathcal{R}(A)$ and $\mathcal{R}(A) = \mathcal{R}(AA^*)$.

In section 2, we give some known characterizations of hypo-*EP* operators and few results which will be used in the sequel. Section 3 deals with a problem of finding conditions, necessary or sufficient or both, such that the product of hypo-*EP* operators is again a hypo-*EP* operator. Finally we conclude the section 4 with few characterizations of hypo-*EP* operators through factorizations.

2 Preliminaries

We start with some known characterizations of hypo-*EP* operators.

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Theorem 1. *[7, 14] Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following statements are equivalent.*
1. *A is hypo-EP ;*
2. *$\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$;*
3. *$\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$;*
4. *$A = A^*C$, for some $C \in \mathcal{B}(\mathcal{H})$;*
5. *for each $x \in \mathcal{H}$, there exists $k > 0$ such that $|\langle Ax, y\rangle| \leq k\|Ay\|$, for all $y \in \mathcal{H}$.*

Example 2. *Let $A : \ell_2 \to \ell_2$ be defined by $A(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$. Then $A^*(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$. Here $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$ and $\mathcal{R}(A)$ is closed. Hence A is a hypo-EP operator. Whereas the operator B on ℓ_2 defined by $B(x_1, x_2, \dots) = (x_2, 0, x_4, 0, \dots)$ is not a hypo-EP operator but it has a closed range.*

Remark 3. *The class of all hypo-EP operators contains the class of all EP operators. Hence it contains all normal, self-adjoint and invertible operators having closed ranges. In the case of finite dimensional settings, EP and hypo-EP are the same.*

3 Product of Hypo-*EP* Operators

Every hypo-*EP* operator is necessarily an operator with a closed range. There is an example in [1] for a bounded operator A in $\mathcal{B}_c(\mathcal{H})$ such that $A^2 \notin \mathcal{B}_c(\mathcal{H})$. But it has been observed that if A is hypo-*EP*, then A^2 has a closed range always. Moreover, any natural power of A has a closed range [9, 14]. We first derive few results on product of operators with closed ranges to analyze closed rangeness of "product of hypo-*EP* operators." We use the notion of angle between a pair of subspaces in a Hilbert space and give some of the basic results.

Definition 4. *[4] Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces of a Hilbert space $\mathcal{H}$. The angle between $\mathcal{M}$ and $\mathcal{N}$ is the angle $\alpha(\mathcal{M}, \mathcal{N})$ in $[0, \pi/2]$ whose cosine is defined by*

$$c(\mathcal{M}, \mathcal{N}) \quad = \quad \sup\left\{|\langle x, y\rangle| : x \in \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^{\perp}, \|x\| \leq 1, \qquad y \in \mathcal{N} \cap (\mathcal{M} \cap \mathcal{N})^{\perp}, \|y\| \leq 1\right\}.$$

We list some consequences of the definition of angle and a result pertaining to the product of operators with a closed range.

Theorem 5. *[4] Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces of a Hilbert space $\mathcal{H}$. Then*
1. *$0 \leq c(\mathcal{M}, \mathcal{N}) \leq 1$.*
2. *$c(\mathcal{M}, \mathcal{N}) = c(\mathcal{N}, \mathcal{M})$ ("Symmetry").*
3. *$|\langle x, y\rangle| \leq c(\mathcal{M}, \mathcal{N})\|x\|\|y\|$, for all $x \in \mathcal{M}$ and $y \in \mathcal{N}$, and at least one of x or y is in $(\mathcal{M} \cap \mathcal{N})^{\perp}$.*
4. *$c(\mathcal{M}, \mathcal{N}) = 0$ if and only if the orthogonal projection onto $\mathcal{M}$ commutes with the orthogonal projection onto $\mathcal{N}$.*
5. *$c(\mathcal{M}, \mathcal{N}) = c(\mathcal{M}^{\perp}, \mathcal{N}^{\perp})$.*

Theorem 6. *[4] Let A and B be bounded operators on $\mathcal{H}$ with closed ranges. Then the following statements are equivalent.*
1. *AB has a closed range ;*
2. *$c(\mathcal{R}(B), \mathcal{N}(A)) < 1$;*
3. *$\mathcal{R}(B) + \mathcal{N}(A)$ is closed.*

The following example illustrates the fact that there are operators A and B in $\mathcal{B}_c(\mathcal{H})$ such that $AB \in \mathcal{B}_c(\mathcal{H})$ but $BA \notin \mathcal{B}_c(\mathcal{H})$. We shall prove that when A and B are *EP* operators, the closed rangeness of AB implies the closed rangeness of BA and vice-versa.

