DE GRUYTER Comput. Methods Appl. Math. 2018; aop

Research Article

Santhosh George* and K. Kanagaraj

Derivative Free Regularization Method for Nonlinear Ill-Posed Equations in Hilbert Scales

https://doi.org/10.1515/cmam-2018-0019
Received January 19, 2018; revised May 18, 2018; accepted June 20, 2018

Abstract: In this paper, we deal with nonlinear ill-posed operator equations involving a monotone operator in the setting of Hilbert scales. Our convergence analysis of the proposed derivative-free method is based on the simple property of the norm of a self-adjoint operator. Using a general Hölder-type source condition, we obtain an optimal order error estimate. Also we consider the adaptive parameter choice strategy proposed by Pereverzev and Schock (2005) for choosing the regularization parameter. Finally, we applied the proposed method to the parameter identification problem in an elliptic PDE in the setting of Hilbert scales and compare the results with the corresponding method in Hilbert space.

Keywords: Nonlinear Ill-Posed Problem, Monotone Operator, Lavrentiev Regularization, Hilbert Scales, Adaptive Parameter Choice Strategy

MSC 2010: 47A52, 65R10, 65J10, 47H09, 49J30

1 Introduction

In this study we consider the problem of approximating a solution $\hat{x}$ of the nonlinear equation

$$F(x) = y \tag{1.1}$$

where $F : D(F) \subset \mathcal{X} \to \mathcal{X}$ is a nonlinear operator and $\mathcal{X}$ is a Hilbert space. Recall [3–5, 13, 15, 28, 30–32] that F is said to be monotone if

$$\langle F(x) - F(y), x - y \rangle \geq 0$$

for all $x, y \in D(F)$. Here and below $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ stand for the inner-product and corresponding norm, respectively, in $\mathcal{X}$.

A typical example of (1.1) is the parameter identification problem in an elliptic PDE [15], i.e., to find the source term q in the elliptic boundary-value problem

$$\begin{aligned} -\Delta u + \xi(u) &= q \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

from measurement of u in Ω. Here $\xi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuously differentiable monotonically increasing function and $\Omega \subseteq \mathbb{R}^3$ is a smooth domain. The corresponding forward operator in this case is $F : H^2(\Omega) \to H^2(\Omega)$ defined by

$$F(q) = u$$

***Corresponding author: Santhosh George,** Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Mangaluru-575 025, India, e-mail: sgeorge@nitk.ac.in. http://orcid.org/0000-0002-3530-5539
K. Kanagaraj, Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Mangaluru-575 025, India, e-mail: kanagaraj102@gmail.com

is monotone. This can be seen as follows:

$$\begin{aligned}\langle F(q_1)-F(q_2), q_1-q_2\rangle &= \int_\Omega (u_1-u_2)(q_1-q_2)\,dx\\ &= \int_\Omega (u_1-u_2)(-\Delta(u_1-u_2)+\xi(u_1)-\xi(u_2))\,dx\\ &= \int_\Omega (|\Delta(u_1-u_2)|^2+(\xi(u_1)-\xi(u_2))(u_1-u_2))\,dx\\ &\geq \|\nabla(u_1-u_2)\|^2_{L^2(\Omega)} \geq 0.\end{aligned}$$

Note that equation (1.1) is in general ill-posed in the sense that the solution $\hat{x}$ is not depending continuously on the data y. We assume that the available data $y^\delta \in \mathcal{X}$ is such that

$$\|y-y^\delta\| \leq \delta$$

and equation (1.1) is ill-posed. Therefore one has to use regularization methods for approximating $\hat{x}$. Since F is monotone, one may use Lavrentiev regularization method [1, 8, 11, 32], in which the solution x_α^δ of the equation

$$F(x)+\alpha(x-x_0)=y^\delta \tag{1.3}$$

is taken as an approximation for $\hat{x}$ where x_0 is some initial guess. Note that a closed form solution for (1.3) is not easy to find for nonlinear F. Therefore, many authors [1, 2, 8, 11, 32] considered iterative methods to find an approximation for x_α^δ. In [12], George and Nair considered a derivative-free iterative method defined for $n = 0, 1, 2, \ldots$ by

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - \beta[F(x^\delta_{n,\alpha}) + \alpha(x^\delta_{n,\alpha}-x_0) - y^\delta], \tag{1.4}$$

where β is a scaling parameter and α is a regularization parameter for approximating x_α^δ. It is known [15, 28, 31, 32] that the optimal order error estimate for Lavrentiev regularization is

$$\|x_\alpha^\delta - \hat{x}\| = O(\delta^{\frac{\nu}{\nu+1}}) \tag{1.5}$$

under the source condition

$$x_0-\hat{x} \in R(F'(x_0)^\nu), \quad 0<\nu\leq 1,$$

or

$$x_0-\hat{x} \in R(F'(\hat{x})^\nu), \quad 0<\nu\leq 1.$$

In order to improve the convergence rate in (1.5), many authors considered iterative regularization method for (1.1) in the setting of Hilbert scales [6, 7, 9, 10, 13, 14, 17, 19–26, 29, 30]. In this paper we consider Lavrentiev regularization method for (1.1) in the setting of Hilbert scales. We also consider an inverse free, derivative-free iterative method for approximating $\hat{x}$ in the setting of a Hilbert scales.

The rest of the paper is organized as follows: Preliminaries are given in Section 2, the method and its convergence analysis are given in Section 3. Error bounds are given in Section 4, parameter strategies are given in Section 5. Implementation of the adaptive parameter choice is given in Section 6 and the numerical experiments are given in Section 7. Finally, the paper ends with a conclusion in Section 8.

2 Preliminaries

First, we recall the definition of Hilbert scale:

Definition 2.1 ([21]). A family $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ of Hilbert spaces is called a Hilbert scale if it satisfies the following conditions:

- For $s<t$, $\mathcal{X}_t \subseteq \mathcal{X}_s$ and $\mathcal{X}_t$ is a dense subset of $\mathcal{X}_s$.
- As Hilbert spaces, the above inclusion is a continuous embedding, i.e., there exists $c_{s,t}>0$ such that

$$\|x\|_s \leq c_{s,t}\|x\|_t \quad \text{for all } x\in\mathcal{X}_t. \tag{2.1}$$

In this study, we consider a Hilbert scale $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ generated by a strictly positive definite, unbounded, densely defined, self-adjoint operator $L : D(L) \subset \mathcal{X} \to \mathcal{X}$. That is, L satisfies

$$\langle Lx, x\rangle > 0,$$

$D(L)$ is dense in $\mathcal{X}$ and

$$\|Lx\| \geq \|x\|, \; x \in D(L).$$

Recall (cf. [9]) that the space $\mathcal{X}_t$ is the completion of $D := \bigcap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t$ induced by the inner product

$$\langle u, v\rangle_t = \langle L^t u, v\rangle, \quad u, v \in D.$$

Moreover, $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ satisfies Definition 2.1 (cf. [9, 10, 29, 30]).

Next we show that the equation

$$F(x) + \alpha L^s(x - x_0) = y^\delta \tag{2.2}$$

has a unique solution $x_{\alpha,s}^\delta$. We need the following definition for our proof.

Definition 2.2 ([2, cf. Definition 1.1.42]). An operator $A : \mathcal{X} \to \mathcal{X}$ is said to be coercive if there exists a function $c(t)$ defined for all $t \geq 0$ such that $c(t) \to \infty$ as $t \to \infty$, and the inequality

$$\langle A(x), x\rangle \geq c(\|x\|)\|x\|$$

holds for all $x \in D(A)$.

