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ON ARITHMETIC GRAPHS

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A (p, q)-graph $G = (V, E)$ is said to be (k, d)-*arithmetic*, where k and d are positive integers if its p vertices admits a labeling of distinct non-negative integers such that the values of the edges obtained as the sums of the labels of their end vertices form the set $\{k, k + d, k + 2d, ..., k + (q - 1)d\}$. In this paper we prove that for all odd n, the generalized web graph $W(t, n)$ and some cycle related graphs are (k, d)-*arithmetic*. Also we prove that a class of trees called T_p-trees and subdivision of T_p-trees are $(k + q - 1)$ (d, d)-*arithmetic* for all positive integers k and d.

Key Words : Arithmetic Labelings; Arithmetic Graphs; Trees

1. INTRODUCTION

For all terminology and notation in graph theory we follow[6].

Graph labelings where the vertices are assigned values subject to certain conditions have been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as Coding theory, including the design of good radar type codes, synch-set codes, missle guidance codes and convolutional codes with optimal autocorrolation properties. They facilitate the optimal nonstandard encodings of integers.

Labeled graphs have also been applied in determining ambiguities in X-ray crystallographic analysis, to design a communication network addressing system, in determining optimal circuit layouts and radio astronomy problems etc. A systematic presentation of diverse applications of graph labelings is presented in[4].

Given a graph $G = (V, E)$, the set N of non-negative integers and a commutative binary operation $* : NXN \to N$, every vertex function $f: V(G) \to N$ induces an edge function $f^* : E(G) \to N$ such that $f^*(u\,v) = f(u) * f(v)$ for all $uv \in E(G)$.

Acharya and Hegde[2] have introduced the notion of (k, d)-arithmetic labelings of graphs. For a non-negative integer k and positive integer d, a (p, q)-graph $G = (V, E)$, a *(k, d)-arithmetic labeling* is an injective mapping $f: V(G) \to N$, where the induced edge function $f^+: E(G) \to \{k, k+d, k+2d, ..., k+(q-1)d\}$ such that $f^+(uv) = f(u) + f(v)$ for all $uv \in E(G)$ is also injective. If a graph G admits such a labeling then the graph G is called *(k, d)-arithmetic graph.*

We recall the following cycle related graphs.

1. The *wheel* $W_n = C_n + K_1$, where C_n cycle of length n.

2. The *helm* H_n, is the graph obtained from the wheel W_n by attaching a pendant edge at each vertex of the n-cycle.

3. The *web graph* $W(2, n)$, is the graph obtained by joining the pendant points of a helm H_n to form a cycle and then adding a single pendant edge to each vertex of the outer cycle.

4. The *generalized web graph* $W(t, n)$, is the graph obtained by iterating the process of constructing web graph $W(2, n)$ from the helm H_n, so that the web has t, n-cycles. (Fig. 1) (Yang's definition in[5]).

5. The *crown* $C_n \odot K_1$, is the graph obtained from a cycle C_n by attaching a pendant edge at each vertex of the cycle.

6. The *generalized web graph* without centre $W_0(t, n)$, is the graph obtained by removing the central vertex of $W(t, n)$.

7. We define the graph *generalized p-web* cone as $W_0(t, n) + \overline{K}_p$ where $\overline{K}_p$ is the complement of complete graph with p vertices.

2. Cycle Related Arithmetic Graphs

In this section, we prove that for all odd n, the cycle related graphs W_n, H_n, $W(2, n)$, $W(t, n)$, $C_n \odot K_1$, $W_0(t, n)$ and $W_0(t, n) + \overline{K}_p$ are (k, d)-arithmetic for $k = [(n-1)/2]d$ and for $k = [(3n-1)/2]\,d$.

***Theorem* 1** — *For all positive integer d and odd n, the generalized web graph W(t, n) is (k, d)-arithmetic for $k = [(n-1)/2]d$.*

PROOF : Label the vertices of $W(t, n)$ as follows :

Denote the vertices of the innermost cycle of $W(t, n)$ successively as $v_{1,1}, v_{1,2}, v_{1,3}, ..., v_{1,n}$. Then denote the vertices adjacent to $v_{1,1}, v_{1,2}, ..., v_{1,n}$ on the second cycle as $v_{2,1}, v_{2,2} ..., v_{2,n}$ respectively and the vertices adjacent to $v_{2,1}, v_{2,2} ..., v_{2,n}$ on the 3rd cycle as $v_{3,1}, v_{3,2}, ..., v_{3,n}$ and the vertices on the tth cycle as $v_{t,1}, v_{t,2}, ..., v_{t,n}$.