Example 7. *[12] Let A be an operator on ℓ_2 defined by $A(x_1, x_2, x_3, \ldots) = (x_1, 0, x_2, 0, \ldots)$ and B be another operator on ℓ_2 defined by $B(x_1, x_2, x_3, \ldots) = (\frac{x_1}{1} + x_2, \frac{x_3}{3} + x_4, \frac{x_5}{5} + x_6, \ldots)$. One can verify that both A and B are bounded operators and are having closed ranges. Also, $\mathcal{R}(AB)$ is closed but $\mathcal{R}(BA)$ is not closed.*

Theorem 8. *Let A and B be EP operators on $\mathcal{H}$. Then $\mathcal{R}(AB)$ is closed if and only if $\mathcal{R}(BA)$ is closed.*

Proof. Suppose that $\mathcal{R}(AB)$ is closed. Then by Theorem 6, $c(\mathcal{R}(B), \mathcal{N}(A)) < 1$. Now using Theorem 5, we get $c(\mathcal{R}(B)^{\perp}, \mathcal{N}(A)^{\perp}) = c(\mathcal{R}(A^*), \mathcal{N}(B^*)) < 1$. Since $\mathcal{R}(A) = \mathcal{R}(A^*)$ and $\mathcal{N}(B) = \mathcal{N}(B^*)$, $c(\mathcal{R}(A), \mathcal{N}(B)) = c(\mathcal{R}(A^*), \mathcal{N}(B^*))$. Therefore $c(\mathcal{R}(A), \mathcal{N}(B)) < 1$. Hence $\mathcal{R}(BA)$ is closed. Converse part of this theorem can be proved similarly. □

Corollary 9. *Let A and B be hypo-EP operators on $\mathcal{H}$ such that $\mathcal{R}(A) \cap \mathcal{N}(B) = \{0\}$ and $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$. Then $\mathcal{R}(AB)$ is closed if and only if $\mathcal{R}(BA)$ is closed.*

Proof. The proof is similar to Theorem 8. □

We now discuss results for the product to be hypo-*EP* if either A or B is hypo-*EP*. We first give an example to show that product AB is not necessarily a hypo-*EP* operator even though A and B are hypo-*EP*.

Example 10. *Let A and B be operators on ℓ_2 defined by $A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$ and $B(x_1, x_2, x_3, \ldots) = (0, x_2, 0, x_4, \ldots)$. Both A and B are hypo-EP operators. Since $\mathcal{R}(AB) = \{(0, 0, x_1, 0, x_2, 0, \ldots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $\mathcal{R}((AB)^*) = \{(0, x_1, 0, x_2, 0, \ldots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, AB is not a hypo-EP operator.*

Theorem 11. *Let A be a hypo-EP operator and P be the orthogonal projection onto $\mathcal{R}(A)$. Then AP is a hypo-EP operator.*

Proof. Since A has a closed range, there is a $k > 0$ such that $\|Ax\| \geq k\|x\|$ for all $x \in \mathcal{N}(A)^{\perp}$. Now let us take $x \in \mathcal{N}(AP)^{\perp}$, then $x \in \mathcal{N}(P)^{\perp} = \mathcal{R}(P) = \mathcal{R}(A) \subseteq \mathcal{R}(A^*) = \mathcal{N}(A)^{\perp}$ and $Px = x$. Hence for $x \in \mathcal{N}(AP)^{\perp}$, we have $\|APx\| = \|Ax\| \geq k\|x\|$. Thus $\mathcal{R}(AP)$ is closed. Now $\mathcal{R}(AP) \subseteq \mathcal{R}(A) = P(\mathcal{R}(A)) \subseteq P(\mathcal{R}(A^*)) = \mathcal{R}(PA^*)$ which implies that AP is hypo-*EP*. □