Next, we prove that the operator $T := F + \alpha L^s$ is coercive. This can be seen as follows:

$$\begin{aligned}
\langle T(x), x\rangle &= \langle F(x) + \alpha L^s(x), x\rangle \\
&= \langle F(x) - F(0) + \alpha L^s(x), x - 0\rangle + \langle F(0), x\rangle \\
&= \langle F(x) - F(0), x - 0\rangle + \langle \alpha L^s(x), x\rangle + \langle F(0), x\rangle \\
&\geq \alpha\|x\|_s^2 - \|F(0)\|\|x\| && \text{(by the monotonicity of } F) \\
&\geq \alpha \frac{1}{c_{0,s}}\|x\|^2 - \|F(0)\|\|x\| && \text{(by (2.1))}
\end{aligned}$$

and hence

$$\lim_{\|x\|\to\infty} \frac{\langle T(x), x\rangle}{\|x\|} \geq \lim_{\|x\|\to\infty} \alpha \frac{1}{c_{0,s}}\|x\| - \|F(0)\| = \infty.$$

That is, $T = F + \alpha L^s$ is coercive. Further,

$$\langle T(x) - T(y), x - y\rangle = \langle F(x) - F(y), x - y\rangle + \alpha\langle L^s(x - y), x - y\rangle \geq \alpha \frac{1}{c_{0,s}}\|x - y\|^2,$$

i.e., T is strongly monotone. So by the Minty–Browder Theorem [2, p. 54], for given $\alpha > 0$, (2.2) has a unique solution $x_{\alpha,s}^\delta$ for any $y^\delta \in \mathcal{X}$.

Let $r_0 = \|x_0 - \hat{x}\|_s$. The following Lemmas is used to prove our main results.

Lemma 2.3. *Let $x_{\alpha,s}^\delta$ be the solution of* (2.2) *and $x_{\alpha,s}$ is the solution of*

$$F(x) + \alpha L^s(x - x_0) = y. \tag{2.3}$$

Then

$$\|x_{\alpha,s}^\delta - x_{\alpha,s}\|_s \leq c_{0,s}\frac{\delta}{\alpha}$$

and

$$\|x_{\alpha,s} - \hat{x}\|_s \leq \|x_0 - \hat{x}\|_s.$$

In particular,

$$\|x_{\alpha,s}^\delta - x_0\|_s \leq c_{0,s}\frac{\delta}{\alpha} + 2r_0. \tag{2.4}$$

Proof. Observe that, by (2.2) and (2.3), we have

$$F(x_{\alpha,s}^{\delta}) - F(x_{\alpha,s}) + \alpha L^s(x_{\alpha,s}^{\delta} - x_{\alpha,s}) = y^{\delta} - y.$$

Hence,

$$\langle F(x_{\alpha,s}^{\delta}) - F(x_{\alpha,s}), x_{\alpha,s}^{\delta} - x_{\alpha,s}\rangle + \alpha\langle L^s(x_{\alpha,s}^{\delta} - x_{\alpha,s}), x_{\alpha,s}^{\delta} - x_{\alpha,s}\rangle = \langle y^{\delta} - y, x_{\alpha,s}^{\delta} - x_{\alpha,s}\rangle,$$

so, by using (2.1) and the monotonicity of F, we have

$$\alpha\|x_{\alpha,s}^{\delta} - x_{\alpha,s}\|_s^2 \leq \delta\|x_{\alpha,s}^{\delta} - x_{\alpha,s}\| \leq \delta c_{0,s}\|x_{\alpha,s}^{\delta} - x_{\alpha,s}\|_s.$$

Thus,

$$\|x_{\alpha,s}^{\delta} - x_{\alpha,s}\|_s \leq c_{0,s}\frac{\delta}{\alpha}.$$

Again, since $y = F(\hat{x})$, we have

$$F(x_{\alpha,s}) + \alpha L^s(x_{\alpha,s} - x_0) = F(\hat{x}),$$

so that

$$F(x_{\alpha,s}) - F(\hat{x}) + \alpha L^s(x_{\alpha,s} - x_0) = 0,$$

i.e.,

$$F(x_{\alpha,s}) - F(\hat{x}) + \alpha L^s(x_{\alpha,s} - \hat{x}) = \alpha L^s(x_0 - \hat{x}).$$

Hence,

$$\langle F(x_{\alpha,s}) - F(\hat{x}), x_{\alpha,s} - \hat{x}\rangle + \alpha\langle L^s(x_{\alpha,s} - \hat{x}), x_{\alpha,s} - \hat{x}\rangle = \alpha\langle L^s(x_0 - \hat{x}), x_{\alpha,s} - \hat{x}\rangle.$$

Again, using the monotonicity of F, we have

$$\alpha\|x_{\alpha,s} - \hat{x}\|_s^2 \leq \alpha\|L^{\frac{s}{2}}(x_{\alpha,s} - \hat{x})\|\|L^{\frac{s}{2}}(x_0 - \hat{x})\| \leq \alpha\|x_{\alpha,s} - \hat{x}\|_s\|x_0 - \hat{x}\|_s.$$

Thus,

$$\|x_{\alpha,s} - \hat{x}\|_s \leq \|x_0 - \hat{x}\|_s.$$

Now (2.4) follows from the triangle inequality:

$$\|x_{\alpha,s}^{\delta} - x_0\|_s \leq \|x_{\alpha,s}^{\delta} - x_{\alpha,s}\|_s + \|x_{\alpha,s} - \hat{x}\|_s + \|\hat{x} - x_0\|_s.$$

This completes the proof. □

Remark 2.4. Note that by (2.1) and (2.4), we have

$$\|x_{\alpha,s}^{\delta} - x_0\| \leq c_{0,s}\|x_{\alpha,s}^{\delta} - x_0\|_s \leq c_{0,s}(c_{0,s}\frac{\delta}{\alpha} + 2r_0),$$

i.e., $x_{\alpha,s}^{\delta} \in B(x_0, R)$, where

$$R = c_{0,s}\Big(c_{0,s}\frac{\delta}{\alpha} + 2r_0\Big).$$

3 The Method and the Convergence Analysis

Let $\rho = c_{0,s}(c_{0,s} + 1)(c_{0,s} + 2r_0)$. We assume that the following conditions hold:
(i) $\bar{B}(x_0, \rho) \subseteq D(F)$,
(ii) F has self-adjoint Fréchet derivative $F'(x)$ for every $x \in \bar{B}(x_0, \rho)$,
(iii) there exists $\beta_0 > 0$ such that

$$\|L^{-\frac{s}{2}}F'(x)L^{-\frac{s}{2}}\| \leq \beta_0 \quad \text{for all } x \in \bar{B}(x_0, \rho).$$

(iv) there exist positive constants d_1, d_2, b such that

$$d_1\|x\|_{-b} \leq \|F'(y)x\| \leq d_2\|x\|_{-b} \quad \text{for all } y \in \bar{B}(x_0, \rho) \text{ and } x \in \mathcal{X}.$$

Let $f(t) := \min\{d_1^t, d_2^t\},\ g(t) := \max\{d_1^t, d_2^t\}$, $t \in \mathbb{R}$, $|t| \leq 1$. Further, let $M_{s,y} := L^{-\frac{s}{2}}F'(y)L^{-\frac{s}{2}}$ for $y \in \bar{B}(x_0, \rho)$.

We shall make use of the following proposition, the proof of which is analogous to the proof of [9, Proposition 3.1].

Proposition 3.1 (cf. [9, Proposition 3.1]). *For $s > 0$ and $|\nu| \le 1$,*

$$f\Big(\frac{\nu}{2}\Big)\|x\|_{-\frac{\nu(s+b)}{2}} \le \|M_{s,y}^{\frac{\nu}{2}}x\| \le g\Big(\frac{\nu}{2}\Big)\|x\|_{-\frac{\nu(s+b)}{2}}, \quad x \in \mathcal{X},\ y \in \bar{B}(x_0,\rho).$$

The Method. Let $\delta \in (0, d]$ and $\alpha \in [\delta, a)$. We define the sequence $\{x_{n,\alpha,s}^\delta\}$ iteratively for $n = 0, 1, 2, 3, \dots$ by

$$x_{n+1,\alpha,s}^\delta = x_{n,\alpha,s}^\delta - \beta[L^{-s}(F(x_{n,\alpha,s}^\delta) - y^\delta) + \alpha(x_{n,\alpha,s}^\delta - x_0)], \tag{3.1}$$

where $x_{0,\alpha,s}^\delta = x_0$ and $\beta := \frac{1}{\beta_0+a}$. We observe that if $\{x_{n,\alpha,s}^\delta\}$ converges as $n \to \infty$, then the limit is $x_{\alpha,s}^\delta$, the solution of (2.2).

