Local convergence analysis of a modified Newton-Jarratt's composition under weak conditions

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Abstract. A. Cordero et. al (2010) considered a modified Newton-Jarratt's composition to solve nonlinear equations. In this study, using decomposition technique under weaker assumptions we extend the applicability of this method. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

Keywords: Newton-Jarratt's method; radius of convergence; local convergence; decomposition techniques; restricted convergence domain

Classification: 65D10, 65D99, 65J20, 49M15, 74G20, 41A25

1. Introduction

In this study, we consider the problem of approximating the solution x^* of nonlinear equation

$$H(x) = 0, \tag{1.1}$$

where $H\colon \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a continuous Fréchet-differentiable operator and Ω is a convex set. Newton-like methods are used widely for obtaining an approximation for the solution x^* of (1.1). Higher order multi-point methods are studied in the literature (see [1], [5]–[20]) for approximating the solution x^* of (1.1).

In the present paper, we consider the following construction considered in [7]

$$\begin{aligned} z_n &= x_n - \frac{2}{3}H'(x_n)^{-1}H(x_n)\\ y_n &= x_n - \frac{1}{2}(3H'(z_n) - H'(x_n))^{-1}(3H'(z_n) + H'(x_n))H'(x_n)^{-1}H(x_n),\\ x_{n+1} &= y_n - \frac{1}{2}(3H'(z_n) - H'(x_n))^{-1}H(y_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point for solving equation (1.1) when, $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i$, i a natural integer. Let $B(a, \varrho)$, $\overline{B}(a, \varrho)$ stand respectively for the open and closed balls in $\mathcal{B}_1$ with center $a \in \mathcal{B}_1$ and of radius $\varrho > 0$. The convergence analysis of iterative methods is usually divided into two categories: semilocal

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and local convergence analysis. The semilocal convergence matter is based on the information around an initial point, to give conditions ensuring the convergence of the method. On the other hand, the local convergence is based on the information around a solution, to find estimates on the radii of the convergence balls. We are concerned with the local convergence analysis of method (1.2) in this study.

Finding solutions for the equation (1.1) is an important problem in mathematics due to its wide applications. In [7] the existence of the Fréchet derivative of H of order up to four was used for the derivation of the convergence order although only the first derivative appears in method (1.2). This assumption on the higher order Fréchet derivatives of the operator H restricts the applicability of method (1.2) to problems where the fourth or higher order derivatives of H exist. For example consider the following:

Example 1.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, $\Omega = \overline{B}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [2], [17], [7]–[11], [15] defined by

$$x(s) = \int_0^1 G(s,t)\Big(x(t)^{3/2} + \frac{x(t)^2}{2}\Big)\,\mathrm{d}t,$$

where the kernel G is the Green function defined on the interval $[0,1]\times[0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.1), where $H\colon C[0,1] \longrightarrow C[0,1]$ is defined by

$$H(x)(s) = x(s) - \int_0^1 G(s,t)\Big(x(t)^{3/2} + \frac{x(t)^2}{2}\Big)\,\mathrm{d}t.$$

Notice that

$$\Big\| \int_0^1 G(s,t)\,\mathrm{d}t \Big\| \le \frac{1}{8}.$$

Then, we have that

$$H'(x)y(s) = y(s) - \int_0^1 G(s,t)\Big(\frac{3}{2}x(t)^{1/2} + x(t)\Big)\,\mathrm{d}t,$$

so since $H'(x^*(s)) = I$,

$$\|H'(x^*)^{-1}(H'(x) - H'(y))\| \le \frac{1}{8}\Big(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|\Big).$$

One can see that, higher order derivatives of H do not exist in this example. Therefore, there is no guarantee that method (1.2) converges under the assumptions in [7] although the method may converge under weaker assumptions.

Our goal is to weaken the assumptions in [7] using only hypotheses on the first derivative and apply the method for solving equation (1.1) in Banach spaces, so that the applicability of the method (1.2) can be extended. To estimate the order of convergence while bypassing the usage of high order derivatives, we use the computational order of convergence and the approximate order of convergence (see Remark 2.2 (d)). So, we find a ball containing the initial guesses and we know how many iterates are needed to obtain a desired error tolerance. Notice that in the studies using Taylor expansions these bounds are not available and the initial guess is a shot in the dark. No uniqueness results are either available in [7].

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result. Special cases and numerical examples are given in the last section.

