A STUDY ON ILL-POSED EQUATIONS AND ITERATIVE METHODS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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Dedicated to the loving memory of my father,

Dr. K. Remesh.

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **"A STUDY ON ILL-POSED EQUATIONS AND ITERATIVE METHODS"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfilment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to *certify* that the Research Thesis entitled **"A STUDY ON ILL-POSED EQUATIONS AND ITERATIVE METHODS"** submitted by **Ms. KRISHNENDU R**, (Register Number: 197MA006) as the record of the research work carried out by her is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

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ABSTRACT

Many problems that arise in various fields of study can be modeled into equations that are well-posed/ill-posed (linear or nonlinear). Especially in science and engineering, most of the inverse problems are ill-posed. The first half of the thesis focuses on finite dimensional realization of regularization methods for ill-posed problems. The second half deals with iterative methods for solving well-posed nonlinear equations.

It is proved in the literature that the Fractional Tikhonov regularization method (FTR) reduces the over smoothing of the solution compared to the usual Tikhonov regularization method for ill-posed problems. In Chapter 2 of the thesis, the FTR method in the finite dimensional setting is studied. The regularization parameter is chosen using Raus and Gfrerer type discrepancy principle in this Chapter.

The choice of regularization parameter and suitable source condition plays an important role in a regularization method. In Chapter 3, an efficient new parameter choice strategy is introduced. The advantage is that this parameter choice strategy computes the regularization parameter before computing the approximate solution and is dependent on the given data of the problem. This new parameter choice also provide the optimal order. The proposed parameter choice strategy is depending on a new source condition.

Higher order iterative methods are used to solve nonlinear equations. The convergence order of these methods uses Taylor's expansion and assumptions on the higher order Fréchet derivative of the operator. In Chapter 4 and Chapter 5, we have eliminated the use of Taylor's expansion and hence assumptions on the higher order Fréchet derivatives of the operator in the problem. Moreover, the desired convergence order of the iterative method is obtained without using assumptions on the higher order Fréchet derivatives and hence the applicability of these iterative methods are extended to problem which were not posible using earlier studies. These iterative methods are also applied to solve nonlinear ill-posed problems.

Keywords: *ill-posed problems,Tikhonov regularization method, Discrepancy principle, Convergence rate, nonlinear ill-posed equations, finite dimension, Lavrentiev regularization, a new parameter-choice strategy, iterative methods, Newton's midpoint method, Cordero's method, order of convergence, Taylor series expansion, Fréchet derivative.*

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CHAPTER 1

INTRODUCTION

The concept of inverse problems (IPs) has been widely recognized in contemporary Applied Mathematics and Engineering. The direct problems have been studied widely. The IPs are relevant in current research and have to be investigated further. There is no proper Mathematical definition for IP in the literature. One can find effects from causes in the direct problems. Conversely, IPs can be used to find an unknown cause from the observed effect. The most inverse problems are the problems that consist of finding an unknown property of an object from the observation of its response of it to a probing signal. Keller (1976), a well-known American Mathematician, gave a broad definition for IP with his oft-quoted statement that "We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other" in Keller (1976). Many research problems in image processing, machine learning, medical image segmentation, oceanography, geophysics, and many other fields are inverse problems. In general, one can model these IPs into an operator equation of the form

$$\mathscr{F}(x) = y, \tag{1.0.1}$$

where $\mathscr{F} : \mathscr{X} \to \mathscr{Y}$ is a linear or non linear operator between appropriate normed linear spaces $\mathscr{X}$ and $\mathscr{Y}$. In practical cases, the difficulty that arises about this operator equation is to find a solution x in $\mathscr{X}$ for a given $y \in \mathscr{Y}$. These kind of problems usually leads to the formulation of ill-posed problems (definition is given in the following paragraph), which does not guarantee a unique solution continuously depends on available data. The first half of this research work focuses on ill-posed problems. Ill-posed problems are problems that do not satisfy Hadamards's postulates (Hadamard (1953)). According to Hadamard (1953), the problem of solving the operator equation (1.0.1) is said to be well-posed if the following conditions hold:

(i) **Existence:** For each $y \in \mathscr{Y}$, there exists a solution $x \in \mathscr{X}$ of (1.0.1).

(ii) **Uniqueness:** The solution x is unique.

(iii) **Stability:** The dependence of x upon $\mathscr{F}$ is continuous.