Next, we prove the main results of this section.

Theorem 3.2. *For each $\delta \in (0, d]$ and $\alpha \in [\delta, a)$ the sequence $\{x_{n,\alpha,s}^\delta\}$ is in $\bar{B}(x_0,\rho)$ and it converges to $x_{\alpha,s}^\delta$ as $n \to \infty$. Further,*

$$\|x_{n,\alpha,s}^\delta - x_{\alpha,s}^\delta\| \le kq_{\alpha,s}^n,$$

where $q_{\alpha,s} := 1 - \beta\alpha$ and $k \ge c_{0,s}(c_{0,s} + 2r_0)$ with $\beta := \frac{1}{\beta_0+a}$.

Proof. Clearly, we have $x_{0,\alpha,s}^\delta = x_0 \in \bar{B}(x_0,\rho)$. Also, since $\rho \ge c_{0,s}R$ by (2.4), we have $x_{\alpha,s}^\delta \in \bar{B}(x_0,\rho)$. By the Fundamental Theorem of Integration, we have

$$F(x) - F(u) = \left[\int_0^1 F'(u + \theta(x-u))\,d\theta\right](x-u)$$

whenever x and u are in a ball contained in $D(F)$. We show iteratively that $x_{n,\alpha,s}^\delta \in \bar{B}(x_0,\rho)$, the operator

$$A_{n,\theta} := \int_0^1 F'(x_{\alpha,s}^\delta + \theta(x_{n,\alpha,s}^\delta - x_{\alpha,s}^\delta))\,d\theta$$

is a well-defined positive self-adjoint operator and

$$\|x_{n+1,\alpha,s}^\delta - x_{\alpha,s}^\delta\|_s \le (1-\beta\alpha)\|x_{n,\alpha,s}^\delta - x_{\alpha,s}^\delta\|_s$$

for $n = 0, 1, 2, \dots$, which will complete the proof, since $\|x_0 - x_{\alpha,s}^\delta\| \le c_{0,s}\|x_0 - x_{\alpha,s}^\delta\|_s \le c_{0,s}R \le \rho$.

Formally, by (2.1), we have

$$x_{n+1,\alpha,s}^\delta - x_{\alpha,s}^\delta = x_{n,\alpha,s}^\delta - x_{\alpha,s}^\delta - \beta[L^{-s}(F(x_{n,\alpha,s}^\delta) - F(x_{\alpha,s}^\delta)) + \alpha(x_{n,\alpha,s}^\delta - x_{\alpha,s}^\delta)].$$

Since

$$F(x_{n,\alpha,s}^\delta) - F(x_{\alpha,s}^\delta) = A_{n,\theta}(x_{n,\alpha,s}^\delta - x_{\alpha,s}^\delta),$$

we have

$$x_{n+1,\alpha,s}^\delta - x_{\alpha,s}^\delta = [I - \beta(L^{-s}A_{n,\theta} + \alpha I)](x_{n,\alpha,s}^\delta - x_{\alpha,s}^\delta). \tag{3.2}$$

Now, let $n = 0$. We have already seen that $\|x_0 - x_{\alpha,s}^\delta\| < \rho$ so that $x_{\alpha,s}^\delta \in \bar{B}(x_0,\rho)$ and $A_{0,\theta}$ is a well-defined positive self-adjoint operator with $\|L^{-\frac{s}{2}}A_{0,\theta}L^{-\frac{s}{2}}\| \le \beta_0$.

Next assume that for some $n \ge 0$, $x_{n,\alpha,s}^\delta \in \bar{B}(x_0,\rho)$ and $A_{n,\theta}$ is a well-defined positive self-adjoint operator with $\|L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}\| \le \beta_0$. Then from (3.2),

$$L^{\frac{s}{2}}(x_{n+1,\alpha,s}^\delta - x_{\alpha,s}^\delta) = [I - \beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}} + \alpha I)]L^{\frac{s}{2}}(x_{n,\alpha,s}^\delta - x_{\alpha,s}^\delta),$$

so

$$\|x_{n+1,\alpha}^\delta - x_{\alpha,s}^\delta\|_s \le \|I - \beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}} + \alpha I)\|\,\|(x_{n,\alpha}^\delta - x_{\alpha,s}^\delta)\|_s.$$

Since $L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}$ and hence $I-\beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}+\alpha I)$ are positive self-adjoint operator, we have (cf. [18])

$$\begin{aligned}\|I-\beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}+\alpha I)\| &= \sup_{\|x\|=1}|\langle[I-\beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}+\alpha I)]x,x\rangle|\\ &= \sup_{\|x\|=1}|(1-\beta\alpha)-\beta\langle L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}x,x\rangle|\end{aligned}$$

and since $\|L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}\|\le\beta_0$ for all $n\in\mathbb{N}$ and $\beta=\frac{1}{\beta_0+a}$, we have

$$0\le\beta\langle L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}x,x\rangle\le\beta\|L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}\|\le\beta\beta_0<1-\beta\alpha$$

for all $\alpha\in(0,a)$. Therefore,

$$\|I-\beta(L^{-\frac{s}{2}}A_{n,\theta}L^{-\frac{s}{2}}+\alpha I)\|\le 1-\beta\alpha.$$

Thus,

$$\|x^\delta_{n+1,\alpha,s}-x^\delta_{\alpha,s}\|_s\le(1-\beta\alpha)\|x^\delta_{n,\alpha}-x^\delta_{\alpha,s}\|_s.$$

Hence,

$$\|x^\delta_{n+1,\alpha,s}-x^\delta_{\alpha,s}\|\le c_{0,s}\|x^\delta_{n+1,\alpha,s}-x^\delta_{\alpha,s}\|_s\le c_{0,s}\|x_0-x^\delta_{\alpha,s}\|_s$$

and

$$\begin{aligned}\|x^\delta_{n+1,\alpha,s}-x_0\| &\le c_{0,s}\|x^\delta_{n+1,\alpha,s}-x_0\|_s\\ &\le c_{0,s}[\|x^\delta_{n+1,\alpha,s}-x^\delta_{\alpha,s}\|_s+\|x^\delta_{\alpha,s}-x_0\|_s]\\ &\le c_{0,s}(c_{0,s}+1)\|x_0-x^\delta_{\alpha,s}\|_s\\ &\le c_{0,s}(c_{0,s}+1)(2r_0+c_{0,s})\le\rho.\end{aligned}$$

Thus, $x^\delta_{n+1,\alpha,s}\in\bar{B}(x_0,\rho)$. Also, for $0\le\theta\le1$,

$$\begin{aligned}\|[x^\delta_{\alpha,s}+\theta(x^\delta_{n+1,\alpha,s}-x^\delta_{\alpha,s})]-x_0\| &= \|(x^\delta_{\alpha,s}-x_0)+\theta(x^\delta_{n+1,\alpha,s}-x^\delta_{\alpha,s})\|\\ &\le\|x^\delta_{\alpha,s}-x_0\|+\theta\|x^\delta_{n+1,\alpha,s}-x^\delta_{\alpha,s}\|\\ &\le c_{0,s}[\|x^\delta_{\alpha,s}-x_0\|_s+\theta\|x^\delta_{n+1,\alpha,s}-x^\delta_{\alpha,s}\|_s]\\ &\le c_{0,s}(c_{0,s}+1)\|x^\delta_{\alpha,s}-x_0\|_s\\ &\le c_{0,s}(c_{0,s}+1)(2r_0+c_{0,s})\\ &\le\rho.\end{aligned}$$

Hence, $A_{n+1,\theta}$ is a well-defined positive self-adjoint operator with $\|L^{-\frac{s}{2}}A_{n+1,\theta}L^{-\frac{s}{2}}\|\le\beta_0$. This completes the proof. □

4 Error Bounds Under Source Conditions

In order to obtain estimate for $\|x^\delta_{\alpha,s}-\hat{x}\|$, we have to impose some nonlinearity conditions on F and assume that $x_0-\hat{x}$ belongs to some source set. We use the following two assumptions to obtain an error estimate for $\|x^\delta_{\alpha,s}-\hat{x}\|$.