Next denote the pendant vertices adjacent to $v_{t,1}, v_{t,2}, ..., v_{t,n}$ as $v_{t+1}, 1, v_{t+1,2}, ..., v_{t+1,n}$ respectively and the centre of the web as $v_{0,0}$.

Define a labeling $f: V(W(t, n)) \to N$ such that

$$f(v_{m,i}) = \begin{cases} \left(\frac{i-1}{2}\right)d + (m-1)\,nd & m \text{ odd}, i \text{ odd}; \quad 1 \le m \le t+1, \quad 1 \le i \le n \\ \left(\frac{n-1}{2}\right)d + \left(\frac{i}{2}\right)d + (m-1)\,nd & m \text{ odd}, i \text{ even}; \quad 1 \le m \le t+1, \quad 2 \le i \le n-1 \\ \left(\frac{3n-1}{2}\right)d + \left(\frac{i-1}{2}\right)d + (m-2)\,nd & m \text{ even}, i \text{ odd}; \quad 2 \le m \le t+1, \quad 1 \le i \le n \\ (n-1)\,d + \left(\frac{i}{2}\right)d + (m-2)\,nd & m \text{ even}, i \text{ even}; \quad 2 \le m \le t+1, \quad 2 \le i \le n-1 \\ \left(\frac{n-1}{2}\right)d + 2t\,nd & m = i = 0. \end{cases} \quad \ldots (1)$$

Because f is injective on every cycle and the maximum vertex value in the m^{th} cycle is less than the minimum vertex value in the $(m + 1)^{\text{th}}$ cycle, it is not hard to verify that f defined above is injective.

Similarly one can see that $f^+(W(t, n)) = \{k, k + d, \ldots, k + [(2t + 1)n - 1]d\}$ where $k = [(n - 1)/2]d$. Hence $W(t, n)$ is (k, d)-*arithmetic* for $k = [(n - 1)/2]d$. ■

For example, (4, 1)-arithmetic labeling of W (4, 9) using Theorem 1 is shown in Fig. 1.