Corollary 12. *Let A be an EP operator and P be the orthogonal projection onto $\mathcal{R}(A)$. Then AP is an EP operator.*

Proof. From the proof of the Theorem 11, we can say $\mathcal{R}(AP)$ is closed. Since P is the orthogonal projection onto $\mathcal{R}(A)$, $\mathcal{R}(AP) = \mathcal{R}(A) = P(\mathcal{R}(A)) = P(\mathcal{R}(A^*)) = \mathcal{R}(PA^*)$. Hence AP is *EP*. □

Theorem 13. *Let A be a hypo-EP operator and $B \in \mathcal{B}_c(\mathcal{H})$. If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$, then AB is hypo-EP.*

Proof. Since $\mathcal{R}(B)$ and $\mathcal{N}(A)$ are closed subspaces of $\mathcal{H}$, the angle between $\mathcal{R}(B)$ and $\mathcal{N}(A)$ is the angle $\alpha \in [0, \pi/2]$ whose cosine is defined by

$$\begin{aligned} c(\mathcal{R}(B), \mathcal{N}(A)) = \sup\Big\{ & |\langle x, y\rangle| : x \in \mathcal{R}(B) \cap (\mathcal{R}(B) \cap \mathcal{N}(A))^{\perp}, \|x\| \leq 1, \\ & y \in \mathcal{N}(A) \cap (\mathcal{R}(B) \cap \mathcal{N}(A))^{\perp}, \|y\| \leq 1 \Big\}. \end{aligned} \tag{1}$$

Since A is hypo-*EP*, $\mathcal{R}(B) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(A^*) = \mathcal{N}(A)^{\perp}$ and hence $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$, so (1) becomes

$$\begin{aligned} c(\mathcal{R}(B), \mathcal{N}(A)) &= \sup\{|\langle x, y\rangle| : x \in \mathcal{R}(B), \|x\| \leq 1, y \in \mathcal{N}(A), \|y\| \leq 1\} \\ &\leq \sup\{|\langle x, y\rangle| : x \in \mathcal{N}(A)^{\perp}, \|x\| \leq 1, y \in \mathcal{N}(A), \|y\| \leq 1\} \\ &= 0. \end{aligned}$$

Hence AB has a closed range. Since A is hypo-*EP*, $\mathcal{N}(B) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(A^*)$ and hence $\mathcal{R}(A) \subseteq \mathcal{R}(B^*)$. Now $\mathcal{R}(AB) = A(\mathcal{R}(B)) \subseteq A(\mathcal{R}(A)) \subseteq A(\mathcal{R}(A^*)) = \mathcal{R}(AA^*) = \mathcal{R}(A) \subseteq \mathcal{R}(B^*) = \mathcal{R}(B^*B) = B^*(\mathcal{R}(B)) \subseteq B^*(\mathcal{R}(A)) \subseteq B^*(\mathcal{R}(A^*)) = \mathcal{R}(B^*A^*)$. Hence AB is hypo-*EP*. □

Corollary 14. *Let A be a hypo-EP operator on $\mathcal{H}$. Then A^n is hypo-EP for any integer $n \geq 1$.*

Proof. The conditions in Theorem 13 are trivial when $A = B$. Hence A^2 is hypo-EP. Continuing this process, we get A^n is hypo-EP for any integer $n \geq 1$. □

Remark 15. *When A and B are EP matrices, the conditions $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ imply that A and B have the same range and null spaces, that is, $\mathcal{R}(A) = \mathcal{R}(B)$ and $\mathcal{N}(A) = \mathcal{N}(B)$. The following examples illustrate that there are hypo-EP operators A and B on an infinite dimensional Hilbert space such that the inclusion relation either in $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ or in $\mathcal{N}(B) \subseteq \mathcal{N}(A)$ is proper.*

