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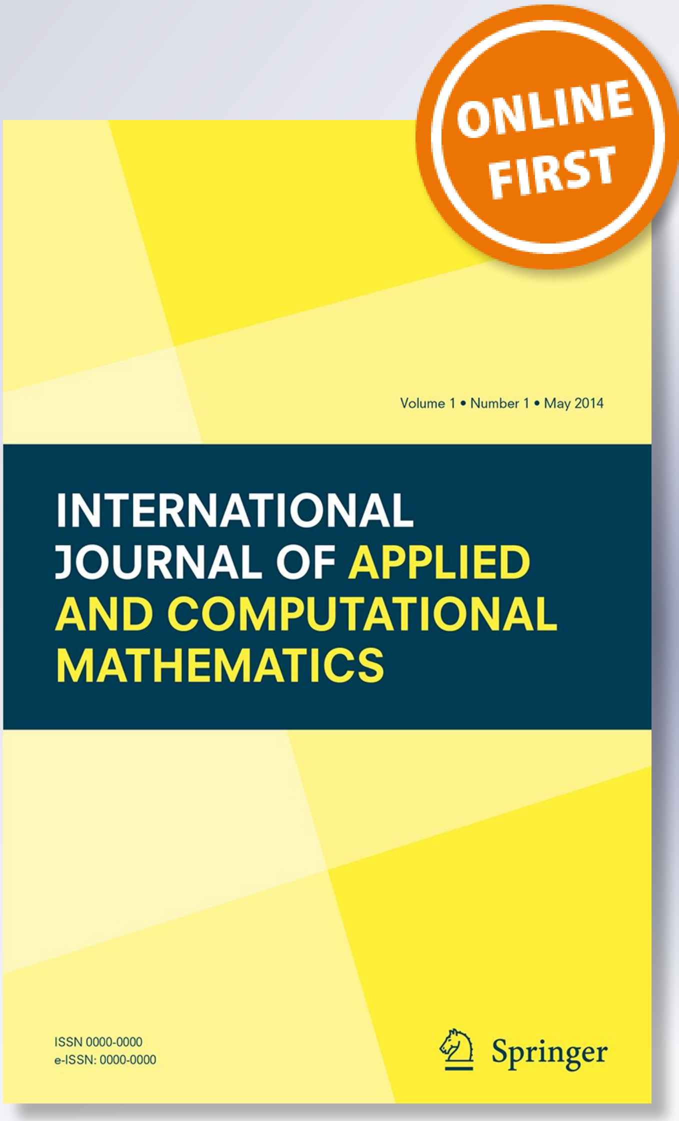
International Journal of Applied and Computational Mathematics

ISSN 2349-5103

Int. J. Appl. Comput. Math
DOI 10.1007/s40819-017-0411-8



Springer

Int. J. Appl. Comput. Math
DOI 10.1007/s40819-017-0411-8

ORIGINAL PAPER

Convergence of a Tikhonov Gradient Type-Method for Nonlinear Ill-Posed Equations

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Abstract In this study Tikhonov Gradient type-method is considered for nonlinear ill-posed operator equations. In our convergence analysis, we use hypotheses only on the first Frec̀het derivative of F in contrast to the higher order Frec̀het derivatives used in the earlier studies. We obtained 'optimal' order error estimate by choosing the regularization parameter according to the adaptive method proposed by Pereverzev and Schock (SIAM J Numer Anal 43(5):2060–2076, 2005).

Keywords Ill-posed equations · Iterative method · Tikhonov regularization · Adaptive method

Mathematics Subject Classification 65J15 · 47H17 · 49M15

Introduction

Inverse problems are wide in range, are important in applied mathematics and other sciences which have been witnessed a rapid growth over past few decades. Inverse problems have wide variety of applications in sciences and engineering. A well known and prominently accepted real world medical application includes tomography, cell detection in various cancer diseases, which helps to calculate the defective cell densities in human body.

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Published online: 05 September 2017

Let X and Y be Hilbert spaces. Consider the ill-posed operator equation

$$F(x) = y \tag{1}$$

where F is a nonlinear operator from a convex domain $D(F)$ in X to Y. In general (1) is ill posed in the sense that its solution need not depend continuously on the data.

The inner product and corresponding norms in X and Y are respectively, denoted by $\langle ., . \rangle$ and $\|.\|$. $B(x, r)$ and $\overline{B(x, r)}$ stand, respectively for open and closed balls in X with center $x \in X$ and radius $r > 0$.

Let $\hat{x}$ be such that $F(\hat{x}) = y$ and let y^δ be the available data with

$$\|y - y^\delta\| \leq \delta. \tag{2}$$

Regularization methods [1,2,4,10] are used to solve non-linear ill-posed equations. Tikhonov regularization is widely used to approximate $\hat{x}$, in which the minimizer of

$$J_\alpha(x) = \min_{x \in D(F)} \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2 \tag{3}$$

is used as an approximation for $\hat{x}$. Here $\alpha > 0$ is a small regularization parameter. It is known [9] that $J_\alpha(x)$ has a unique solution x_α^δ if F is weakly (sequentially) closed, continuous, and Fréchet differentiable with convex domain $D(F)$.