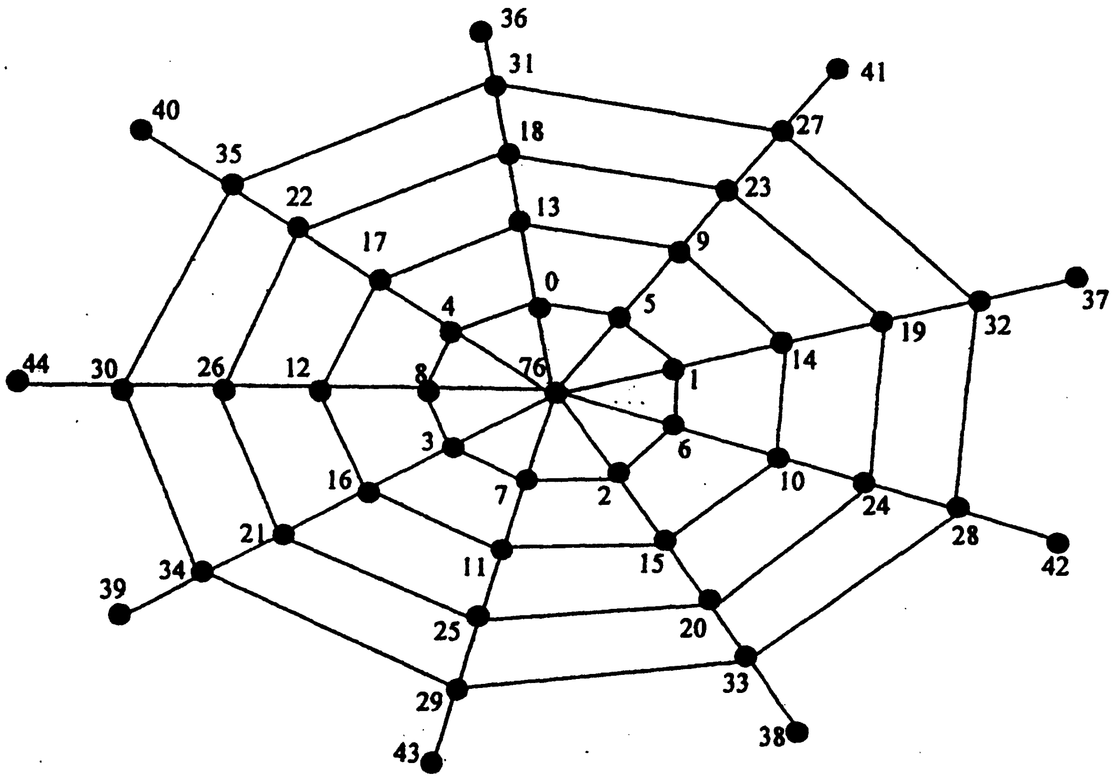


FIG. 1. (4, 1)-arithmetic labeling of $W(4, 9)$

Remark : For all positive integrs d and odd n, the generalized web graph $W(t, n)$, is (k, d)-arithmetic for $k = [(3n - 1)/2]d$. (Proof is analogous to Theorem 1). Therefore the value of k is not unique for $W(t, n)$, to be (k, d)-arithmetic.

For example, using $k = [(3n - 1) / 2]d$, a (13, 1)-arithmetic labeling of $W(4, 9)$ is shown in Fig. 2.