Assumption 4.1. There exists a constant $k_0\ge0$ such that for every $x\in\bar{B}(x_0,\rho)$ and $v\in\mathcal{X}$ there exists an element $\Phi(x,x_0,v)\in X$ such that

$$[F'(x)-F'(x_0)]v=F'(x_0)\Phi(x,x_0,v)$$

and

$$\|\Phi(x,x_0,v)\|\le k_0\|v\|\|x-x_0\|$$

for all $x,v\in\bar{B}(x_0,\rho)$.

Assumption 4.2. There exists some $E > 0$, $t > 0$ such that $x_0 - \hat{x} \in \mathcal{X}_t$ and $\|x_0 - \hat{x}\|_t \le E$.

Theorem 4.3. *Let $x_{\alpha,s}^\delta$ be the solution of (2.2), let $x_{\alpha,s}$ be the solution of (2.3), and let Assumption 4.1 and Assumption 4.2 with $t \le s + b$ hold. Further, suppose*

$$\rho k_0 < \frac{f(\frac{s}{s+b})}{g(\frac{s}{s+b})}.$$

Then we have the following estimates:

(a) *We have*

$$\|x_{\alpha,s}^\delta - x_{\alpha,s}\| \le \frac{c_{-s,0}}{f(\frac{s}{s+b}) - g(\frac{s}{s+b})\rho k_0} \frac{\delta}{\alpha^{\frac{b}{s+b}}}.$$

(b) *We have*

$$\|x_{\alpha,s} - \hat{x}\| \le \frac{g(\frac{s-t}{s+b})}{f(\frac{s}{s+b}) - g(\frac{s}{s+b})k_0\rho} E\alpha^{\frac{t}{s+b}}.$$

(c) *In particular, for $\alpha = \delta^{\frac{s+b}{b+t}}$, we have*

$$\|x_{\alpha,s}^\delta - \hat{x}\| = O(\delta^{\frac{t}{t+b}}).$$

Proof. Let $A_s = \int_0^1 F'(x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}))\,d\theta$. Then, since

$$F(x_{\alpha,s}^\delta) - F(x_{\alpha,s}) + \alpha L^s(x_{\alpha,s}^\delta - x_{\alpha,s}) = y^\delta - y,$$

we have

$$(A_s + \alpha L^s)(x_{\alpha,s}^\delta - x_{\alpha,s}) = y^\delta - y.$$

In particular,

$$(F'(x_0) + \alpha L^s)(x_{\alpha,s}^\delta - x_{\alpha,s}) = y^\delta - y + (F'(x_0) - A_s)(x_{\alpha,s}^\delta - x_{\alpha,s}).$$

Therefore, we have

$$\begin{aligned} x_{\alpha,s}^\delta - x_{\alpha,s} &= (F'(x_0) + \alpha L^s)^{-1}[y^\delta - y + (F'(x_0) - A_s)(x_{\alpha,s}^\delta - x_{\alpha,s})] \\ &= (F'(x_0) + \alpha L^s)^{-1}\Big[y^\delta - y - F'(x_0)\int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})\,d\theta\Big] \end{aligned}$$

and hence

$$\begin{aligned} \|x_{\alpha,s}^\delta - x_{\alpha,s}\| &\le \|(F'(x_0) + \alpha L^s)^{-1}(y^\delta - y)\| \\ &\quad + \Big\|(F'(x_0) + \alpha L^s)^{-1}F'(x_0)\int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})\,d\theta\Big\| \\ &= \Gamma_1 + \Gamma_2, \end{aligned}$$

where

$$\begin{aligned} \Gamma_1 &= \|(F'(x_0) + \alpha L^s)^{-1}(y^\delta - y)\|, \\ \Gamma_2 &= \Big\|(F'(x_0) + \alpha L^s)^{-1}F'(x_0)\int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})\,d\theta\Big\|. \end{aligned}$$

Note that by Proposition 3.1, we have

$$\begin{aligned} \Gamma_1 &= \|(F'(x_0) + \alpha L^s)^{-1}(y^\delta - y)\| \\ &= \|L^{-\frac{s}{2}}(L^{-\frac{s}{2}}F'(x_0)L^{-\frac{s}{2}} + \alpha I)^{-1}L^{-\frac{s}{2}}(y^\delta - y)\| \\ &\le \frac{1}{f(\frac{s}{s+b})}\|B_s^{\frac{s}{s+b}}(B_s + \alpha I)^{-1}L^{-\frac{s}{2}}(y^\delta - y)\| \\ &\le \frac{c_{-s,0}}{f(\frac{s}{s+b})}\frac{\delta}{\alpha^{\frac{b}{s+b}}}, \end{aligned}$$

where here and below

$$B_s = L^{-\frac{s}{2}} F'(x_0) L^{-\frac{s}{2}}.$$

Again, by Proposition 3.1, we have

$$\begin{aligned}
\Gamma_2 &= \left\| L^{-\frac{s}{2}} (B_s + \alpha I)^{-1} L^{-\frac{s}{2}} F'(x_0) \int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})\, d\theta \right\| \\
&\le \frac{1}{f(\frac{s}{s+b})} \left\| B_s^{\frac{s}{s+b}} (B_s + \alpha I)^{-1} B_s L^{\frac{s}{2}} \int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})\, d\theta \right\| \\
&\le \frac{1}{f(\frac{s}{s+b})} \left\| (B_s + \alpha I)^{-1} B_s B_s^{\frac{s}{s+b}} L^{\frac{s}{2}} \int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})\, d\theta \right\| \\
&\le \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})} \left\| \int_0^1 \Phi(x_0, x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}), x_{\alpha,s}^\delta - x_{\alpha,s})\, d\theta \right\| \\
&\le \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})} k_0 \int_0^1 \|x_0 - x_{\alpha,s} - \theta(x_{\alpha,s}^\delta - x_{\alpha,s})\| \|x_{\alpha,s}^\delta - x_{\alpha,s}\|\, d\theta \\
&\le \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})} \rho k_0 \|x_{\alpha,s}^\delta - x_{\alpha,s}\|.
\end{aligned}$$

The last step follows from the fact that $x_{\alpha,s}, x_{\alpha,s}^\delta \in B(x_0, \rho)$ and hence

$$x_{\alpha,s} + \theta(x_{\alpha,s}^\delta - x_{\alpha,s}) \in B(x_0, \rho).$$

This proves (a). To prove (b), we notice that since $y = F(\hat{x})$, we have by (2.3)

$$F(x_{\alpha,s}) - F(\hat{x}) + \alpha L^s (x_{\alpha,s} - x_0) = 0. \tag{4.1}$$

Let

$$A = \int_0^1 F'(\hat{x} + \theta(x_{\alpha,s} - \hat{x}))\, d\theta$$

Then by (4.1),

$$(A + \alpha L^s)(x_{\alpha,s} - \hat{x}) = \alpha L^s (x_0 - \hat{x})$$

or

$$(F'(x_0) + \alpha L^s)(x_{\alpha,s} - \hat{x}) = (F'(x_0) - A)(x_{\alpha,s} - \hat{x}) + \alpha L^s (x_0 - \hat{x}).$$