In [8], Ramlau considered iterative method

$$x_{n+1}^\delta = x_n + \beta_n \left(F'\left(x_n^\delta\right)^* \left(y^\delta - F\left(x_n^\delta\right)\right) + \alpha_n \left(x_n^\delta - x_0\right)\right) \tag{4}$$

where β_n is a scaling parameter and α_n is the regularization parameter to obtain approximation for $\hat{x}$. The following assumptions $(\mathcal{A})$ are used in [8].

$(\mathcal{A}_1)$ F is twice differentiable in the sense of Frechet with continuous second derivative.
$(\mathcal{A}_2)$ F^1 satisfies the Lipschitz continuous:

$$\|F'(x_1) - F'(x_2)\| \leq L\|x_1 - x_2\|.$$

$(\mathcal{A}_3)$ $\hat{x} - x_0 = (F'(\hat{x})^* F'(\hat{x}))$ for some $w \in Y$.
$(\mathcal{A}_4)$ $\|w\| \leq q$ and $Lq \leq 0.241$.

And also assumed that Lipschitz constant L is given. Note that β_n in (4) is depending on the Lipschitz constant L and satisfying (see (4.15) in [8])

$$\beta_n \leq \min\left\{\frac{\gamma\alpha}{\|\bigtriangledown J_\alpha(x_n)\|^2}, \frac{4\gamma\alpha}{4K^2 + 4\alpha + 12q\alpha L + 4KL + L^2} \frac{(J_\alpha(x_n) - \phi_{min,n})}{\|\bigtriangledown J_\alpha(x_n)\|^2}\right\}.$$

Here $\phi_{min,n} = \min\{J_\alpha(x_n) + t \bigtriangledown J_\alpha(x_n) : t \in \mathbb{R}^+\}$, $K = \max\{L\|y^\delta - F(x_0)\| + \|F'(x_0)\|, \frac{2Lq}{\sqrt{1-Lq}} + (\frac{1}{1-Lq} + 1)\|F'(x_0)\|\}$ (see (3.14) in [8]) and γ is such that $\gamma + 3L\|w\| \leq \gamma + 3Lq < 1$ (see (3.20) in [8]).

The purpose of this paper is to consider the local convergence of the modified form of the iterative procedure (4), but with a fixed β (a generic constant) and α instead of β_n and α_n. Here β is depending only on α and $\|F'(\cdot)\|$. Precisely we consider the iteration defined for $n = 0, 1, 2, 3\ldots$ by

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - \beta\left(F'\left(x_{n,\alpha}^\delta\right)^* \left(F(x_n^\delta) - y^\delta\right) + \alpha\left(x_{n,\alpha}^\delta - x_0\right)\right). \tag{5}$$

The iterative procedure in (4) is a bit cumbersome than (5). Our approach in this paper is two fold: (i) using hypothesis $(\mathcal{A}_2)$ we prove the convergence of $\{x_{n,\alpha}^\delta\}$ in (5) to x_α^δ (ii) instead of

$(\mathcal{A}_2)$, using two additional Assumptions (2.5 and 2.6) we prove the convergence of $\{x_{n,\alpha}^{\delta}\}$ in (5) to x_{α}^{δ}. We also obtained an error estimate for $\|x_{n,\alpha}^{\delta}-\hat{x}\|$ using source condition on $\hat{x}-x_0$ involving the operator $F'(\hat{x})$ (see Assumptions 3.1). Furthermore, our analysis is simpler than the analysis in [8]. One of the main differences of our approach to that of [8] is, that we fix the scaling and regularization parameters during the iteration.

The rest of the paper is structured as follows: In "Method and Convergence Analysis" section the convergence analysis of (5) is given. In "Error Estimates" section we provide error bounds. Finally the paper ends with a conclusion in "Conclusion" section.

Method and Convergence Analysis

We present local convergence analysis of method (5) in this section. Let $\delta_0 > 0$, $a_0 > 0$, $r_0 > 0$ and $r > 0$ be some constants with $L\delta_0 < a_0$, $\|x_0-\hat{x}\| \le r_0$ with $r_0 < \frac{\sqrt{\alpha}}{L} - \frac{\delta}{\sqrt{\alpha}}$ and

$$2(r_0+1) \le r. \tag{6}$$

Let $M > 0$ be such that

$$\|F'(x)\| \le M, \quad \forall x \in \overline{B(x_0, r)}, \tag{7}$$

$\delta \in (0, \delta_0]$ and $\alpha \in [\min\{L\delta, \delta^2\}, a_0]$. Let

$$\beta = \frac{1}{M^2+a_0} \tag{8}$$

and

$$q_{\alpha,\beta} = 1-\alpha\beta+\beta L(\delta+\sqrt{\alpha}r_0)+\frac{\beta M L}{2}r. \tag{9}$$

Hereafter for simplicity we use the notation $x_n := x_{n,\alpha}^{\delta}$. The following Lemmas are used to prove our main results.