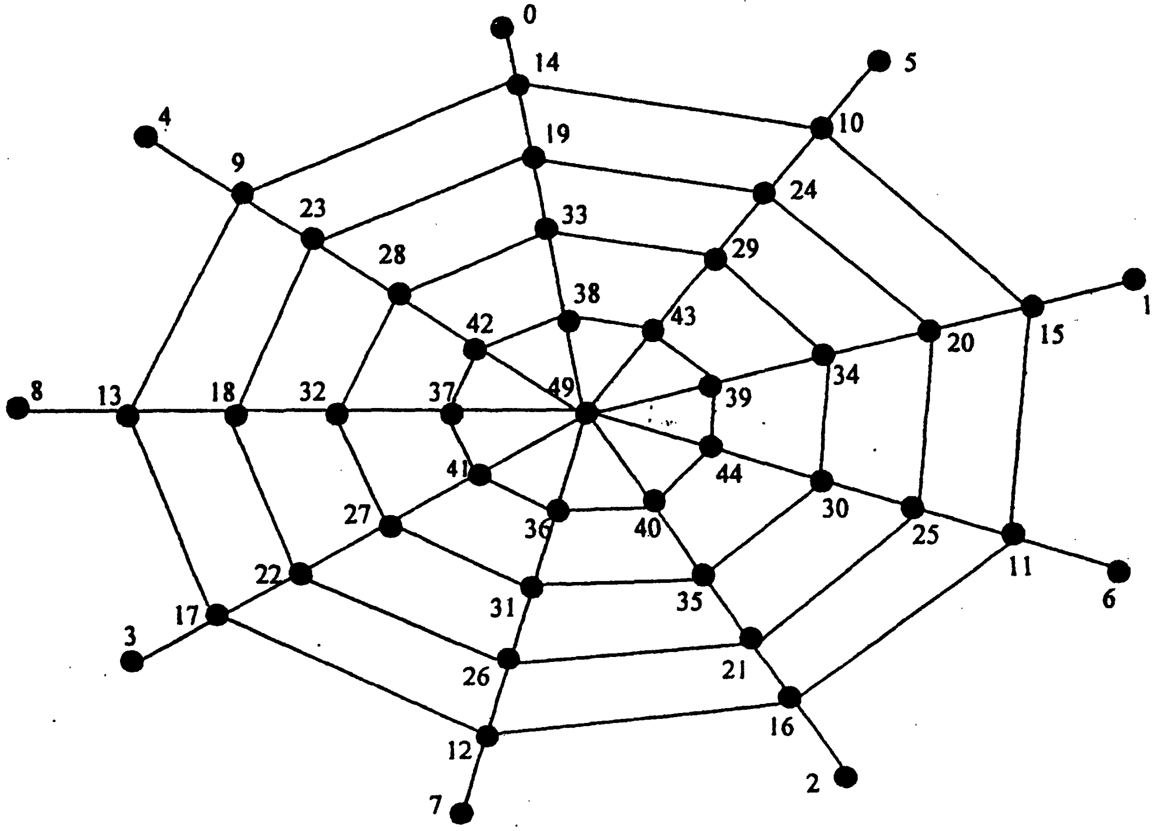


FIG. 2. (13, 1)-arithmetic labeling of $W(4, 9)$

Corollary 1 — For n odd, the helm H_n and the web graph $W(2, n)$ are (k, d)-arithmetic for $k = [(n - 1) / 2]d$.

PROOF : By taking $t = 1$ and $t = 2$ respectively in (1), the proof is trivial. ■

Corollary 2 — For n odd, the crown $C_n \odot K_1$ is (k, d)-arithmetic for $k = [(n - 1) / 2]d$.

PROOF : Taking $t = 1$ in (1) and removing the centre of the helm H_n the proof follows. ■

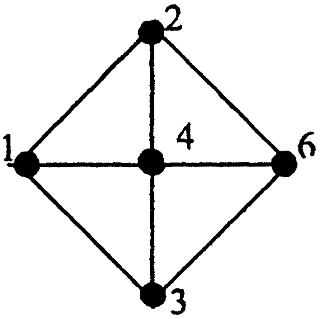
Strongly k-Indexable Graph

A (p, q) -graph $G = (V, E)$ is said to be *strongly k-indexable* if there exists a bijective labeling $f: V(G) \to \{0, 1, 2, ..., (p - 1)\}$ such that induced edge labeling $f^+: E(G) \to \{k, k + 1, k + 2, ..., k + (q - 1)\}$ is also bijective. (Acharya and Hegde[2]).

Corollary 3 — For n odd, the generalized web without centre $W_0(t, n)$, is strongly k-indexable for $k = (n - 1) / 2$. ■

Corollary 4 — For n odd, the generalized p-web cone $W_0(t, n) + \overline{K}_p$ is (k, d)-arithmetic for $k = [(n - 1)/2]\, d$. ■

We have examples for arithmetic wheels W_n and arithmetic helms H_n when n is even (Fig. 3). But whether the generalized web $W(t, n)$ is (k, d)-arithmetic or not when n even is an open problem.



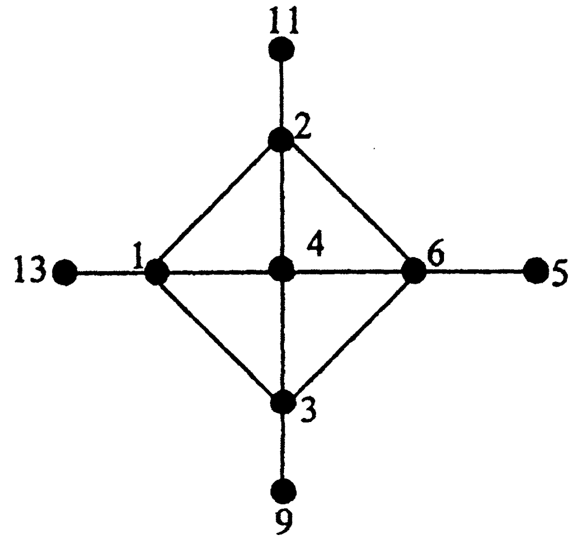


FIG. 3. (3, 1)-arithmetic labeling of W_4 *and* H_4

3. A CLASS OF ARITHMETIC TREES

In this section we prove that a class of trees called T_p-*trees* (*transformed trees*) are $(k+(q-1)d, d)$-*arithmetic* for all positive integers k and d. Also we prove that the subdivision $S(T)$ of a T_p-tree T, obtained by subdividing every edge of T exactly once is $(k + (q - l) d, d)$-*arithmetic* for all positive integers k and d. (Note that q is the number of edges of T and that the subdivision $S(T)$ of a T_p-tree T is not necessarily a T_p-tree)