2. Ball convergence

We introduce some functions and parameters for the ball convergence analysis of method (1.2). Let $w_0\colon [0,R_0) \longrightarrow \mathbb{R}_+$ be a continuous and non-decreasing function with $w_0(0)=0$. Define the parameter R_0 by

$$R_0 = \sup\{t \geq 0\colon w_0(t) < 1\}. \tag{2.1}$$

Let also $w\colon [0,R_0) \longrightarrow \mathbb{R}_+, v\colon [0,R_0) \longrightarrow \mathbb{R}_+$ be continuous and nondecreasing functions with $w(0)=0$. Moreover, define functions $\varphi_1, \psi_1, q, \psi_q$ on the interval $[0,R_0)$ by

$$\varphi_1(t) = \frac{\int_0^1 w((1-\theta)t)\,\mathrm{d}\theta + \frac{1}{3}\int_0^1 v(\theta t)\,\mathrm{d}\theta}{1-w_0(t)},$$
$$\psi_1(t) = \varphi_1(t) - 1,$$
$$q(t) = \frac{1}{2}(3w_0(\varphi_1(t)t) + w_0(t)),$$

and

$$\psi_q(t) = q(t) - 1.$$

Suppose that

$$v(0) < 3. \tag{2.2}$$

We have that $\psi_1(0) = -1 < 0$, $\psi_q(0) - 1 < 0$ and $\psi_1(t) \to \infty$, $\psi_q(t) \longrightarrow \infty$ as $t \to R_0^-$. Then, the intermediate value theorem guarantees the existence of zeros in the interval $(0,R_0)$ for functions ψ_1 and ψ_q. Let R_1, R_q stand for the smallest zeros of functions ψ_1 and ψ_q on the interval $(0,R_0)$, respectively. Furthermore, define functions φ_2 and ψ_2 on the interval $[0,R_q)$ by

$$\varphi_2(t) = \frac{\int_0^1 w((1-\theta)t)\,\mathrm{d}\theta}{1-w_0(t)} + \frac{3}{4}\,\frac{(w_0(\varphi 1(t)t) + w_0(t))\int_0^1 v(\theta t)\,\mathrm{d}\theta}{(1-q(t))(1-w_0(t))},$$

and

$$\psi_2(t) = \varphi_2(t) - 1.$$

We get that $\psi_2(0) = -1 < 0$ and $\psi_2(t) \longrightarrow \infty$ as $t \longrightarrow R_q^-$. Denote by R_2 the smallest zero of function ψ_2 on the interval $(0, R_q)$. Finally, define functions φ_3 and ψ_3 on the interval $[0, R_q)$ by

$$\varphi_3(t) = \left(1 + \frac{\int_0^1 v(\theta\varphi_2(t)t)\,\mathrm{d}\theta}{4(1-q(t))}\right)\varphi_2(t),$$

and

$$\psi_3(t) = \varphi_3(t) - 1.$$

We obtain that $\psi_3(0) = -1 < 0$ and $\psi_3(R_2) = \int_0^1 v(\theta R_2)\,\mathrm{d}\theta/(4(1-q(R_2))) > 0$. Denote by R_3 the smallest zero of function ψ_3 on the interval $(0, R_2)$. Define the radius of convergence R by

$$R = \min\{R_1, R_3\}. \tag{2.3}$$

Then, we have for each $t \in [0, R)$

$$0 \le \varphi_i(t) < 1, \qquad i = 1, 2, 3, \tag{2.4}$$

and

$$0 \le q(t) < 1. \tag{2.5}$$

We shall show next the ball convergence result for method (1.2) using the preceding notation.

Theorem 2.1. *Let $H\colon \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a continuously Fréchet-differentiable operator. Assume there exist $x^* \in \Omega$ and function $w_0\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ continuous non-decreasing such that for each $x \in \Omega$*

$$H(x^*) = 0, \qquad H'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \tag{2.6}$$

and

$$\|H'(x^*)^{-1}(H'(x) - H'(x^*)\| \le w_0(\|x - x^*\|), \tag{2.7}$$

there exist functions $w\colon [0, R_0) \longrightarrow \mathbb{R}_+, v\colon [0, R_0) \longrightarrow \mathbb{R}_+$ continuous nondecreasing with $w(0) = 0$ and v satisfying (2.2) such that for each $x, y \in \Omega_0 = \Omega \cap B(x^*, R)$

$$\|H'(x^*)^{-1}(H'(x) - H'(y)\| \le w(\|x - y\|), \tag{2.8}$$

$$\|H'(x^*)^{-1}H'(x)\| \le v(\|x - y\|), \tag{2.9}$$

and

$$\overline{B}(x^*, R) \subseteq \Omega, \tag{2.10}$$

where the radii R_0 *and* R *are given in* (2.1) *and* (2.3), *respectively. Then, the sequence* $\{x_n\}$ *generated for* $x_0 \in U(x^*, R) - \{x^*\}$ *by method* (1.2) *is well defined in* $U(x^*, R)$, *remains in* $U(x^*, R)$ *for each* $n = 0, 1, 2, \dots$ *and converges to* x^* *so that*