In brief, the operator equation (1.0.1) is well posed if and only if the operator $\mathscr{F}$ is bijective and the inverse operator $\mathscr{F}^{-1} : \mathscr{Y} \to \mathscr{X}$ is continuous. An operator equation of the form (1.0.1) which is not well-posed is known as an **ill-posed** problem. First, we look into the linear ill-posed problems which had been widely studied in the literature (Engl et al. (1996); Groetsch (1984)). We use the notation T in the case of linear ill-posed problems in this report. Also, we assume that $\mathscr{X}$ and $\mathscr{Y}$ are Hilbert spaces.

Throughout this thesis we use the following notations;

- Let $\langle , \rangle$ and $\| \|$, respectively stand for inner product and norm.
- $D(T), R(T)$ and $N(T)$ denote the Domain of T, Range of T, Null space of T respectively.
- $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ denote the set of all bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$.
- $B(x,r)$ denote the open ball centered at x with radius $r > 0$.
- T^* denotes the adjoint of an operator T.

1.1 LINEAR ILL-POSED PROBLEM

A most common example for the linear ill posed problem is Fredholm intergral operator equation of the first kind defined by

$$(Tx)(s) = \int_a^b k(s,t)u(t)dt = y(s), a \leq s \leq b \tag{1.1.1}$$

where $x, y \in L^2[a,b]$ and $k(s,t)$ is a non-degenerate kernel. Next we discuss examples of the linear ill-posed problems.

Example 1.1.1. *(Interpretation of measurement data) (Kabanikhin (2008)). The operation of many measurement devices that register nonstationary fields can be described as follows; a signal $q(t)$ arrives at the input of the device, and a function $f(t)$ is registered at the output. In the simplest case, the functions $q(t)$ and $f(t)$ are related by the formula*

$$\int_0^t g(t-\tau)q(\tau)d\tau = f(t). \tag{1.1.2}$$

In this case, $g(t)$ is called the impulse response function of the device. In theory, $g(t)$ is the output of the device in the case where the input is the generalized function $\delta(t)$, i.e., Dirac's delta function: $\int_0^t g(t-\tau)\delta(t)d\tau = g(t)$. In practice, in order to obtain $g(t)$, a sufficiently short and powerful impulse is provided as an input. The resulting output function represents the impulse response function.
Thus, the problem of interpreting measurement data, i.e., determining the form of the input signal $q(t)$ is reduced to solving the integral equation of the first kind (1.1.2)*. For a linear device, this has the form*

$$\int_0^t K(t,\tau)q(\tau)d\tau = f(t).$$

This model describes the operation of devices that register alternate electromagnetic fields, pressure and tension modes in a continuous medium, seismographs, which record vibrations of the Earth's surface, and many other kind of devices.

Example 1.1.2. *(Steady State Heat Distributions)(Groetsch (2007)). Consider the problem of determining the temperature flux (cause) on the left edge of a semi - infinite strip from observation of the temperature on that face (effect) when the temperature in the strip is at steady state. The problem can be stated mathematically as follows. Let*

$$\Omega = \{(x,y) : x > 0, 0 < y < \pi\}$$

and suppose $u = u(x,y)$ is a function defined on the closure of Ω and satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in Ω and $u(x,0) = u(x,\pi) = 0$ for $x > 0$. Suppose we wish to determine the temperature flux

$$f(y) = \frac{\partial u}{\partial x}(0,y), 0 < y < \pi$$

given the temperature distribution $g(y) = u(0,y)$. By elementary seperation of variables techniques,

$$u(x,y) = \sum_{n=1}^{\infty} a_n e^{-nx} \sin ny.$$

Then we will get,

$$f(y) = \sum_{n=1}^{\infty} (-na_n) \sin ny$$

hence,

$$a_n = -\frac{2}{n\pi}\int_0^{\pi} f(\zeta)\sin n\zeta d\zeta.$$

while,

$$\begin{aligned} g(y) &= \sum_{n=1}^{\infty} \sin ny \\ &= -\sum_{n=1}^{\infty} \frac{2}{n\pi}\int_0^{\pi} f(\zeta)\sin n\zeta d\zeta \sin ny \\ &= \int_0^{\pi} k(y,\zeta)f(\zeta)d\zeta, \end{aligned}$$

where

$$k(y,\zeta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin ny \sin n\zeta.$$

Again the inverse problem is modeled by an integral equation of the first kind.