Transformed Tres (T_p-trees)

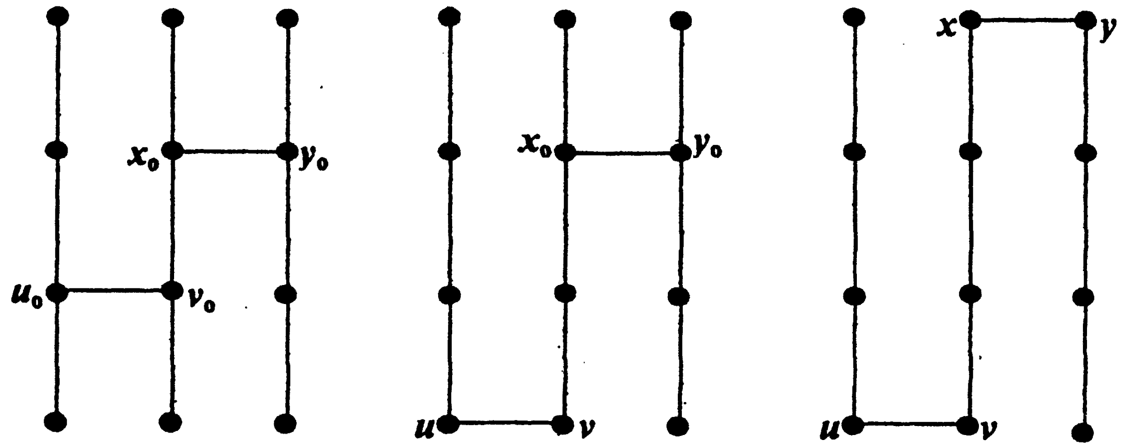
Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Let there be two pendant vertices u and v in T such that the length of u_0-u path is equal to the length of v_0-v path. If the edge $u_0 v_0$ is deleted from T and u and v are joined by an edge uv, then such a transformation of T is called an *elementary parallel transformation* (or an ept) and the edge $u_0 v_0$ is called a *transformable edge*. (Acharya[1]).

If by a sequence of ept's T can be reduced to a path then T is called a T_p-tree (*transformed tree*) and any such sequence regarded as a composition of mappings (ept's) denoted by P, is called a *parallel transformation of* T. The path, the image of T under P, is denoted as $P(T)$.

A T_p-tree and a sequence of two ept's reducing it to a path are illustrated in Fig. 4.

Theorem 2 — *Every T_p-tree T is $(k + (q - 1)d, d)$-arithmetic for all positive integers k and d.*

PROOF : Let T be a T_p-tree with $n + 1$ vertices. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$, we have



a)A T_P-tree *T*. *b*) An ept $P_1(T)$ *c*)Second ept $P_2(T)$

FIG. 4.

(i) $V(P\ (T)) = V(T)$

(ii) $E\,(P\,(T)) = (E\,(T) - E_d) \cup E_p$

where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, ..., P_k)$ of the ept's P_i used to arrive at the path $P(T)$. Clearly E_d and E_p have the same number of edges.

Then denote the vertices of $P(T)$ successively as $v_1, v_2, \ldots, v_p$ starting from one pendant vertex of $P(T)$ right up to other.

Define $f: V(P\,(T)) \to N$ by

$$f(v_i) = \begin{cases} [(i-1)/2]\, d & \text{for odd } i, \quad 1 \le i \le n+1 \\ k + (q-1)\,d + [(i-2)/2]d & \text{for even } \ i, 2 \le i \le n+1 \end{cases}$$

where k and d are positive integers and q is the number of edges of T. Clearly f is a $(k + (q - 1)\ d,\ d)$-arithmetic labeling of $P(T)$.

Let $v_i\,v_j$ be an edge in T for some indices i and j, $1 < i < j \le n+1$ and let P_1 be the *ept* obtained by deleting this edge and adding the edge $v_{i+t}\,v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent ept's. Since $v_{i+t}\,v_{j-t}$ is an edge in the path $P(T)$ it follows that $i + t + 1 = j - t \Rightarrow j = i + 2t + 1$. Therefore i and j are of opposite parity.