$$\|z_n - x^*\| \le \varphi_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < R, \tag{2.11}$$

$$\|y_n - x^*\| \le \varphi_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{2.12}$$

and

$$\|x_{n+1} - x^*\| \le \varphi_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{2.13}$$

where for $i = 1, 2, 3$ *the functions* φ_i *are defined previously. Moreover, if there exists* $R^* \ge R$ *such that*

$$\int_0^1 w_0(\theta R^*)\,\mathrm{d}\theta < 1, \tag{2.14}$$

then the limit point x^* *is the only solution of equation* $H(x) = 0$ *in* $\Omega_1 = \Omega \cap \overline{B}(x^*, R^*)$.

Proof: We shall show that point x_0 is well defined by the first substep of method (1.2) for $n = 0$. Using the hypothesis $x_0 \in B(x^*, R) - \{x^*\}$, (2.1), (2.3) and (2.7) we have in turn that

$$\|H'(x^*)^{-1}(H'(x_0) - H'(x^*)\| \le w_0(\|x_0 - x^*\|) \le w_0(R) < 1. \tag{2.15}$$

It follows by (2.15) and the Banach lemma on invertible operators [2], [10], [20] that $H'(x_0)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|H'(x_0)^{-1}H'(x^*)\| \le \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{2.16}$$

Hence, y_0 is well defined. By the first substep of method (1.2) for $n = 0$ we can write the identity

$$z_0 - x^* = x_0 - x^* - H'(x_0)^{-1}H(x_0) + \frac{1}{3}H'(x_0)^{-1}H(x_0). \tag{2.17}$$

In view of (2.3), (2.4) for $i = 1$, (2.6), (2.8), (2.16) and (2.17) we get in turn that

$$\begin{aligned}\|z_0 - x^*\| \le \|H'(x_0)^{-1}H'(x^*)\| &\\ &\times \left\| \int_0^1 H'(x^*)^{-1}(H'(x^* + \theta(x_0 - x^*)) - H'(x_0))(x_0 - x^*)\,\mathrm{d}\theta \right\| \\ &+ \frac{1}{3}\|H'(x_0)^{-1}H'(x^*)\|\|H'(x^*)^{-1}H(x_0)\|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\int_0^1 w((1-\theta)\|x_0-x^*\|)\,\mathrm{d}\theta\,\|x_0-x^*\|}{1-w_0(\|x_0-x^*\|)}\\
&\quad+\frac{1}{3}\frac{\int_0^1 v(\theta\|x_0-x^*\|)\,\mathrm{d}\theta\,\|x_0-x^*\|}{1-w_0(\|x_0-x^*\|)}\\
&=\varphi_1(\|x_0-x^*\|)\|x_0-x^*\|\le\|x_0-x^*\|<R,
\end{aligned}
\tag{2.18}
$$

which shows (2.11) for $n=0$ and $z_0\in B(x^*,R)$, where we also used the estimate

$$
\begin{aligned}
\|H'(x^*)^{-1}H(x_0)\|&=\left\|\int_0^1 H'(x^*)^{-1}H'(x^*+\theta(x_0-x^*))(x_0-x^*)\,\mathrm{d}\theta\right\|\\
&\le\int_0^1 v(\theta\|x_0-x^*\|)\,\mathrm{d}\theta\|x_0-x^*\|\qquad\text{by (2.9).}
\end{aligned}
\tag{2.19}
$$

Next, we must show that $(3H'(z_0)-H'(x_0))^{-1}\in L(\mathcal{B}_2,\mathcal{B}_1)$. In view of (2.3), (2.5) and (2.7) we obtain in turn that

$$
\begin{aligned}
&\|(2H'(x^*))^{-1}(3(H'(z_0)-H'(x^*))+(H'(x^*)-H'(x_0)))\|\\
&\quad\le\frac{1}{2}[3\|H'(x^*)^{-1}3(H'(z_0)-H'(x^*))\|\\
&\qquad+\|H'(x^*)^{-1}(H'(x^*)-H'(x_0))\|]\\
&\quad\le\frac{1}{2}(3w_0(\|z_0-x^*\|)+w_0(\|x_0-x^*\|))\\
&\quad\le\frac{1}{3}(3w_0(g_1(\|x_0-x^*\|)\|x_0-x^*\|)+w_0(\|x_0-x^*\|))\\
&\quad=q(\|x_0-x^*\|)\le q(R)<1,
\end{aligned}
$$

so

$$
\|(3H'(z_0)-H'(x_0))^{-1}H'(x^*)\|\le\frac{1}{2(1-q(\|x_0-x^*\|))}.
\tag{2.20}
$$