Therefore,

$$x_{\alpha,s} - \hat{x} = (F'(x_0) + \alpha L^s)^{-1} [(F'(x_0) - A)(x_{\alpha,s} - \hat{x}) + \alpha L^s (x_0 - \hat{x})].$$

Hence, using Assumptions 4.1 and 4.2, we have

$$\begin{aligned}
x_{\alpha,s} - \hat{x} &= L^{-\frac{s}{2}} (B_s + \alpha I)^{-1} L^{-\frac{s}{2}} \left[-F'(x_0) \int_0^1 \Phi(x_0, \hat{x} + \theta(x_{\alpha,s} - \hat{x}), x_{\alpha,s} - \hat{x})\, d\theta \right] \\
&\quad + \alpha L^{-\frac{s}{2}} (B_s + \alpha I)^{-1} L^{-\frac{s}{2}} L^s (x_0 - \hat{x}) \\
&= L^{-\frac{s}{2}} (B_s + \alpha I)^{-1} L^{-\frac{s}{2}} \left[-F'(x_0) \int_0^1 \Phi(x_0, \hat{x} + \theta(x_{\alpha,s} - \hat{x}), x_{\alpha,s} - \hat{x})\, d\theta \right] \\
&\quad + \alpha L^{-\frac{s}{2}} (B_s + \alpha I)^{-1} L^{\frac{s}{s}} (x_0 - \hat{x}).
\end{aligned}$$

So

$$
\begin{aligned}
\|x_{\alpha,s} - \hat{x}\| \le& \left\| L^{-\frac{s}{2}}(B_s + \alpha I)^{-1} L^{-\frac{s}{2}} \left[F'(x_0) \int_0^1 \Phi(x_0, \hat{x} + \theta(x_{\alpha,s} - \hat{x}), x_{\alpha,s} - \hat{x})\, d\theta \right] \right\| \\
&+ \alpha \|L^{-\frac{s}{2}}(B_s + \alpha I)^{-1} L^{\frac{s}{2}}(x_0 - \hat{x})\| \\
\le& \frac{1}{f(\frac{s}{s+b})} \left\| B_s^{\frac{s}{s+b}}(B_s + \alpha I)^{-1} B_s L^{\frac{s}{2}} \int_0^1 \Phi(x_0, \hat{x} + \theta(x_{\alpha,s} - \hat{x}), x_{\alpha,s} - \hat{x})\, d\theta \right\| \\
&+ \frac{1}{f(\frac{s}{s+b})} \alpha \|B_s^{\frac{s}{s+b}}(B_s + \alpha I)^{-1} L^{\frac{s}{2}}(x_0 - \hat{x})\| \\
\le& \frac{1}{f(\frac{s}{s+b})} \left\| (B_s + \alpha I)^{-1} B_s B_s^{\frac{s}{s+b}} L^{\frac{s}{2}} \int_0^1 \Phi(x_0, \hat{x} + \theta(x_{\alpha,s} - \hat{x}), x_{\alpha,s} - \hat{x})\, d\theta \right\| \\
&+ \frac{1}{f(\frac{s}{s+b})} \alpha \|B_s^{\frac{s}{s+b}}(B_s + \alpha I)^{-1} L^{\frac{s}{2}}(x_0 - \hat{x})\| \\
\le& \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})} \left\| \int_0^1 \Phi(x_0, \hat{x} + \theta(x_{\alpha,s} - \hat{x}), x_{\alpha,s} - \hat{x})\, d\theta \right\| + \frac{1}{f(\frac{s}{s+b})} \alpha \|(B_s + \alpha I)^{-1} B_s^{\frac{t}{s+b}} B_s^{\frac{s-t}{s+b}} L^{\frac{s}{2}}(x_0 - \hat{x})\| \\
\le& \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})} k_0 \|x_0 - \hat{x} - \theta(x_{\alpha,s} - \hat{x})\| \|x_{\alpha,s} - \hat{x}\| + \frac{1}{f(\frac{s}{s+b})} \alpha \|(B_s + \alpha I)^{-1} B_s^{\frac{t}{s+b}}\| \|B_s^{\frac{s-t}{s+b}} L^{\frac{s}{2}}(x_0 - \hat{x})\| \\
\le& \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})} k_0 \rho \|x_{\alpha,s} - \hat{x}\| + \frac{g(\frac{s-t}{s+b})}{f(\frac{s}{s+b})} \alpha^{\frac{t}{s+b}} \|x_0 - \hat{x}\|_t \\
\le& \frac{g(\frac{s}{s+b})}{f(\frac{s}{s+b})} k_0 \rho \|x_{\alpha,s} - \hat{x}\| + \frac{g(\frac{s-t}{s+b})}{f(\frac{s}{s+b})} E \alpha^{\frac{t}{s+b}}.
\end{aligned}
$$

This completes the proof of (b). Now (c) follows from (a) and (b). □

5 A Priori Choice of the Parameter

Note that by (a) and (b) of Theorem 4.3, we have

$$\|x_{\alpha,s}^{\delta} - \hat{x}\| \le C\Big(\frac{\delta}{\alpha^{\frac{b}{s+b}}} + \alpha^{\frac{t}{s+b}}\Big), \tag{5.1}$$

where

$$C = \max\Big\{\frac{c_{-s,0}}{f(\frac{s}{s+b}) - g(\frac{s}{s+b})\rho k_0}, \frac{g(\frac{s-t}{s+b})E}{f(\frac{s}{s+b}) - g(\frac{s}{s+b})k_0\rho}\Big\}. \tag{5.2}$$

Further observe that the error $\frac{\delta}{\alpha^{\frac{b}{s+b}}} + \alpha^{\frac{t}{s+b}}$ in (5.1) is of optimal order if $\alpha_\delta := \alpha(t, \delta)$ satisfies,

$$\frac{\delta}{\alpha^{\frac{b}{s+b}}} = \alpha^{\frac{t}{s+b}}.$$

That is, $\alpha_\delta = \delta^{\frac{s+b}{t+b}}$. Hence, by (5.1) we have the following theorem.

Theorem 5.1. *Let the assumptions in Theorem 3.2 and Theorem 4.3 hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \delta^{\frac{s+b}{t+b}}$. Let n_δ be such that*

$$n_\delta := \min\Big\{n : q_{\alpha,s}^n \le \frac{\delta}{\alpha^{\frac{b}{s+b}}}\Big\}.$$

Then

$$\|x_{n_\delta,\alpha,s}^{\delta} - \hat{x}\| = O(\delta^{\frac{t}{t+b}}).$$

5.1 Adaptive Scheme and Stopping Rule

In [27], Pereverzev and Schock introduced the adaptive selection of the parameter strategy. We modified the adaptive method suitably for the situation for choosing the regularization parameter α. For convenience, take $x^\delta_{i,\alpha,s} := x^\delta_{n_i,\alpha_i,s}$. Let $i \in \{0, 1, 2, \dots, N\}$ and $\alpha_i = \mu^i \alpha_0$, where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$l := \max\Big\{i : \alpha_i^{\frac{t}{s+b}} \le \frac{\delta}{\alpha^{\frac{b}{s+b}}}\Big\} < N, \tag{5.3}$$

$$k := \max\Big\{i : \|x^\delta_{i,\alpha,s} - x^\delta_{j,\alpha,s}\| \le 4\bar{C}\frac{\delta}{\alpha_j^{\frac{b}{s+b}}},\ j = 0, 1, 2, \dots, i-1\Big\}, \tag{5.4}$$

where $\bar{C} = C + k$ with C is as in (5.2) and k is as in Theorem 3.2. Now we have the following theorem.