Example 16. *Let A and B be operators on ℓ_2 defined by $A(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ and $B(x_1, x_2, \ldots) = (0, x_1, 0, x_2, \ldots)$. Here both A and B are hypo-EP operators. Also $\mathcal{R}(B) \subsetneq \mathcal{R}(A)$ and $\mathcal{N}(A) = \mathcal{N}(B) = \{0\}$.*

Example 17. *Let A and B be operators on ℓ_2 defined by $A(x_1, x_2, \ldots) = (x_1, 0, x_3, 0, \ldots)$ and $B(x_1, x_2, \ldots) = (x_1, 0, x_2, 0, \ldots)$. Even though both A and B are hypo-EP operators with $\mathcal{R}(A) = \mathcal{R}(B)$ but $\mathcal{N}(B) \subsetneq \mathcal{N}(A)$.*

Remark 18. *If one of the sufficient conditions in Theorem 13 is not true, then the product of hypo-EP operator and an operator with a closed range need not be a hypo-EP operator. The operators A and B given in Example 10 are hypo-EP operators and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ but AB is not hypo-EP. Note that $\mathcal{N}(B) \nsubseteq \mathcal{N}(A)$.*

Theorem 19. *Let A and B be EP operators on $\mathcal{H}$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. Then AB is EP if and only if $\mathcal{R}(AB^*) = \mathcal{R}(B^*A)$.*

Proof. Suppose that A and B are EP operators. Then the following equality relations are true.

$$\mathcal{R}(AB) = A(\mathcal{R}(B)) = A(\mathcal{R}(B^*)) = \mathcal{R}(AB^*) \text{ and}$$
$$\mathcal{R}(B^*A) = B^*(\mathcal{R}(A)) = B^*(\mathcal{R}(A^*)) = \mathcal{R}(B^*A^*).$$

Hence AB is EP if and only if $\mathcal{R}(AB^*) = \mathcal{R}(B^*A)$. □

Corollary 20. *Let A and B be EP operators on $\mathcal{H}$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. Then AB is hypo-EP if and only if $A(\mathcal{R}(B^*)) \subseteq B^*(\mathcal{R}(A))$.*

Corollary 21. *Let A and B be hypo-EP operators on $\mathcal{H}$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. If*

$$A(\mathcal{R}(B^*)) \subseteq B^*(\mathcal{R}(A)), \tag{2}$$

then AB is hypo-EP.

Proposition 22. *Let $A \in \mathcal{B}(\mathcal{H})$ be hypo-EP and $B \in \mathcal{B}(\mathcal{H})$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. If there is a $k > 0$ such that*

$$\|Ax\| \leq k\|ABx\| \text{ for all } x \in \mathcal{H} \tag{3}$$

then AB is hypo-EP.

Proof. Let $x \in \mathcal{H}$. Since A is hypo-EP, for $ABx \in \mathcal{R}(A)$ there exists $z \in \mathcal{H}$ such that $ABx = A^*z$. Hence for each $y \in \mathcal{H}$,

$$|\langle ABx, y\rangle| = |\langle A^*z, y\rangle| = |\langle z, Ay\rangle| \leq \|z\|\|Ay\| \leq k\|z\|\|ABy\|. \tag{4}$$

Take $\ell = k\|z\|$. Therefore for each $x \in \mathcal{H}$, there exists $\ell > 0$ such that $|\langle ABx, y\rangle| \leq \ell\|ABy\|$ for all $y \in \mathcal{H}$. Hence by Theorem 1, AB is hypo-EP. □

Remark 23. *The condition (3) is equivalent to $\mathcal{N}(AB) \subseteq \mathcal{N}(A)$. Also this condition is not necessary for AB to be hypo-EP. For example $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Here $\mathcal{N}(AB) \nsubseteq \mathcal{N}(A)$. But A, B and AB are all hypo-EP.*

Proposition 24. *Let $A \in \mathcal{B}_c(\mathcal{H})$ and B be hypo-EP operator. If $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and A is injective, then AB is hypo-EP.*