It also follows from the second and third substep of method (1.2), respectively that y_0 and x_1 are well defined. In particular, we can write

$$
\begin{aligned}
y_0-x^*&=x_0-x^*-H'(x_0)^{-1}H(x_0)\\
&\quad+\Big[I-\frac{1}{2}(3H'(z_0)-H'(x_0))^{-1}(3H'(z_0)\\
&\quad+H'(x_0))\Big]H'(x_0)^{-1}H(x_0)\\
&=x_0-x^*-H'(x_0)^{-1}H(x_0)\\
&\quad+\frac{3}{2}(3H'(z_0)-H'(x_0))^{-1}[(H'(z_0)-H'(x^*))\\
&\quad+(H'(x^*)-H'(x_0))]H'(x_0)^{-1}H(x_0).
\end{aligned}
\tag{2.21}
$$

Next, by using (2.3), (2.4) for $i = 2$, (2.16), (2.18), (2.20) and (2.21), we have in turn that

$$
\begin{aligned}
\|y_0 - x^*\| &\leq \|x_0 - x^* - H'(x_0)^{-1}H(x_0)\| \\
&\quad + \frac{3}{2}\|(3H'(z_0) - H'(x_0))^{-1}H'(x^*)\| \\
&\quad \times [\|H'(x^*)^{-1}(H'(z_0) - H'(x^*))\| \\
&\quad + \|H'(x^*)^{-1}(H'(x^*) - H'(x_0))\|] \\
&\quad \times \|H'(x_0)^{-1}H'(x^*)\|\|H'(x^*)^{-1}H(x_0)\| \\
&\leq \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)\,\mathrm{d}\theta}{1 - w_0(\|x_0 - x^*\|)}\|x_0 - x^*\| \\
&\quad + \frac{3(w_0(\|z_0 - x^*\|) + w_0(\|x_0 - x^*\|))}{4(1 - q(\|x_0 - x^*\|))} \\
&\quad \times \frac{\int_0^1 v(\theta\|x_0 - x^*\|)\,\mathrm{d}\theta\|x_0 - x^*\|}{(1 - w_0(\|x_0 - x^*\|))} \\
&\leq \varphi_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < R,
\end{aligned}
\tag{2.22}
$$

which shows (2.12) for $n = 0$ and $y_0 \in B(x^*, R)$. Then from the third substep of method (1.2) for $n = 0$, (2.3), (2.4) for $i = 3$, (2.18), (2.19) for $x_0 = y_0$ and (2.22), we get in turn that

$$
\begin{aligned}
\|x_1 - x^*\| &\leq \|y_0 - x^*\| \\
&\quad + \frac{1}{2}\|(3H'(z_0) - H'(x_0))^{-1}H'(x^*)\|\|H'(x^*)^{-1}H(y_0)\| \\
&\leq \|y_0 - x^*\|) + \frac{\int_0^1 v(\theta\|y_0 - x^*\|)\|y_0 - x^*\|\,\mathrm{d}\theta}{4(1 - q(\|x_0 - x^*\|))} \\
&\leq \left(1 + \frac{\int_0^1 v(\theta\varphi_2(\|x_0 - x^*\|)\|x_0 - x^*\|)\,\mathrm{d}\theta}{4(1 - q(\|x_0 - x^*\|))}\right) \\
&\quad \times \varphi_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= \varphi_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < R,
\end{aligned}
\tag{2.23}
$$

which shows (2.13) for $n = 0$ and $x_1 \in B(x^*, R)$. By simply replacing x_0, z_0, y_0, x_1 by x_k, z_k, y_k, x_{k+1} in the preceding estimates, we arrive at estimates (2.11)–(2.13). Then, from the estimates

$$
\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < R, \tag{2.24}
$$

where $c = \varphi_3(\|x_0 - x^*\|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in B(x^*, R)$. Finally to show the uniqueness part, let $T = \int_0^1 H'(x^* + \theta(y^* - x^*))\,\mathrm{d}\theta$

where $y^* \in \Omega_2$ with $H(y^*) = 0$. Using (2.7), we obtain that

$$\|H'(x^*)^{-1}(T - H'(x^*))\| \le \int_0^1 w_0(\theta\|x^* - y^*\|)\,\mathrm{d}\theta \le \int_0^1 w_0(\theta R^*)\,\mathrm{d}\theta < 1. \tag{2.25}$$