The value of the edge $v_i\,v_j$ is $f^+(v_i\,v_j) = f^+(v_i\,v_{i+2t+1})$

$$= f(v_i) + f(v_{i+2t+1}) \qquad \text{... (2)}$$

If i is odd and $1 \le i \le n$, then

$$f(v_i) + f(v_{i+2t+1}) = [(i-1)/2]\, d + k + (q-1)d + [(i+2t+1-2)/2]\, d$$

$$= k + (q-1)\, d + (i+t-1)\, d. \qquad \text{... (3)}$$

If i is even and $2 \leq i \leq n$, then

$$f(v_i) + f(v_{i+2t+1}) = k + (q-1)\, d + [(i-2)/2]\, d$$

$$+ \ [(i+2t+1-1)/2]\, d$$

$$= k + (q-1)\, d + (i+t-1)\, d \qquad \text{... (4)}$$

Therefore, from (2), (3), (4), we get

$$f^+(v_i\, v_j) = k + (q-1)\, d + (i+t-1)\, d. \text{ for all } i,\ 1 \leq i \leq n \qquad \text{... (5)}$$

The value of the edge $v_{i+t}\, v_{j-t}$ is

$$f^+(v_{i+t}\, v_{j-t}) = f(v_{i+t}) + f(v_{j-t})$$

$$= f(v_{i+t}) + f(v_{i+t+1}) \qquad \text{... (6)}$$

If $i + t$ is odd, then

$$f(v_{i+t}) + f(v_{i+t+1}) = [(i+t-1)/2]\, d + k + (q-1)\, d$$

$$+ \ [(i+t+1-2)/2]\, d$$

$$= k + (q-1)\, d + (i+t-1)\, d \qquad \text{... (7)}$$

If $i + t$ is even, then

$$f(v_{i+t} + f(v_{i+t+1}) = [(i+t+1-1)/2]\, d + k + (q-1)\, d + [(i+t-2)/2]\, d$$

$$= k + (q-1)\, d + (i+t-1)\, d \qquad \text{... (8)}$$

Therefore, from (6), (7), (8), we get

$$f^+(v_{i+t}\, v_{j-t}) = k + (q-1)\, d + (i+t-1)\, d \qquad \text{... (9)}$$

From (5) and (9), $f^+(v_i\, v_j) = f^+(v_{i+1}\, v_{j-t})$.

Hence, f is a $(k + (q-1)d,\ d)$ - arithmetic labeling of T. ■

For example, a (14, 1)-arithmetic labeling of a T_p-tree, using theorem 2 is shown in Fig. 5.

Theorem 3 — *If T is a T_p-tree with q edges then the subdivision tree $S(T)$ is $(k + (q - 1)d,\ d)$-arithmetic for all positive integers k and d.*

PROOF : Let T be a T_p-tree with n vertices and q edges. By the definition of a T_p-tree there exists a parallel transformation P of T so that we get $P(T)$. Denote the succession vertices of $P(T)$