Theorem 5.2. *Assume that there exists $i \in \{0, 1, \dots, N\}$ such that*

$$\alpha_i^{\frac{t}{s+b}} \le \frac{\delta}{\alpha^{\frac{b}{s+b}}}.$$

Let the assumptions of Theorem 3.2 and Theorem 4.3 be fulfilled, and let l and k be as in (5.3) and (5.4), respectively. Let

$$n_i = \min\Big\{n : q^n_{\alpha_i,s} \le \frac{\delta}{\alpha_i^{\frac{b}{s+b}}}\Big\}.$$

Then $l \le k$ and

$$\|x^\delta_{n_k,\alpha,s} - \hat{x}\| \le 6\bar{C}\mu^{\frac{b}{s+b}}\delta^{\frac{t}{t+b}}.$$

Proof. To prove $l \le k$, it is enough to show that, for each $i \in \{1, 2, \dots, N\}$,

$$\alpha_i^{\frac{t}{s+b}} \le \frac{\delta}{\alpha^{\frac{b}{s+b}}} \implies \|x^\delta_{n_i,\alpha,s} - x^\delta_{n_j,\alpha,s}\| \le 4\bar{C}\frac{\delta}{\alpha^{\frac{b}{s+b}}} \quad \text{for all } j = 0, 1, 2, \dots, i-1.$$

For $j < i$, we have

$$\begin{aligned}\|x^\delta_{n_i,\alpha,s} - x^\delta_{n_j,\alpha,s}\| &\le \|x^\delta_{i,\alpha,s} - \hat{x}\| + \|\hat{x} - x^\delta_{j,\alpha,s}\| \\ &\le \bar{C}\Big(\alpha_i^{\frac{t}{s+b}} + \frac{\delta}{\alpha_i^{\frac{b}{s+b}}}\Big) + \bar{C}\Big(\alpha_j^{\frac{t}{s+b}} + \frac{\delta}{\alpha_j^{\frac{b}{s+b}}}\Big) \le 2\bar{C}\alpha_i^{\frac{t}{s+b}} + 2\bar{C}\frac{\delta}{\alpha_j^{\frac{b}{s+b}}} \le 4\bar{C}\frac{\delta}{\alpha_j^{\frac{b}{s+b}}}.\end{aligned}$$

Thus, the relation $l \le k$ is proved. Observe that

$$\|\hat{x} - x^\delta_{n_k,\alpha,s}\| \le \|\hat{x} - x^\delta_{n_l,\alpha,s}\| + \|x^\delta_{n_k,\alpha,s} - x^\delta_{n_l,\alpha,s}\|,$$

where

$$\|\hat{x} - x^\delta_{n_l,\alpha,s}\| \le \bar{C}\Big(\alpha_l^{\frac{t}{s+b}} + \frac{\delta}{\alpha_l^{\frac{b}{s+b}}}\Big) \le 2\bar{C}\frac{\delta}{\alpha_l^{\frac{b}{s+b}}}.$$

Now since $l \le k$, we have

$$\|x^\delta_{n_k,\alpha,s} - x^\delta_{n_l,\alpha,s}\| \le 4\bar{C}\frac{\delta}{\alpha_l^{\frac{b}{s+b}}}.$$

Hence,

$$\|\hat{x} - x^\delta_{n_k,\alpha,s}\| \le 6\bar{C}\frac{\delta}{\alpha_l^{\frac{b}{s+b}}}.$$

Now, since $\alpha_\delta^{\frac{b}{s+b}} = \delta^{\frac{b}{t+b}} \le \alpha_{l+1}^{\frac{b}{s+b}} \le \mu^{\frac{b}{s+b}}\alpha_l^{\frac{b}{s+b}}$, it follows that

$$\frac{\delta}{\alpha_l^{\frac{b}{s+b}}} \le \mu^{\frac{b}{s+b}}\frac{\delta}{\alpha_\delta^{\frac{b}{s+b}}} = \mu^{\frac{b}{s+b}}\delta^{\frac{t}{t+b}}.$$

This completes the proof. □

6 Implementation of Adaptive Choice Rule

The balancing algorithm associated with the choice of the parameter specified in Theorem 5.2 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0$, $i = 0, 1, 2, \ldots, N$.

6.1 Algorithm

(1) Set $i = 0$.
(2) Choose

$$n_i := \min\Big\{n : q^n_{\alpha_i,s} \leq \frac{\delta}{\alpha_i^{\frac{b}{s+b}}}\Big\}.$$

(3) Solve $x_{i,\alpha,s} := x^\delta_{n_i,\alpha_i,s}$ by using the iteration (3.1).
(4) If

$$\|x_{i,\alpha,s} - x_{j,\alpha,s}\| > 4\bar{C}\frac{\delta}{\alpha_j^{\frac{b}{s+b}}}, \quad j < i,$$

then take $k = i - 1$ and return $x_{k,\alpha,s}$.
(5) Else set $i = i + 1$ and go to (2).

7 Numerical Experiments

In this section we present a numerical experiment for the elliptic boundary-value problem (1.2) and compare the results of method (3.1) with that of method (1.4). Let us define the linear operator

$$L : H^2 \cap H^1_0[0,1] \subset L^2[0,1] \to L^2[0,1]$$

by $Lx = -x''$. Then L is densely defined, self-adjoint and positive definite [16] and the Hilbert scale $\{\mathcal{X}\}_s$ generated by L is given by

$$\mathcal{X}_s = \Big\{x \in H^s[0,1] : x^{(2l)}(0) = x^{(2l)}(1) = 0,\ l = 0, 1, \ldots, \Big[\frac{s}{2} - \frac{1}{4}\Big]\Big\}$$

for any $s \in \mathbb{R}$, where $H^s[0,1]$ is the usual Sobolev space and

$$\|x\|_s = \int_0^1 |x^{(s)}(t)|\,dt$$

for all $s = 0, 1, 2, \ldots$. We have taken $s = b = 2$ in our computation. Tables 1 and 2 gives the number of iterations, alpha and the relative error.

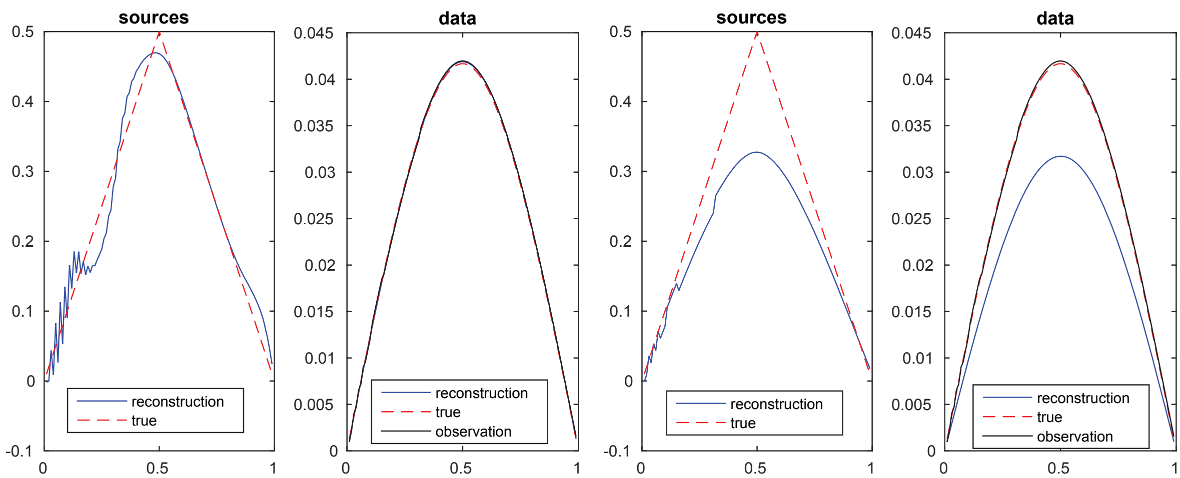
Remark 7.1. From the tables and figures, one can see that method (3.1) gives a better approximation than method (1.4).