The operator equation (1.0.1) has a solution if and only if $y \in R(T)$. If $y \notin R(T)$, then we look for an element $x \in \mathscr{X}$ such that Tx is close to y. i.e,

Definition 1.1.3. *(Nair (2009)) Let $T : \mathscr{X} \to \mathscr{Y}$ be a bounded linear operator and*

$$T(x) = y, \tag{1.1.3}$$

where $x \in \mathscr{X}$ and $y \in \mathscr{Y}$. Then $x \in \mathscr{X}$ is called ***least residual norm solution (LRN) or least square solution of equation*** (1.1.3) *if*

$$||Tx - y|| = inf\{||Tz - y||, z \in \mathscr{X}\}.$$

Note that if $y \in R(T)$, then every solution of $Tx = y$ is an LRN solution. If $y \notin R(T)$, then LRN solution need not exist. For example, consider $\mathscr{X} = (C[a,b], \|\|_2)$, $\mathscr{Y} = \left(L^2[a,b], \|\|_2\right)$ and $T : \mathscr{X} \to \mathscr{Y}$ be the identity map. Then we can see that, $\|Tx - y\|_2 = \|x - y\|_2, \forall x \in \mathscr{X}, \forall y \in \mathscr{Y}$. Since $C[a,b]$ is dense in $L^2[a,b]$, there exists a sequence (x_n) in $\mathscr{X}$ converges to y. Therefore, we get, $\inf\{\|Tx - y\|_2, x \in \mathscr{X}\} = 0$. But there does not exists a $x \in \mathscr{X}$ such that $T(x) = y$. The following theorems give characterizations of the LRN solution.

Theorem 1.1.4. *(cf.Nair (2009), Theorem 4.2) Let $T : \mathscr{X} \to \mathscr{Y}$ be a linear operator. Let $P : \mathscr{Y} \to \mathscr{Y}$ be the orthogonal projection onto $clR(T)$, the closure of $R(T)$. For $y \in \mathscr{Y}$, the following are equivalent.*

(i) The equation $Tx = y$ has an LRN solution.

(ii) $y \in R(T) + R(T)^{\perp}$.

(iii) The equation $Tx = Py$ *has a solution.*

Theorem 1.1.5 (cf. Nair (2009), Theorem 4.5)**.** *Let* $T \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$*, and* $y \in R(T) + R(T)^{\perp}$*. Then* $x \in \mathscr{X}$ *is an LRN-solution of (1.1.3) if and only if* $T^*Tx = T^*y$*.*

Note that an LRN solution need not be unique. If x is an LRN solution, then $x + x'$ is also an LRN solution for all $x' \in N(T)$. Therefore, if T is injective and $y \in R(T) + R(T)^{\perp}$, then only (1.1.3) has a unique a LRN solution. We denote the set of all LRN solution of (1.1.2) by

$$S_y = \{x \in \mathscr{X} : \|Tx - y\| = \inf_{z \in \mathscr{X}} \|Tz - y\|\}.$$

By Theorem 1.1.5, One can see that $S_y = \{x \in \mathscr{X} : T^*Tx = T^*y\}$, whenever $T \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$, and $y \in R(T) + R(T)^{\perp}$. By using continuity and linearity of T, one can prove that S_y is a closed convex set. Since closed convex set in a Hilbert space has unique element of minimal norm (cf. Groetsch (1977), Theorem 1.1.4), the set S_y has a unique minimal norm element, say $\hat{x}$.

Definition 1.1.6. *Let* $T \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$ *and* $y \in R(T) + R(T)^{\perp}$*. Then* $\hat{x} \in \mathscr{X}$ *is called* ***best - approximate solution*** *of* (1.1.3) *if* $\hat{x} \in S_y$ *and*

$$\|\hat{x}\| = \inf\{\|x\| : x \in S_y\}.$$

Definition 1.1.7. *Let* $T \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$*. The operator* $T^{\dagger} : D(T^{\dagger}) \subseteq Y \to \mathscr{X}$*, where* $D(T^{\dagger}) = R(T) + R(T)^{\perp}$*, defined by* $T^{\dagger}y = \hat{x}$*, where* $\hat{x}$ *is the best approximate solution of the equation* (1.1.3)*, is called* ***Generalized inverse or Moore - Penrose inverse*** *of* T*. And* $\hat{x} = T^{\dagger}y$ *is called the* ***generalized solution*** *of* (1.1.3).

1.2 REGULARIZATION

Our goal is to approximate the generalised solution $\hat{x} = T^{\dagger}y$ of equation (1.1.3) for a particular y in the situation that the exact data y is not known precisely, but that only an approximation y^{δ} with

$$\|y - y^{\delta}\| \leq \delta, \tag{1.2.1}$$

is available. In general, in equation (1.1.3) if $y \in R(T)$, then one can say that y is attainable, and if not one consider the generalized solution $T^{\dagger}y$ instead of the actual solution x of (1.1.3). But if $R(T)$ is not closed, then $T^{\dagger}$ does not exists and the problem becomes ill-posed. Even though a generalized inverse $T^\dagger$ exists, $T^\dagger y^\delta$ need not be a good approximation of $T^\dagger y$, because of the discontinuity of $T^\dagger$.