Lemma 2.1 [11, Proposition 2.1] *Let x_{α}^{δ} be the minimizer of* (3). *Then,*

$$\|x_{\alpha}^{\delta}-x_0\| \le \frac{\delta}{\sqrt{\alpha}} + \|x_0-\hat{x}\|.$$

Lemma 2.2 [6, Lemma 2.3] *Let a_n be the sequence satisfying $0 \le a_n \le a$ and $\lim a_n \le a$. Moreover, we assume that γ_n be the sequence satisfying*

$$0 \le \gamma_{n+1} \le a_n + b\gamma_n + c\gamma_n^2 \tag{10}$$

with $n \in \mathbb{N}$ and $\gamma_0 \ge 0$ that holds for some $b, c \ge 0$. Let γ' and $\bar{\gamma}$ be defined as $\gamma' = \frac{2a}{1-b+\sqrt{(1-b)^2-4ac}}$ and $\bar{\gamma} = \frac{1-b+\sqrt{(1-b)^2-4ac}}{2c}$. If $b+2\sqrt{ac} < 1$ and if $\gamma_0 \le \bar{\gamma}$, then

$$\gamma_n \le \max\{\gamma_0, \gamma'\}.$$

Theorem 2.3 *Let x_n be as in* (5) *and let $r < \frac{2(\alpha-L(\delta+\sqrt{\alpha}r_0))}{ML}$. Then for each $\delta \in (0, \delta_0]$, $\alpha \in [\min\{L\delta, \delta^2\}, a_0]$, the sequence $\{x_n\}$ is in $B(x_0, r)$ and $\lim_{n\to\infty} x_n = x_{\alpha}^{\delta}$. Further*

$$\|x_{n+1}-x_{\alpha}^{\delta}\| \le q_{\alpha,\beta}^{n+1}\|x_0-x_{\alpha}^{\delta}\| \tag{11}$$

where $q_{\alpha,\beta}$ is as in (9).

Proof Clearly, $x_0 \in \overline{B(x_0, r)}$. Let $A_n := \int_0^1 F'(x_\alpha^\delta + t(x_n - x_\alpha^\delta))dt$. By Lemma 2.1, we have $x_\alpha^\delta \in B(x_0, r)$, hence A_0 is well defined and $\|A_0\| \leq M$. Assume that for some $n > 0$, $x_n \in B(x_0, r)$ and A_n is well defined. Then, since x_α^δ satisfies the Euler equation

$$F'\left(x_\alpha^\delta\right)^*\left(F\left(x_\alpha^\delta\right) - y^\delta\right) + \alpha\left(x_\alpha^\delta - x_0\right) = 0 \tag{12}$$

of the Tikhonov's functional $J_\alpha(x)$, we have,

$$\begin{aligned} x_{n+1} - x_\alpha^\delta &= x_n - x_\alpha^\delta - \beta\left[F'(x_n)^*\left(F(x_n) - F\left(x_\alpha^\delta\right)\right) + \alpha\left(x_n - x_\alpha^\delta\right)\right] \\ &\quad + \beta\left[F'\left(x_\alpha^\delta\right)^* - F'(x_n)^*\right]\left(F\left(x_\alpha^\delta\right) - y^\delta\right) \\ &= x_n - x_\alpha^\delta - \beta\left[F'(x_n)^* A_n + \alpha I\right]\left(x_n - x_\alpha^\delta\right) \\ &\quad + \beta\left[F'\left(x_\alpha^\delta\right)^* - F'(x_n)^*\right]\left(F\left(x_\alpha^\delta\right) - y^\delta\right) \\ &= x_n - x_\alpha^\delta - \beta\left[F'(x_n)^*\left(A_n - F'(x_n)\right)\right]\left(x_n - x_\alpha^\delta\right) \\ &\quad - \beta\left[F'(x_n)^* F'(x_n) + \alpha I\right]\left(x_n - x_\alpha^\delta\right) \\ &\quad + \beta\left[F'\left(x_\alpha^\delta\right)^* - F'(x_n)^*\right]\left(F\left(x_\alpha^\delta\right) - y^\delta\right) \\ &= \left[I - \beta\left(F'(x_n)^* F'(x_n) + \alpha I\right)\right]\left(x_n - x_\alpha^\delta\right) \\ &\quad - \beta\left[F'(x_n)^*\left(A_n - F'(x_n)\right)\right]\left(x_n - x_\alpha^\delta\right) \\ &\quad + \beta\left[F'\left(x_\alpha^\delta\right)^* - F'(x_n)^*\right]\left(F\left(x_\alpha^\delta\right) - y^\delta\right). \end{aligned} \tag{13}$$

Now since $I - \beta(F'(x_n)^* F'(x_n) + \alpha I)$ is a positive self-adjoint operator,

$$\begin{aligned} \|I - \beta(F'(x_n)^* F'(x_n) + \alpha I)\| &= \sup_{\|x\|=1} \left|\left\langle\left(I - \beta\left(F'(x_n)^* F'(x_n) + \alpha I\right)\right)x, x\right\rangle\right| \\ &= \sup_{\|x\|=1} |(1 - \beta\alpha)\langle x, x\rangle - \beta\langle F'(x_n)^* F'(x_n)x, x\rangle| \\ &\leq 1 - \alpha\beta. \end{aligned} \tag{14}$$

The last step follows from relation

$$\beta|\langle F'(x_n)^* F'(x_n)x, x\rangle| \leq \beta\|F'(x_n)\|^2 \leq \beta M^2 \leq \frac{1}{M^2 + \alpha} M^2 = 1 - \frac{\alpha}{M^2 + \alpha} \leq 1 - \beta\alpha.$$