Proof. Since B is hypo-*EP*, by Theorem 1, for each $x \in \mathcal{H}$, there is $k_1 > 0$ such that $|\langle Bx, y\rangle| \leq k_1\|By\|$ for all $y \in \mathcal{H}$. Let $x \in \mathcal{H}$. Since $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $ABx \in \mathcal{R}(A)$, there exists $x' \in \mathcal{H}$ such that $ABx = Bx'$. Hence for each $y \in \mathcal{H}$,

$$|\langle ABx, y\rangle| = |\langle Bx', y\rangle| \leq k_1\|By\|. \tag{5}$$

Since A is injective and $\mathcal{R}(A)$ is closed, there exists $k_2 > 0$ such that $\|ABy\| \geq k_2\|By\|$ for all $y \in \mathcal{H}$. Therefore $|\langle ABx, y\rangle| \leq k_1\frac{1}{k_2}\|ABy\|$ for all $y \in \mathcal{H}$. Hence AB is hypo-*EP*. □

4 Factorizations of Hypo-*EP* Operators

In this section we give some characterizations of hypo-*EP* operators through factorizations. Pearl [10] showed that a matrix A is *EP* if and only if A can be expressed as $U(B \oplus 0)U^*$ with U unitary and B an invertible matrix. Drivalliaris [5] extended the results to *EP* operators on Hilbert spaces. Here we extend the results to hypo-*EP* operators on Hilbert spaces. We extend Pearl's characterizations of matrices to hypo-*EP* operators through factorizations. The direct sum of linear operators A and B is denoted by $A \oplus B$. One may refer section 1.8 in [8] for more details about direct sum of linear operators.

Lemma 25. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{B}_c(\mathcal{H})$ and $B \in \mathcal{B}_c(\mathcal{K})$. Then $A \oplus B$ is hypo-EP if and only if A and B are hypo-EP.*

Proof. Suppose that $A \oplus B$ is hypo-*EP* and $x \in \mathcal{N}(A)$. Then $(x, 0) \in \mathcal{N}(A \oplus B) \subseteq \mathcal{N}(A^* \oplus B^*)$ and $x \in \mathcal{N}(A^*)$. Hence A is hypo-*EP*. Similarly B is also hypo-*EP*. Conversely, suppose that A, B are hypo-*EP* and $(x, y) \in \mathcal{N}(A \oplus B)$, then $Ax = 0$ and $By = 0$. This implies $A^*x = 0$, $B^*y = 0$. Hence $(x, y) \in \mathcal{N}(A^* \oplus B^*)$. Therefore $A \oplus B$ is hypo-*EP*. □

Lemma 26. *Let $A \in \mathcal{B}_c(\mathcal{H})$, $B \in \mathcal{B}_c(\mathcal{K})$ and $U \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ injective such that $A = UBU^*$. Then A is hypo-EP if and only if B is hypo-EP.*

Proof. Suppose that B is hypo-*EP* and $x \in \mathcal{N}(A)$. Then $UBU^*x = 0$. Since U is injective, $BU^*x = 0$ implies that $B^*U^*x = 0$ (B is hypo-EP), which in turn implies that $UB^*U^*x = 0$, equivalently $x \in \mathcal{N}(A^*)$. Hence A is hypo-*EP*. Conversely, suppose that A is hypo-*EP* and $x \in \mathcal{N}(B)$. Therefore $Bx = 0$. Since U is injective, U^* is surjective. Hence for $x \in \mathcal{K}$ there exists $y \in \mathcal{H}$ such that $U^*y = x$. Therefore $BU^*y = 0$ implies that $UBU^*y = Ay = 0$. Since A is hypo-*EP*, $A^*y = UB^*U^*y = 0$. Using injectivity of U and $U^*y = x$, we get $x \in \mathcal{N}(B^*)$. Hence B is hypo-*EP*. □