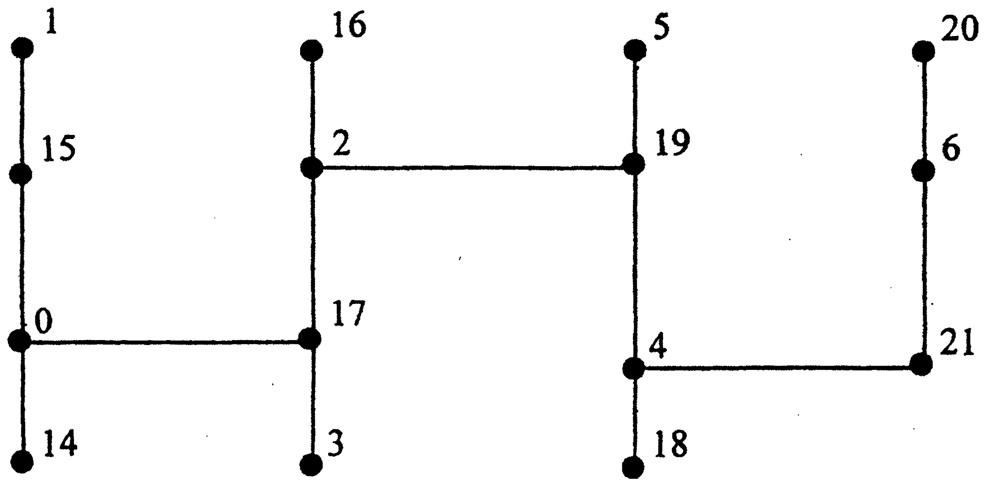


FIG. 5. (14, 1)-arithmetic labeling of a T_p-tree

as $v_1, v_2, \ldots, v_n$ starting from one pendant vertex of $P(T)$ right up to other and preserve the same for T.

Now construct the subdivision tree $S(T)$ of T by introducing exactly one vertex between every edge $v_i v_j$ of T and denote the vertex as $v_{i,j}$. Let $v_{m^x} v_{h^x}, x = 1, 2, \ldots, z$ be the z transformable edges of T with $m^x < m^x + 1$ for all x. Let t_x be the path length from the vertex v_{m^x} to the corresponding pendant vertex decided by the transformable edge $v_{m^x} v_{h^x}$ of T.

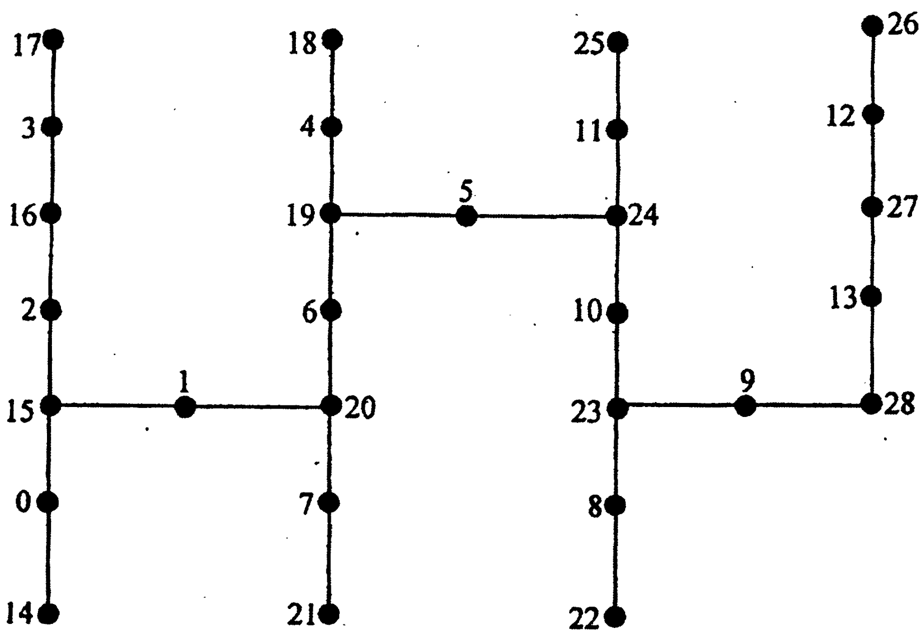


FIG. 6. (14, 1)-arithmetic labeling of subdivision of a T_p-tree

Define a labeling $f: V(S(T)) \to N$ by

$$f(v_i) = k + (q-1)d + (i-1)d \text{ if } i = 1, 2, \ldots, n.$$

$$f(v_{i,j}) = (i-1)d \text{ if } j \neq i+1.$$

$$f(v_{i,j}) = i\,d \text{ if } j = i + 1 \text{ and } i = m^c, m^c + 1, \ldots, m^c + t_c - 1,\ c = 1, 2, \ldots, z$$

$$f(v_{ij}) = +(i-1)\,d \text{ if } j = i + 1 \text{ and } i \neq m^c,\ m^c + 1, \ldots, m^c + t_c - 1,\ c = 1, 2, \ldots, z,$$

where k and d are positive integers and $2q$ is the number of edges of $S(T)$. Clearly f is a $(k + (q - 1)d,\ d)$-arithmetic labeling of $S(T)$. ■

For example a (14, 1)-arithmetic labeling of subdivision of a T_P-tree, using Theorem 3 is shown in Fig. 6.

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