Using $(\mathcal{A}_2)$, we have

$$\begin{aligned} &\|\left[\beta F'(x_n)^*\left(A_n - F'(x_n)\right)\right]\left(x_n - x_\alpha^\delta\right)\| \\ &\leq \beta F'(x_n)^* \int_0^1 \left(F'\left(x_\alpha^\delta + t\left(x_n - x_\alpha^\delta\right)\right) - F'(x_n)\right) dt \left(x_n - x_\alpha^\delta\right)\| \\ &\leq \beta \frac{ML}{2} \|x_n - x_\alpha^\delta\|^2 \end{aligned}$$

and

$$\begin{aligned} &\left\|\beta\left[F'\left(x_\alpha^\delta\right)^* - F'(x_n)^*\right]\left(F\left(x_\alpha^\delta\right) - y^\delta\right)\right\| \\ &\leq \beta\left\|F'\left(x_\alpha^\delta\right)^* - F'(x_n)^*\right\|\left\|F\left(x_\alpha^\delta\right) - y^\delta\right\| \\ &= \beta\left\|F'\left(x_\alpha^\delta\right) - F'(x_n)\right\|\left\|F\left(x_\alpha^\delta\right) - y^\delta\right\| \\ &\leq \beta L\left\|x_n - x_\alpha^\delta\right\|\left\|F\left(x_\alpha^\delta\right) - y^\delta\right\|. \end{aligned}$$

Now using (3), we have

$$\left\|F\left(x_\alpha^\delta\right)-y^\delta\right\| \leq \delta+\sqrt{\alpha} r_0. \tag{15}$$

Hence,

$$\left\|x_{n+1}-x_\alpha^\delta\right\| \leq\left(1-\alpha \beta+\beta L(\delta+\sqrt{\alpha} r_0)\right)\left\|x_n-x_\alpha^\delta\right\|+\frac{\beta M L}{2}\left\|x_n-x_\alpha^\delta\right\|^2 .$$

The above expression is of the form (10), where $a_n=0$, $b=1-\alpha\beta+\beta L(\delta+\sqrt{\alpha}r_0)$, $\gamma_n=\|x_n-x_\alpha^\delta\|$ and $c=\frac{\beta M L}{2}$. We have by the condition on r_0, $b+2\sqrt{ac}=b<1$ and

$$\gamma_0=\left\|x_0-x_\alpha^\delta\right\| \leq \frac{1-b}{c}=\bar{\gamma}.$$

Hence by Lemma 2.2, we have

$$\begin{aligned}
\left\|x_{n+1}-x_\alpha^\delta\right\| & \leq\left(1-\alpha \beta+\beta L(\delta+\sqrt{\alpha} r_0)\right)\left\|x_n-x_\alpha^\delta\right\|+\frac{\beta M L}{2}\left\|x_0-x_\alpha^\delta\right\|\left\|x_n-x_\alpha^\delta\right\| \\
& \leq\left(1-\alpha \beta+\beta L(\delta+\sqrt{\alpha} r_0)\right)\left\|x_n-x_\alpha^\delta\right\|+\frac{\beta M L}{2} r\left\|x_n-x_\alpha^\delta\right\| \\
& \leq q_{\alpha, \beta}\left\|x_n-x_\alpha^\delta\right\| .
\end{aligned} \tag{16}$$

Thus, since $r<\frac{2(\alpha-L(\delta+\sqrt{\alpha}r_0))}{ML}$, we have $q_{\alpha,\beta}<1$ and

$$\left\|x_{n+1}-x_\alpha^\delta\right\|<\left\|x_0-x_\alpha^\delta\right\| \leq r$$

and

$$\left\|x_{n+1}-x_0\right\|<2\left\|x_0-x_\alpha^\delta\right\| \leq 2(r_0+1) \leq r$$

i.e., $x_{n+1}\in B(x_0,r)$. Also, for $0\leq t\leq 1$,

$$\left\|x_\alpha^\delta+t(x_{n+1}-x_\alpha^\delta)-x_0\right\|=\left\|x_\alpha^\delta-x_0+t(x_{n+1}-x_\alpha^\delta)\right\|<2(r_0+1) \leq r .$$

Hence, $x_\alpha^\delta+t(x_{n+1}-x_\alpha^\delta)\in B(x_0,r)$ and A_{n+1} is well defined with $\|A_{n+1}\|\leq M$. Thus, by induction x_n is well defined and remains in $B(x_0,r)$ for each $n=0,1,2,\ldots$ By letting $n\to\infty$ in (5), we obtain the convergence of x_n to x_α^δ. The estimate (11) now follows from (16). □

Remark 2.4 Note that the condition $r_0<\frac{\sqrt{\alpha}}{L}-\frac{\delta}{\sqrt{\alpha}}$ is too restrictive. We can avoid this restriction by imposing some additional assumptions (see Assumptions 2.5 and 2.6 below). We also prove the convergence of (5) using the assumptions below.

Assumption 2.5 Suppose there exists a constant $K_1>0$ such that for all $u,v\in B(x_0,r)\subseteq D(F)$ and $w\in X$, there exists element $\phi_1(u,v,w)\in X$ such that $[F'(u)-F'(v)]w=F'(v)\phi_1(u,v,w)$, $\|\phi_1(u,v,w)\|\leq K_1\|w\|\|u-v\|$.