Theorem 27. *Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent.*

1. *A is hypo-EP ;*
2. *There exist Hilbert spaces $\mathcal{K}_1$ and $\mathcal{L}_1$, $U_1 \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{L}_1, \mathcal{H})$ unitary and $B_1 \in \mathcal{B}(\mathcal{K}_1)$ injective such that $A = U_1(B_1 \oplus 0)U_1^*$;*
3. *There exist Hilbert spaces $\mathcal{K}_2$ and $\mathcal{L}_2$, $U_2 \in \mathcal{B}(\mathcal{K}_2 \oplus \mathcal{L}_2, \mathcal{H})$ isomorphism and $B_2 \in \mathcal{B}(\mathcal{K}_2)$ injective such that $A = U_2(B_2 \oplus 0)U_2^*$;*
4. *There exist Hilbert spaces $\mathcal{K}_3$ and $\mathcal{L}_3$, $U_3 \in \mathcal{B}(\mathcal{K}_3 \oplus \mathcal{L}_3, \mathcal{H})$ injective and $B_3 \in \mathcal{B}(\mathcal{K}_3)$ injective such that $A = U_3(B_3 \oplus 0)U_3^*$.*

Proof. It is enough to prove ($1 \Rightarrow 2$) and ($4 \Rightarrow 1$). All other implications follow trivially. Let $\mathcal{K}_1 = \mathcal{R}(A^*)$ and $\mathcal{L}_1 = \mathcal{N}(A)$. Define $U_1 : \mathcal{K}_1 \oplus \mathcal{L}_1 \to \mathcal{H}$ by $U_1(y, z) = y + z$ for $y \in \mathcal{R}(A^*)$, $z \in \mathcal{N}(A)$. Direct calculation shows that $U_1^*x = (P_{\mathcal{R}(A^*)}x, P_{\mathcal{N}(A)}x)$, for all $x \in \mathcal{H}$ and U_1 is unitary. Take $B_1 = A|_{\mathcal{R}(A^*)} : \mathcal{R}(A^*) \to \mathcal{R}(A^*)$ which is injective. Since $AP_{\mathcal{R}(A^*)} = A$, $A = U_1(B_1 \oplus 0)U_1^*$. Hence the implication $(1 \Rightarrow 2)$ is proved. Lemma 25 and Lemma 26 give $(4 \Rightarrow 1)$. □

Theorem 28. *Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent.*
1. *A is hypo-EP ;*
2. *There exist Hilbert spaces $\mathcal{K}_1$ and $\mathcal{L}_1$, $V_1 \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{L}_1, \mathcal{H})$ injective, $W_1 \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{L}_1, \mathcal{H})$, $S_1 \in \mathcal{B}(\mathcal{H}, \mathcal{K}_1 \oplus \mathcal{L}_1)$, $B_1 \in \mathcal{B}(\mathcal{K}_1)$ injective and $C_1 \in \mathcal{B}(\mathcal{K}_1)$ such that $A = V_1(B_1 \oplus 0)S_1$ and $A^* = W_1(C_1 \oplus 0)S_1$.*

Proof. Suppose that A is hypo-EP. Then (2) follows from Theorem 27. Now assume (2), then from $A = V_1(B_1 \oplus 0)S_1$ and injectivity of V_1 and B_1, we get $\mathcal{N}(A) = S_1^{-1}(\{0\} \oplus \mathcal{L}_1)$. From $A^* = W_1(C_1 \oplus 0)S_1$, we get $S_1^{-1}(\{0\} \oplus \mathcal{L}_1) \subseteq \mathcal{N}(A^*)$. Therefore $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$. Hence A is hypo-EP. □

Theorem 29. *Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent.*
1. *A is hypo-EP ;*
2. *There exist Hilbert spaces $\mathcal{K}_1$ and $\mathcal{L}_1$, $U_1 \in \mathcal{B}(\mathcal{K}_1 \oplus \mathcal{L}_1, \mathcal{H})$ isomorphism, $B_1 \in \mathcal{B}(\mathcal{K}_1)$ injective and $C_1 \in \mathcal{B}(\mathcal{K}_1)$ such that $A = U_1(B_1 \oplus 0)U_1^{-1}$ and $A^* = U_1(C_1 \oplus 0)U_1^{-1}$.*