	Method (3.1)				Method (1.4)			
Function	k	n_k	$\alpha(k)$	$\frac{\|\hat{x}-x^\delta_{n,\alpha_k,s}\|}{\|x^\delta_{n,\alpha_k,s}\|}$	k	n_k	$\alpha(k)$	$\frac{\|\hat{x}-x^\delta_{n,\alpha_k}\|}{\|x^\delta_{n,\alpha_k}\|}$
$\hat{x} = \min\{x, 1-x\}$, $x \in [0,1]$	5	31	0.0829	0.0074	7	24	0.0312	0.3021
$\hat{x} = x^2$ if $0.2 < x < 0.7$, $\hat{x} = x$ else	5	32	0.0954	0.0093	6	27	0.0474	0.4655

Table 1: The number of iterations, alpha and the error for $\mu = 1.15$, $\delta = \frac{1}{153}$, $\beta = 0.25$.

Function	Method (3.1)				Method (1.4)			
	k	n_k	$\alpha(k)$	$\frac{\|\hat{x}-x^{\delta}_{n,\alpha_k,s}\|}{\|x^{\delta}_{n,\alpha_k,s}\|}$	k	n_k	$\alpha(k)$	$\frac{\|\hat{x}-x^{\delta}_{n,\alpha_k}\|}{\|x^{\delta}_{n,\alpha_k}\|}$
$\hat{x} = \min\{x, 1-x\}, x \in [0, 1]$	23	15	0.0080	0.0085	20	16	0.0100	0.0926
$\hat{x} = x^2$ if $0.2 < x < 0.7$, $\hat{x} = x$ else	12	20	0.0245	0.0086	15	18	0.0157	0.1548

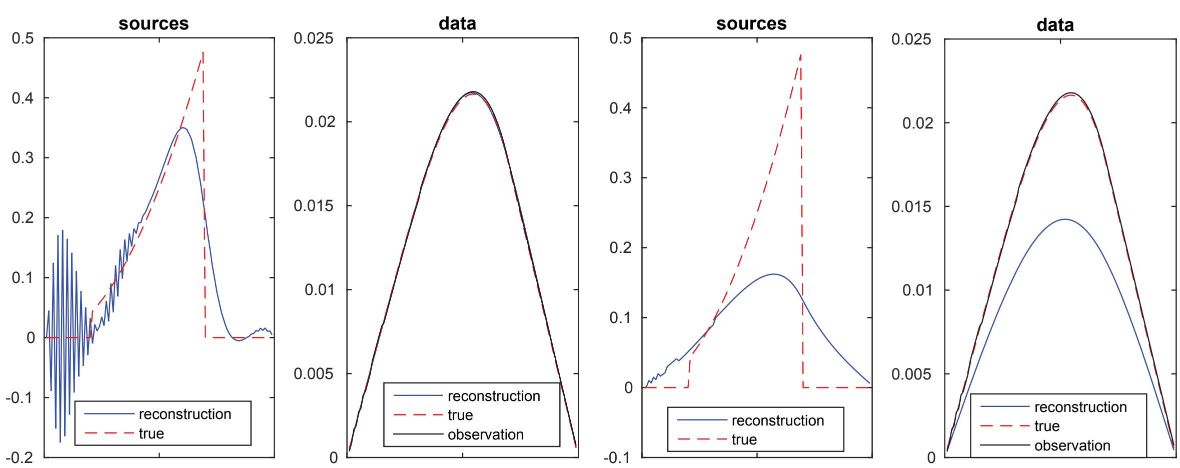
Table 2: The number of iterations, alpha and the error for $\mu = 1.25$, $\delta = \frac{1}{590}$, $\beta = 0.25$.



(a) Exact and approximate data and solution of method (3.1) for $\hat{x} = \min\{x, 1-x\}$.

(b) Exact and approximate data and solution of method (1.4) for $\hat{x} = \min\{x, 1-x\}$.

Figure 1: Exact solution and approximated solution for $\mu = 1.15$, $\delta = \frac{1}{153}$, $\beta = 0.25$.



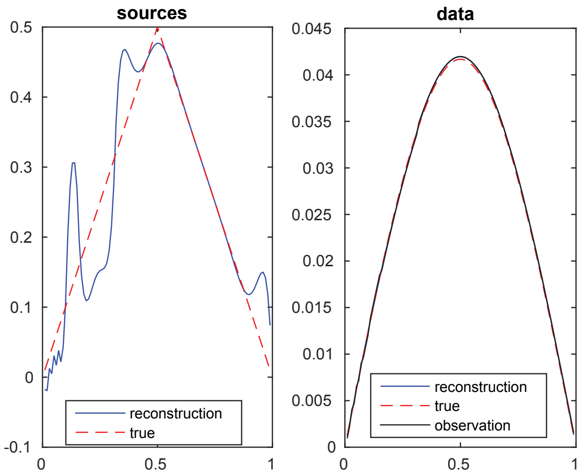
(a) Exact and approximate data and solution of method (3.1) for $\hat{x} = x^2$ if $0.2 < x < 0.7$ and $\hat{x} = x$ else.

(b) Exact and approximate data and solution of method (1.4) for $\hat{x} = x^2$ if $0.2 < x < 0.7$ and $\hat{x} = x$ else.

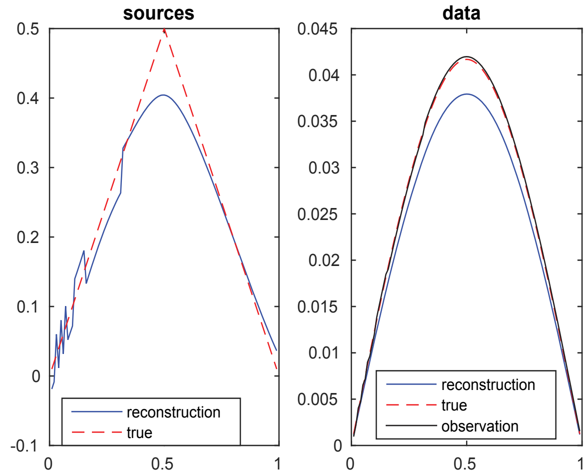
Figure 2: Exact solution and approximated solution for $\mu = 1.15$, $\delta = \frac{1}{153}$, $\beta = 0.25$.

8 Conclusion

In this paper we considered a derivative-free iterative method for approximately solving ill-posed equations involving a monotone operator in the setting of Hilbert scales. We obtained an optimal order error estimate under a general Hölder-type source condition. Also we considered the adaptive parameter choice strategy considered by Pereverzev and Schock [27] for choosing the regularization parameter.

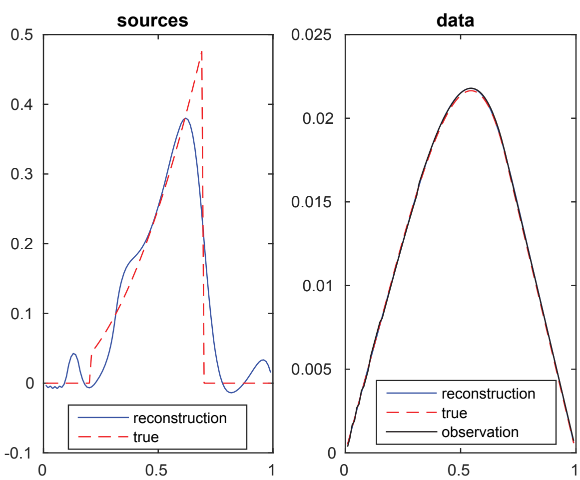


(a) Exact and approximate data and solution of method (3.1) for $\hat{x} = \min\{x, 1 - x\}$.