Assumption 2.6 [9] There exists K_2 such that for every $x,y\in B(x_0,r)\subseteq D(F)$ and $h\in X$, there exists element $\phi_2(x,y,h)\in X$ such that $[F'(x)^*-F'(y)^*]h=\phi_2(x,y,F'(y)^*h)$ with $\|\phi_2(x,y,F'(y)^*h)\|\leq K_2\|x-y\|\|F'(z)^*h\|$.

Next, we shall give an example satisfying Assumptions 2.5 and 2.6.

Example 2.7 [3,9]. Consider the nonlinear Hammerstein operator

$$(Fx)(t) = \int_0^1 k(t,\tau) g(\tau, x(\tau))\, d\tau,$$

with k continuous and g sufficiently smooth so that $F : H^1((0,1)) \to L^2((0,1))$ is Frečhet differentiable with respect to x and

$$F'(x)h(t) = \int_0^1 k(t,\tau) g_x(\tau, x(\tau))\, h(\tau) d\tau.$$

Then F satisfies Assumptions 2.5 and 2.6 (see [9, Lemma 2.8]).

Let $\delta_1 > 0,\ b_0 > 0$ and $\bar{r} > 0$ be some constants with $\delta_1^2 < b_0$ and

$$2(r_0 + 1) \le \bar{r}. \tag{17}$$

Let $\delta \in (0, \delta_1]$ and $\alpha \in [\delta^2, b_0]$. Let

$$\beta = \frac{1}{M^2 + b_0} \tag{18}$$

and

$$\bar{q}_{\alpha,\beta} = 1 - \alpha\beta + \alpha\beta K_2 \bar{r} + \frac{\beta M^2 K_1}{2}\bar{r}. \tag{19}$$

Theorem 2.8 *Let x_n be as in (5) and let $\bar{r} < \frac{2\alpha}{2\alpha K_2 + M^2 K_1}$. Then for each $\delta \in (0, \delta_1]$, $\alpha \in [\delta^2, b_0]$, the sequence $\{x_n\}$ is in $B(x_0, \bar{r})$ and $\lim_{n\to\infty} x_n = x_\alpha^\delta$. Further*

$$\left\| x_{n+1} - x_\alpha^\delta \right\| \le \bar{q}_{\alpha,\beta}^{n+1} \left\| x_0 - x_\alpha^\delta \right\| \tag{20}$$

where $\bar{q}_{\alpha,\beta}$ is as in (19).

Proof Clearly, $x_0 \in \overline{B(x_0, \bar{r})}$. Let $A_n := \int_0^1 F'(x_\alpha^\delta + t(x_n - x_\alpha^\delta))dt$. By Lemma 2.1, we have $x_\alpha^\delta \in B(x_0, \bar{r})$, hence A_0 is well defined and $\|A_0\| \le M$. Assume that for some $n > 0$, $x_n \in B(x_0, \bar{r})$ and A_n is well defined. Using (13), Assumptions 2.5 and 2.6 we have

$$\begin{aligned}
x_{n+1} - x_\alpha^\delta &= \left[I - \beta\left(F'(x_n)^* F'(x_n)\right) + \alpha I\right]\left(x_n - x_\alpha^\delta\right) \\
&\quad - \beta\left[F'(x_n)^* \int_0^1 F'(x_n)\phi_1\left(x_\alpha^\delta + t\left(x_n - x_\alpha^\delta\right), x_n, x_n - x_\alpha^\delta\right)\right] dt \\
&\quad - \beta\phi_2\left(x_n, x_\alpha^\delta, F'\left(x_\alpha^\delta\right)^*\left(F\left(x_\alpha^\delta\right) - y^\delta\right)\right) \\
&= \left[I - \beta\left(F'(x_n)^* F'(x_n) + \alpha I\right)\right]\left(x_n - x_\alpha^\delta\right) \\
&\quad - \beta\left[F'(x_n)^* F'(x_n) \int_0^1 \phi_1\left(x_\alpha^\delta + t\left(x_n - x_\alpha^\delta\right), x_n, x_n - x_\alpha^\delta\right)\right] dt \\
&\quad - \beta\phi_2\left(x_n, x_\alpha^\delta, -\alpha\left(x_\alpha^\delta - x_0\right)\right).
\end{aligned}$$

Hence, using (14) we have

$$\begin{aligned}\left\|x_{n+1}-x_\alpha^\delta\right\| &\le (1-\alpha\beta)\left\|x_n-x_\alpha^\delta\right\| + \beta M^2 K_1 \left\|x_n-x_\alpha^\delta\right\|^2 \int_0^1 (1-t)dt \\ &\quad +\beta K_2 \alpha \left\|x_n-x_\alpha^\delta\right\| \left\|x_\alpha^\delta - x_0\right\| \\ &\le \left(1-\alpha\beta+\alpha\beta K_2\left\|x_\alpha^\delta-x_0\right\|\right)\left\|x_n-x_\alpha^\delta\right\| + \frac{\beta M^2 K_1}{2}\left\|x_n-x_\alpha^\delta\right\|^2 \\ &\le (1-\alpha\beta+\alpha\beta K_2\bar{r})\left\|x_n-x_\alpha^\delta\right\| + \frac{\beta M^2 K_1}{2}\left\|x_n-x_\alpha^\delta\right\|^2. \end{aligned} \tag{21}$$