Proof. Suppose that A is hypo-EP. Then (2) follows from Theorem 27. The proof of $(2 \Rightarrow 1)$ follows from the proof $(2 \Rightarrow 1)$ of Theorem 28. □

Definition 30. *[6] If $A \in \mathcal{B}_c(\mathcal{H}, \mathcal{K})$, then $A^\dagger$ is the unique linear operator in $\mathcal{B}_c(\mathcal{K}, \mathcal{H})$ satisfying*
1. $AA^\dagger A = A$
2. $A^\dagger AA^\dagger = A^\dagger$
3. $AA^\dagger = (AA^\dagger)^*$
4. $A^\dagger A = (A^\dagger A)^*$.

The operator $A^\dagger$ is called the Moore-Penrose inverse of A.

Next we are going to prove another characterization through the factorization of the form $A = BC$ which involves the Moore-Penrose inverse of an operator. Let $A \in \mathcal{B}_c(\mathcal{H})$. Then $A = A|_{\mathcal{R}(A^*)}P_{\mathcal{R}(A^*)}$, where $A|_{\mathcal{R}(A^*)}$ is the restriction of the operator A to $\mathcal{R}(A^*)$ and $P_{\mathcal{R}(A^*)}$ is the projection onto $\mathcal{R}(A^*)$. Here $B = A|_{\mathcal{R}(A^*)}$ and $C = P_{\mathcal{R}(A^*)}$ in the factorization $A = BC$. Also, B is an injective operator with a closed range and C is a surjective operator. The factorization of the form $A = BC$ is not unique because of the following reason.

Suppose that $U \in \mathcal{B}(\mathcal{K}, \mathcal{R}(A^*))$ is an isomorphism, $BU \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is injective with a closed range and $U^{-1}C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is surjective. Thus $A = (BU)(U^{-1}C)$ is also a factorization of the same type. Thus if $A \in \mathcal{B}_c(\mathcal{H})$, then there exists a Hilbert space $\mathcal{K}$ such that $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ injective and $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ surjective with $A = BC$. Therefore the factorization $A = BC$ is not unique. Moreover, $\mathcal{R}(A) = \mathcal{R}(B)$, $\mathcal{R}(A^*) = \mathcal{R}(C^*)$, $B^\dagger B = I_{\mathcal{K}}$, $CC^\dagger = I_{\mathcal{H}}$ and $A^\dagger = C^\dagger B^\dagger$.

Theorem 31. *[2] Let $A, B \in \mathcal{B}_c(\mathcal{H})$ such that $AB \in \mathcal{B}_c(\mathcal{H})$. Then $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.*

Theorem 32. *Let $A \in \mathcal{B}_c(\mathcal{H})$ and $A = BC$ be a factorization. Then the following are equivalent.*
1. *A is hypo-EP ;*
2. $C^\dagger C \geq BB^\dagger$;
3. $\mathcal{R}(B) \subseteq \mathcal{R}(C^*)$;
4. $B = C^\dagger CB$;
5. $B^\dagger = B^\dagger C^\dagger C$;
6. $AA^* = BCC^*B^*C^*(C^*)^\dagger$;
7. $A^*A = C^*B^*C^\dagger CBC$.