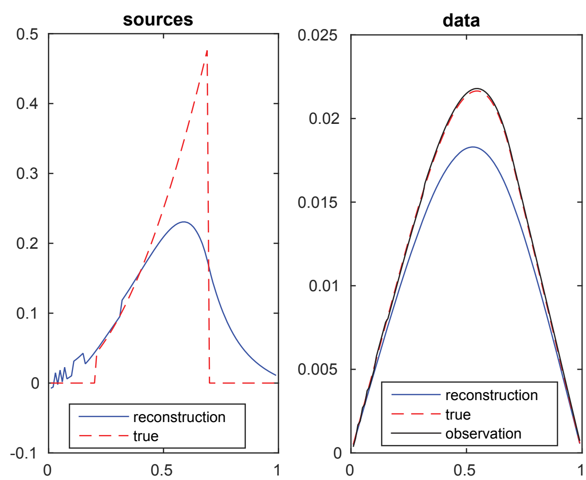


(b) Exact and approximate data and solution of method (1.4) for $\hat{x} = \min\{x, 1 - x\}$.

Figure 3: Exact solution and approximated solution for $\mu = 1.25$, $\delta = \frac{1}{590}$, $\beta = 0.25$.



(a) Exact and approximate data and solution of method (3.1) for $\hat{x} = x^2$ if $0.2 < x < 0.7$and $\hat{x} = x$ else.



(b) Exact and approximate data and solution of method (1.4) for $\hat{x} = x^2$ if $0.2 < x < 0.7$ and $\hat{x} = x$ else.

Figure 4: Exact solution and approximated solution for $\mu = 1.25$, $\delta = \frac{1}{590}$, $\beta = 0.25$.

Funding: The work of Santhosh George is supported by the Core Research Grant by SERB, Department of Science and Technology, Government of India, EMR/2017/001594. K. Kanagaraj would like to thank National Institute of Technology Karnataka, India, for the financial support.

References

[1] J. I. Al'ber, The solution by the regularization method of operator equations of the first kind with accretive operators in a Banach space, *Differ. Uravn.* **11** (1975), no. 12, 2242–2248, 2302.
[2] Y. Alber and I. Ryazantseva, *Nonlinear Ill-Posed Problems of Monotone Type*, Springer, Dordrecht, 2006.
[3] N. Buong, Convergence rates in regularization for nonlinear ill-posed equations under accretive perturbations, *Zh. Vychisl. Mat. Mat. Fiz.* **44** (2004), no. 3, 397–402.
[4] N. Buong, On nonlinear ill-posed accretive equations, *Southeast Asian Bull. Math.* **28** (2004), no. 4, 595–600.
[5] N. Buong and N. T. H. Phuong, Convergence rates in regularization for nonlinear ill-posed equations involving *m*-accretive mappings in Banach spaces, *Appl. Math. Sci. (Ruse)* **6** (2012), no. 61–64, 3109–3117.

[6] H. Egger and A. Neubauer, Preconditioning Landweber iteration in Hilbert scales, *Numer. Math.* **101** (2005), no. 4, 643–662.
[7] H. W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Math. Appl. 375, Kluwer Academic, Dordrecht, 1996.
[8] S. George and M. Kunhanandan, An iterative regularization method for ill-posed Hammerstein type operator equation, *J. Inverse Ill-Posed Probl.* **17** (2009), no. 9, 831–844.
[9] S. George and M. T. Nair, Error bounds and parameter choice strategies for simplified regularization in Hilbert scales, *Integral Equations Operator Theory* **29** (1997), no. 2, 231–242.
[10] S. George and M. T. Nair, An optimal order yielding discrepancy principle for simplified regularization of ill-posed problems in Hilbert scales, *Int. J. Math. Math. Sci.* (2003), no. 39, 2487–2499.
[11] S. George and M. T. Nair, A modified Newton–Lavrentiev regularization for nonlinear ill-posed Hammerstein-type operator equations, *J. Complexity* **24** (2008), no. 2, 228–240.
[12] S. George and M. T. Nair, A derivative-free iterative method for nonlinear ill-posed equations with monotone operators, *J. Inverse Ill-Posed Probl.* **25** (2017), no. 5, 543–551.
[13] S. George, S. Pareth and M. Kunhanandan, Newton Lavrentiev regularization for ill-posed operator equations in Hilbert scales, *Appl. Math. Comput.* **219** (2013), no. 24, 11191–11197.
[14] A. Goldenshluger and S. V. Pereverzev, Adaptive estimation of linear functionals in Hilbert scales from indirect white noise observations, *Probab. Theory Related Fields* **118** (2000), no. 2, 169–186.
[15] B. Hofmann, B. Kaltenbacher and E. Resmerita, Lavrentiev's regularization method in Hilbert spaces revisited, *Inverse Probl. Imaging* **10** (2016), no. 3, 741–764.
[16] Q.-N. Jin, Error estimates of some Newton-type methods for solving nonlinear inverse problems in Hilbert scales, *Inverse Problems* **16** (2000), no. 1, 187–197.
[17] S. G. Kreĭn and J. I. Petunin, Scales of Banach spaces, *Russian Math. Surveys* **21** (1966), 85–160.
[18] M. T. Nair, *Functional Analysis: A First Course*, 4th print, Prentice-Hall, New Delhi, 2014.
[19] F. Liu and M. Z. Nashed, Tikhonov regularization of nonlinear ill-posed problems with closed operators in Hilbert scales, *J. Inverse Ill-Posed Probl.* **5** (1997), no. 4, 363–376.
[20] S. Lu, S. V. Pereverzev, Y. Shao and U. Tautenhahn, On the generalized discrepancy principle for Tikhonov regularization in Hilbert scales, *J. Integral Equations Appl.* **22** (2010), no. 3, 483–517.
[21] P. Mahale, Simplified iterated Lavrentiev regularization for nonlinear ill-posed monotone operator equations, *Comput. Methods Appl. Math.* **17** (2017), no. 2, 269–285.
[22] P. Mathé and S. V. Pereverzev, Geometry of linear ill-posed problems in variable Hilbert scales, *Inverse Problems* **19** (2003), no. 3, 789–803.
[23] F. Natterer, Error bounds for Tikhonov regularization in Hilbert scales, *Appl. Anal.* **18** (1984), no. 1–2, 29–37.
[24] A. Neubauer, An a posteriori parameter choice for Tikhonov regularization in Hilbert scales leading to optimal convergence rates, *SIAM J. Numer. Anal.* **25** (1988), no. 6, 1313–1326.
[25] A. Neubauer, Tikhonov regularization of nonlinear ill-posed problems in Hilbert scales, *Appl. Anal.* **46** (1992), no. 1–2, 59–72.
[26] A. Neubauer, On Landweber iteration for nonlinear ill-posed problems in Hilbert scales, *Numer. Math.* **85** (2000), no. 2, 309–328.
[27] S. Pereverzev and E. Schock, On the adaptive selection of the parameter in regularization of ill-posed problems, *SIAM J. Numer. Anal.* **43** (2005), no. 5, 2060–2076.
[28] E. V. Semenova, Lavrentiev regularization and balancing principle for solving ill-posed problems with monotone operators, *Comput. Methods Appl. Math.* **10** (2010), no. 4, 444–454.
[29] U. Tautenhahn, Error estimates for regularization methods in Hilbert scales, *SIAM J. Numer. Anal.* **33** (1996), no. 6, 2120–2130.
[30] U. Tautenhahn, On a general regularization scheme for nonlinear ill-posed problems. II. Regularization in Hilbert scales, *Inverse Problems* **14** (1998), no. 6, 1607–1616.
[31] U. Tautenhahn, On the method of Lavrentiev regularization for nonlinear ill-posed problems, *Inverse Problems* **18** (2002), no. 1, 191–207.
[32] V. Vasin and S. George, An analysis of Lavrentiev regularization method and Newton type process for nonlinear ill-posed problems, *Appl. Math. Comput.* **230** (2014), 406–413.