Note that (21) is of the form (10), where $a_n = 0$, $b = 1-\alpha\beta+\alpha\beta K_2\bar{r}$, $\gamma_n = \|x_n - x_\alpha^\delta\|$ and $c = \frac{\beta M^2 K_1}{2}$. So by 17, $b + 2\sqrt{ac} = b < 1$ and

$$\gamma_0 = \left\|x_0 - x_\alpha^\delta\right\| \le \frac{1-b}{c} = \bar{\gamma}.$$

Hence by Lemma 2.2, we have

$$\begin{aligned}\left\|x_{n+1}-x_\alpha^\delta\right\| &\le \left(1-\alpha\beta+\alpha\beta K_2\left\|x_0-x_\alpha^\delta\right\| + \frac{\beta M^2 K_1}{2}\left\|x_0-x_\alpha^\delta\right\|\right)\left\|x_n-x_\alpha^\delta\right\| \\ &\le \left(1-\alpha\beta+\alpha\beta K_2\bar{r} + \frac{\beta M^2 K_1}{2}\bar{r}\right)\left\|x_n-x_\alpha^\delta\right\| \\ &\le \bar{q}_{\alpha,\beta}\left\|x_n-x_\alpha^\delta\right\|. \end{aligned} \tag{22}$$

Thus, since $\bar{q}_{\alpha,\beta} < 1$, we have

$$\left\|x_{n+1}-x_\alpha^\delta\right\| < \left\|x_0-x_\alpha^\delta\right\| \le \bar{r}$$

and

$$\|x_{n+1}-x_0\| < 2\left\|x_0-x_\alpha^\delta\right\| \le 2(r_0+1) \le \bar{r}$$

i.e., $x_{n+1} \in B(x_0, \bar{r})$. Also, for $0 \le t \le 1$,

$$\left\|x_\alpha^\delta + t\left(x_{n+1}-x_\alpha^\delta\right) - x_0\right\| = \left\|x_\alpha^\delta - x_0 + t\left(x_{n+1}-x_\alpha^\delta\right)\right\| < 2(r_0+1) \le \bar{r}.$$

Hence, $x_\alpha^\delta + t(x_{n+1} - x_\alpha^\delta) \in B(x_0, \bar{r})$ and A_{n+1} is well defined with $\|A_{n+1}\| \le M$. Thus, x_n is well defined and remains in $B(x_0, \bar{r})$ for each $n = 0, 1, 2, \ldots$ by induction. By letting $n \to \infty$ in (5), we obtain the convergence of x_n to x_α^δ. The estimate (20) now follows from (22). □

Error Estimates

For the convenience of the convergence analysis that follows, we use the following well known Assumption [10].

Assumption 3.1 There exists a continuous, strictly monotonically increasing function $\phi : (0, \bar{a}] \to (0, \infty)$ with $\bar{a} \ge \|F'(\hat{x})\|^2$ satisfying

(i) $\lim_{\lambda\to 0} \phi(\lambda) = 0$.
(ii) $\sup_{\lambda\ge 0} \frac{\alpha\phi(\lambda)}{\lambda+\alpha} \le \phi(\alpha), \forall\lambda \in (0, \bar{a}]$.

(iii) There exists $v \in X$ such that

$$x_0 - \hat{x} = \phi(F'(\hat{x})^* F'(\hat{x}))v.$$

Theorem 3.2 *Let x_α^δ be the minimizer of (3) and let*

$$\bar{r} < \min\left\{\frac{2\alpha}{2\alpha K_2 + M^2 K_1}, \frac{2(1+K_1+K_2)}{2K_1+K_2}\right\}.$$

Then

$$\left\|x_\alpha^\delta - \hat{x}\right\| \leq \frac{1}{1+K_1+K_2-\frac{\bar{r}}{2}(2K_1+K_2)}\left[\frac{\delta}{\sqrt{\alpha}} + \|v\|\phi(\alpha)\right].$$

Proof Let $\hat{M} = \int_0^1 F'(\hat{x} + t(x_\alpha^\delta - \hat{x}))dt$ and $A = F'(x_\alpha^\delta)$. Then from (12) we have

$$\left(A^*\hat{M} + \alpha I\right)\left(x_\alpha^\delta - \hat{x}\right) = A^*(y^\delta - y) + \alpha(x_0 - \hat{x})$$

and

$$\begin{aligned} x_\alpha^\delta - \hat{x} &= (A^*A + \alpha I)^{-1}A^*(A - \hat{M})\left(x_\alpha^\delta - \hat{x}\right) + (A^*A + \alpha I)^{-1}A^*(y^\delta - y) \\ &\quad + (A^*A + \alpha I)^{-1}\alpha(x_0 - \hat{x}). \end{aligned} \tag{23}$$