Proof. Since $A^\dagger = C^\dagger B^\dagger$, $CC^\dagger = I$ and $B^\dagger B = I$, A is hypo-EP if and only if $A^\dagger A \geq AA^\dagger$ if and only if $C^\dagger C \geq BB^\dagger$. Hence (1) and (2) are equivalent. The equivalence of (1) and (3) are trivial from the relation $\mathcal{R}(A) = \mathcal{R}(B), \mathcal{R}(A^*) = \mathcal{R}(C^*)$. Now assume $\mathcal{R}(B) \subseteq \mathcal{R}(C^*)$, then $B = P_{\mathcal{R}(B)}B = P_{\mathcal{R}(C^*)}B = C^\dagger CB$. Assume $B = C^\dagger CB$. Since the conditions for Theorem 31 are satisfied for $C^\dagger C$ and B, taking the Moore-Penrose inverse on both sides gives (5). Suppose that $B^\dagger = B^\dagger C^\dagger C$, then $\mathcal{N}(C) \subseteq \mathcal{N}(B^\dagger C^\dagger C) = \mathcal{N}(B^\dagger)$. Since $\mathcal{N}(B^\dagger) = \mathcal{N}(B^*)$, we have $\mathcal{N}(C) \subseteq \mathcal{N}(B^*)$. Hence $\mathcal{R}(B) \subseteq \mathcal{R}(C^*)$. Suppose that A is hypo-EP, then (6) and (7) follow from (4). Suppose that $AA^* = BCC^*B^*C^*(C^*)^\dagger$, then $\mathcal{N}(A) = \mathcal{N}(C) = \mathcal{N}(C^*)^\dagger \subseteq \mathcal{N}(AA^*)$. Since $\mathcal{N}(AA^*) = \mathcal{N}(A^*)$, we have $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$. Hence A is hypo-EP. Finally if $A^*A = C^*B^*C^\dagger CBC$, then $A^*A = A^*P_{\mathcal{R}(A^*)}A$. This implies $\|Ax\|^2 = \|P_{\mathcal{R}(A^*)}Ax\|^2$. Therefore $Ax = P_{\mathcal{R}(A^*)}Ax$ and hence $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$. Thus A is hypo-EP. □

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References

[1] Barnes, B.A.: Restrictions of bounded linear operators: closed range. Proc. Amer. Math. Soc. **135**(6), 1735–1740 (electronic) (2007). 10.1090/S0002-9939-06-08624-2. URL http://dx.doi.org/10.1090/S0002-9939-06-08624-2

[2] Bouldin, R.H.: Generalized inverses and factorizations, *Res. Notes in Math.*, vol. 66. Pitman, Boston, Mass.-London (1982)

[3] Campbell, S.L., Meyer, C.D.: EP operators and generalized inverses. Canad. Math. Bull **18**(3), 327–333 (1975)

[4] Deutsch, F.: The angle between subspaces of a Hilbert space. In: Approximation theory, wavelets and applications (Maratea, 1994), *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, vol. 454, pp. 107–130. Kluwer Acad. Publ., Dordrecht (1995)

[5] Drivaliaris, D., Karanasios, S., Pappas, D.: Factorizations of EP operators. Linear Algebra Appl. **429**(7), 1555–1567 (2008). 10.1016/j.laa.2008.04.026. URL http://dx.doi.org/10.1016/j.laa.2008.04.026

[6] Groetsch, C.W.: Generalized inverses of linear operators: representation and approximation. Res. Notes in Math. Marce Dekker, Inc., New York-Basel (1977). Monographs and Textbooks in pure and Applied Mathematics No. 37

[7] Itoh, M.: On some EP operators. Nihonkai Math. J. **16**(1), 49–56 (2005)

[8] Megginson, R. E.: An introduction to Banach space theory. Springer-Verlag, New York (1998)

[9] Patel, A.B., Shekhawat, M.P.: Hypo-EP operators. Indian J. Pure Appl. Math. **47**(1), 73–84 (2016). 10.1007/s13226-015-0168-x. URL http://dx.doi.org/10.1007/s13226-015-0168-x

[10] Pearl, M.: On normal and EP_r matrices. Michigan Math. J. **6**, 1–5 (1959). URL http://projecteuclid.org/euclid.mmj/1028998132

[11] Pearl, M.H.: On generalized inverses of matrices. Proc. Cambridge Philos. Soc. **62**, 673–677 (1966)

[12] Sam Johnson, P., Ganesa Moorthy, C.: Composition of operators with closed range. J. Anal. **14**, 79–80 (2006)

[13] Schwerdtfeger, H.: Introduction to Linear Algebra and the Theory of Matrices. P. Noordhoff, Groningen (1950)

[14] Vinoth, A., Sam Johnson, P.: On sum and restriction of hypo-EP operators. Funct. Anal. Approx. Comput. **9**(1), 37–41 (2017)