Hence, we have that $T^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity $0 = H(y^*) - H(x^*) = T(y^* - x^*)$, we conclude that $x^* = y^*$. □

Remark 2.2. (a) In the case when $w_0(t) = L_0 t$, $w(t) = Lt$ and $\Omega_0 = \Omega$, the radius $r_A = 2/(2L_0 + L)$ was obtained by I. K. Argyros in [2] as the convergence radius for Newton's method under condition (2.6)–(2.8). Notice that the convergence radius for Newton's method given independently by W. C. Rheinboldt in [16] and J. F. Traub in [20] is given by

$$\varrho = \frac{2}{3L} < r_A.$$

As an example, let us consider the function $H(x) = \mathrm{e}^x - 1$. Then $x^* = 0$. Set $\Omega = B(0,1)$. Then, we have that $L_0 = \mathrm{e} - 1 < L = \mathrm{e}$, so $\varrho = 0.24252961 < r_A = 0.324947231$.

Moreover, the new error bounds, see [2], are:

$$\|x_{n+1} - x^*\| \le \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones, see [4], [7]

$$\|x_{n+1} - x^*\| \le \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (1.2) given by r_3 to be larger than r_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy, see [2]–[4].

(c) The results can be also used to solve equations where the operator H' satisfies the autonomous differential equation [2]–[4]:

$$H'(x) = P(H(x)),$$

where $P\colon \mathcal{B}_2 \longrightarrow \mathcal{B}_2$ is a known continuous operator and say $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$. Since $H'(x^*) = P(H(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. As an example, let $H(x) = \mathrm{e}^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) are not changing if we use the new instead of the old conditions [7]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \ln \frac{\|x_{n+2} - x^*\|}{\|x_{n+1} - x^*\|} \Big/ \ln \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} \qquad \text{for each} \quad n = 1, 2, \ldots,$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|} \Big/ \ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|} \qquad \text{for each} \quad n = 0, 1, 2, \ldots.$$

(e) In view of (2.4) and the estimate

$$\begin{aligned}\|H'(x^*)^{-1}H'(x)\| &= \|H'(x^*)^{-1}(H'(x) - H'(x^*)) + I\| \\ &\leq 1 + \|H'(x^*)^{-1}(H'(x) - H'(x^*))\| \leq 1 + w_0(\|x - x^*\|)\end{aligned}$$

condition (2.6) can be dropped and can be replaced by

$$v(t) = 1 + w_0(t),$$

or

$$v(t) = 1 + w_0(R_0),$$

since $t \in [0, R_0)$.

3. Numerical examples

We present two examples in this section.

Example 3.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3$, $D = \overline{U}(0,1)$, $x^* = (0,0,0)^T$. Define function H on D for $w = (x, y, z)^T$ by

$$H(w) = \Big(\mathrm{e}^x - 1, \frac{\mathrm{e} - 1}{2} y^2 + y, z\Big)^T.$$

Then the Fréchet-derivative is given by

$$H'(v) = \begin{bmatrix} \mathrm{e}^x & 0 & 0 \\ 0 & (\mathrm{e} - 1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using $x^* = (0,0,0)^T$ and (2.5)–(2.7), we can choose $w_0(t) = L_0 t$, $w(t) = \mathrm{e}^{1/L_0} t$, $v(t) = \mathrm{e}^{1/L_0}$, $L_0 = \mathrm{e} - 1$. Then, the radius of convergence R is given by

$$R = R_1 = 0.1544, \qquad R_3 = 0.2321.$$

The iterates are given in Table 1 and $\xi = 1.9989$.

n	x_n
1	(0.1400, 0.1200, 0.1400)
2	(−0.0792, −0.1154, 0)
3	(−0.0300, −0.0926, 0)
4	(−0.0042, −0.0614, 0)
5	(−0.0001, −0.0278, 0)
6	(−0.0000, −0.0059, 0)
7	(−0.0000, −0.0003, 0)
8	(0.0000, −0.0000, 0)

Table 1. Iterates.

Example 3.2. Returning back to the motivational example given at the introduction of this study, we can choose (see also Remark 2.2 (e) for function v) $x^* = 0$, $w_0(t) = w(t) = \frac{1}{8}\big(\frac{3}{2}\sqrt{t} + t\big)$ and $v(t) = 1 + w_0(R_0)$, $R_0 \backsimeq 4.7354$. Then, the radius of convergence R is given by

$$R_1 = 1.2246, \qquad R_3 = 1.1185 = R.$$

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