Therefore

$$\|x_\alpha^\delta - \hat{x}\| \leq \|\Gamma_1\| + \frac{\delta}{\sqrt{\alpha}} + \|\Gamma_2\| \tag{24}$$

where $\Gamma_1 = (A^*A + \alpha I)^{-1}A^*(A - \hat{M})(x_\alpha^\delta - \hat{x})$ and $\Gamma_2 = (A^*A + \alpha I)^{-1}\alpha(x_0 - \hat{x})$. Using definition of $\hat{M}$ and Assumption 2.6, we have in turn

$$\begin{aligned} \Gamma_1 &= (A^*A + \alpha I)^{-1}A^*\left[F'\left(x_\alpha^\delta\right) - \int_0^1 F'\left(\hat{x} + t\left(x_\alpha^\delta - \hat{x}\right)\right)dt\right]\left(x_\alpha^\delta - \hat{x}\right) \\ &= \left(A^*A + \alpha I\right)^{-1}A^*\left[\int_0^1 F'\left(x_\alpha^\delta\right) - F'\left(\hat{x} + t(x_\alpha^\delta - \hat{x})\right)dt\right]\left(x_\alpha^\delta - \hat{x}\right) \\ &= -\left(A^*A + \alpha I\right)^{-1}A^*\int_0^1 A\phi_1\left(\hat{x} + t\left(x_\alpha^\delta - \hat{x}\right), x_\alpha^\delta, x_\alpha^\delta - \hat{x}\right)dt. \end{aligned} \tag{25}$$

Now, by using triangle inequality, Lemma 2.1 and the definition of $\bar{r}$, we have

$$\begin{aligned} \|\Gamma_1\| &\leq \frac{K_1}{2}\left\|x_\alpha^\delta - \hat{x}\right\|^2 \\ &\leq \frac{K_1\bar{r}}{2}\left\|x_\alpha^\delta - \hat{x}\right\|. \end{aligned} \tag{26}$$

Let $\hat{A} := F'(\hat{x})$. Then using Assumptions 2.5, 2.6 and 3.1, we have in turn

$$\begin{aligned}
\|\Gamma_2\| &= \left\|\left[(A^*A+\alpha I)^{-1} - (\hat{A}^*\hat{A}+\alpha I)^{-1}\right]\alpha(x_0-\hat{x}) \right. \\
&\quad \left. + (\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(x_0-\hat{x})\right\| \\
&\leq \left\|\left[(A^*A+\alpha I)^{-1}(\hat{A}^*\hat{A} - A^*A)(\hat{A}^*\hat{A}+\alpha I)^{-1}\right]\alpha(x_0-\hat{x})\right\| \\
&\quad + \left\|(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(x_0-\hat{x})\right\| \\
&\leq \left\|\left[(A^*A+\alpha I)^{-1}((\hat{A}^*-A^*)\hat{A} - A^*(A-\hat{A}))(\hat{A}^*\hat{A}+\alpha I)^{-1}\right]\alpha(x_0-\hat{x})\right\| \\
&\quad + \left\|(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(x_0-\hat{x})\right\| \qquad (27)\\
&\leq \left\|(A^*A+\alpha I)^{-1}(\hat{A}^*-A^*)\hat{A}(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(x_0-\hat{x})\right\| \\
&\quad + \left\|(A^*A+\alpha I)^{-1}A^*(A-\hat{A})(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(x_0-\hat{x})\right\| \\
&\quad + \left\|(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(x_0-\hat{x})\right\| \\
&\leq \|(A^*A+\alpha I)^{-1}\| \left\|\varphi_2\left(x_\alpha^\delta, \hat{x}, \hat{A}^*\hat{A}(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(x_0-\hat{x})\right)\right\| \\
&\quad + \|(A^*A+\alpha I)^{-1}A^*A\| \left\|\varphi_1\left(x_\alpha^\delta, \hat{x}, (\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(x_0-\hat{x})\right)\right\| \\
&\quad + \left\|(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(x_0-\hat{x})\right\| \\
&\leq K_2\left\|x_\alpha^\delta - \hat{x}\right\| \|x_0-\hat{x}\| + K_1\|x_\alpha^\delta-\hat{x}\|\|x_0-\hat{x}\| + \phi(\alpha)\|v\| \\
&\leq (K_1+K_2)r_0\|x_\alpha^\delta-\hat{x}\| + \phi(\alpha)\|v\| \\
&\leq (K_1+K_2)\left(\frac{r}{2}-1\right)\|x_\alpha^\delta-\hat{x}\| + \phi(\alpha)\|v\|. \qquad (28)
\end{aligned}$$

The result now follows from (24), (26) and (28). This completes the proof of the theorem. □

Remark 3.3 If we use $(\mathcal{A}_2)$, instead of Assumptions 2.5 and 2.6, then by (25) we have $\|\Gamma_1\| \leq \frac{Lr}{2\sqrt{\alpha}}\|x_\alpha^\delta - \hat{x}\|$ and by (27) we have $\|\Gamma_2\| \leq \frac{2Lr_0}{\sqrt{\alpha}}\|x_\alpha^\delta - \hat{x}\| + \phi(\alpha)\|v\|$. Hence in this case we have

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{1}{1-\frac{4Lr_0+Lr}{2\sqrt{\alpha}}}\phi(\alpha)\|v\|$$

provided $4Lr_0 + Lr < 2\sqrt{\alpha}$.

Using Theorems 2.8 and 3.2 we have the following theorem.

Remark 3.4 Similar result, as in Theorem 3.2 can be obtained, if we use Theorem 2.3 instead of Theorem 2.8 in the above theorem.

Theorem 3.5 *Let x_n be as in (5) and let the assumptions in Theorems 2.8 and 3.2 be satisfied. Then we have*

$$\left\|x_{n+1,\alpha}^\delta - \hat{x}\right\| \leq \bar{q}_{\alpha,\beta}^{n+1}\bar{r} + \frac{1}{1+K_1+K_2-\frac{\bar{r}}{2}(2K_1+K_2)}\left(\frac{\delta}{\sqrt{\alpha}} + \|v\|\phi(\alpha)\right). \qquad (29)$$

Let

$$n_\delta = \min\left\{n : \bar{q}^n_{\alpha,\beta} \le \frac{\delta}{\sqrt{\alpha}}\right\}. \tag{30}$$

Theorem 3.6 *Let x_n be as in (5) and let the assumptions in Theorem 3.5 be satisfied. Let n_δ be as in (30). Then*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| \le \bar{C}\left(\frac{\delta}{\sqrt{\alpha}} + \phi(\alpha)\right) \tag{31}$$

where $\bar{C} = \max\{\bar{r} + \frac{1}{1+K_1+K_2-\frac{\bar{r}}{2}(2K_1+K_2)}, \frac{\|v\|}{1+K_1+K_2-\frac{\bar{r}}{2}(2K_1+K_2)}\}$.

A Priori Choice of the Parameter

Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \le \|F'(\hat{x})\|^2$. Then for $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ we have $\frac{\delta}{\sqrt{\alpha_\delta}} = \varphi(\alpha_\delta)$, i.e., α_δ is the optimal choice for α. Hence we have the following.

Theorem 3.7 *Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$ for $0 < \lambda \le \|F'(\hat{x})\|^2$, and let the assumptions in Theorem 3.6 holds. For $\delta \in (0, \delta_0]$, let $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ and let n_δ be as in (30). Then*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| = O\left(\psi^{-1}(\delta)\right).$$

Balancing Principle

Observe that the a priori choice of the parameter could be achieved only in the ideal situation when the function ψ is known. To overcome this difficulty Pereverzev and Schock [7] considered the adaptive method in which the parameter $\alpha = \alpha_i$ is chosen from the finite set

$$D := \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N < 1\},$$

and corresponding elements $x^\delta_{n,\alpha_i}, i = 1, 2, \ldots, N$ are studied. Let

$$n_i := \min\left\{n : \bar{q}^n_{\alpha,\beta} \le \frac{\delta}{\sqrt{\alpha_i}}\right\}$$

and let $x^\delta_{\alpha_i} := x^\delta_{n_i,\alpha_i}$. Then from Theorem 3.6, we have

$$\|x^\delta_{\alpha_i} - \hat{x}\| \le \bar{C}\left(\frac{\delta}{\sqrt{\alpha_i}} + \phi(\alpha_i)\right), \quad \forall i = 1, 2, \ldots N.$$

In this study the regularization parameter α from the set D_N defined by

$$D_N := \{\alpha_i = \mu^i\alpha_0 < 1, i = 1, 2, \ldots N\},$$

where $\alpha_0 = \delta^2$ (see [10]) and $\mu > 1$.

Now we state the main result of this section, proof of which is similar to the proof of Theorem 4.4 in [5].

Theorem 3.8 *Suppose there exists $i \in \{0, 1, \ldots N\}$ such that $\phi(\alpha_i) \le \frac{\delta}{\sqrt{\alpha_i}}$. Let assumptions of Theorem 3.5 be satisfied and let*

$$l := \max\left\{i : \phi(\alpha_i) \le \frac{\delta}{\sqrt{\alpha_i}}\right\} < N,$$

$$k := \max\left\{i : \forall j = 1, 2, \ldots, i;\ \left\|x^\delta_{\alpha_i} - x^\delta_{\alpha_j}\right\| \le 4\bar{C}\frac{\delta}{\sqrt{\alpha_j}}\right\}$$

where $\bar{C}$ is as in Theorem 3.6. Then $l \leq k$ and

$$\|x_{\alpha_i}^\delta - \hat{x}\| \leq 6\bar{C}\mu\psi^{-1}(\delta).$$

The choice of the regularization parameter in Theorem 3.8 involves the following steps:

- Choose $\alpha_0 = \delta^2$
- Choose $\alpha_i := \mu^{2i}\alpha_0,\ i = 0, 1, 2, \ldots, N$ with $\mu > 1$.

Algorithm

1. Set $i = 0$.
2. Choose $n_i := \min\left\{n : \sqrt{\alpha_i} q^n \leq \delta\right\}$.
3. Solve $x_i^\delta := x_{n_i,\alpha_i}^\delta$ by using the iteration (5).
4. If $\|x_i^\delta - x_j^\delta\| > 4\bar{C}\frac{1}{\mu^j},\ j < i$, then take $k = i - 1$ and return to x_k.
5. Else set $i = i + 1$ and go to 2.

Conclusion

In this study, presented convergence analysis of a modified Tikhonov gradient [8] type-method for approximately solving the nonlinear ill-posed equations $F(x) = y$. The assumptions used for the convergence analysis in Theorem 2.3 (we used only $(\mathcal{A}_2)$ with an additional assumption on the initial guess) is weaker than that of assumptions in [8].

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