One can not make an unstable problem stable and so has to use the regularization techniques. Regularization is the procedure of approximating an ill-posed problem by a family of neighboring well-posed problems. Generally, a regularization method consists of two steps; (i) choosing a regularization operator R_α and (ii) choosing an appropriate regularization parameter α. Thus, we use the regularization method to obtain an approximation of $\hat{x}$, say $x_\alpha^\delta = R_\alpha y^\delta$, such that $x_\alpha^\delta \to \hat{x}$ as $\delta \to 0$ and with a suitable choice of $\alpha > 0$.

Definition 1.2.1. *(Engl et al. (1996)) Let $T \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$ and satisfies (1.1.3), (1.2.1) and $\alpha_0 \in (0, +\infty]$. For every $\alpha \in (0, \alpha_0)$, let the set $\{R_\alpha\}$ be a regularized family such that if for all $y \in D(T^\dagger)$ there exists a parameter choice rule $\alpha = \alpha(\delta, y^\delta)$ satisfies*

$$\limsup_{\delta \to 0} \{||R_\alpha y^\delta - T^\dagger y|| : y^\delta \in Y, ||y - y^\delta|| \le \delta\} = 0. \tag{1.2.2}$$

Here the parameter choice rule is a function, $\alpha : \mathbb{R}^+ \times \mathscr{Y} \to (0, \alpha_0)$ such that

$$\limsup_{\delta \to 0} \{\alpha(\delta, y^\delta) : y^\delta \in \mathscr{Y}, ||y - y^\delta|| \le \delta\} = 0. \tag{1.2.3}$$

For a particular $y \in D(T^\dagger)$, a pair (R_α, α) is called a **regularization method** if (1.2.2) and (1.2.3) hold. A choice α of the regularization parameter can be made in either an apriori or a posteriori way (Groetsch (1993)).

Definition 1.2.2. *(Engl et al. (1996)) Let α be a parameter choice rule according to the definition (1.2.2). If α depends only on δ, i.e., $\alpha = \alpha(\delta)$, then α is called an **a-priori parameter choice rule**. Otherwise, i.e., α depends on δ and y^δ, then it is called an **a-posteriori parameter choice rule**.*

In a posteriori parameter methods, the parameter α is determined during the computation of x_α^δ. There are many posteriori parameter choice rules in the literature of linear ill-posed problems (Engl et al. (1996); Tautenhahn and Jin (2003)).

The well known regularization method to solve linear ill-posed problems is **Tikhonov regularization** (Engl and Neubauer (1985b); Engl (1987a,b); Engl et al. (1996)) (Engl and Neubauer (1987); Groetsch (1984); George and Nair (1998)). The unique minimizer $x_\alpha^\delta \in \mathscr{X}$ of the Tikhonov functional

$$J_\alpha(x) = ||Tx - y^\delta||^2 + \alpha||x||^2, x \in X, \alpha > 0 \tag{1.2.4}$$

is known as Tikhonov regularized solution of linear ill posed problem (1.1.3) when the available date is y^δ. It is known that if T is a bounded linear operator, the unique minimizer x_α^δ is the solution of the well-posed equation

$$(T^*T+\alpha I)x_\alpha^\delta = T^*y^\delta, \alpha > 0, \tag{1.2.5}$$

and if $x \in R(T^*T)^\nu, 0 < \nu \leq 1$, then

$$\|\hat{x} - x_\alpha^\delta\| \leq c_1\alpha^\nu + c_2\left(\frac{\delta}{\sqrt{\alpha}}\right),$$

where c_1 and c_2 are constants (Nair (2009)). Recall Engl et al. (1996); Groetsch (1984); Nair (2009), if $\alpha(\delta) = c\delta^{\frac{2}{2\nu+1}}$, then

$$\|\hat{x} - x_\alpha^\delta\| \leq O\left(\delta^{\frac{2\nu}{2\nu+1}}\right), \tag{1.2.6}$$

and this is the best possible error estimate of Tikhonov regularization. But since ν is unknown, this choice of $\alpha = \alpha(\delta)$ is not practical. Therefore one needs to consider a-posteriori parameter choice strategy, i.e., $\alpha = \alpha(\delta, y^\delta)$ to choose a suitable parameter α. We use discrepancy principles for this purpose. (i.e., choose α as the solution of the equation given in the discrepancy principles). Next we list some of the discrepancy principles from the literature.

1. Morozov (Morozov (1968); Groetsch (1983)) $\|Tx_\alpha^\delta - y^\delta\| = \delta$.
2. Arcangeli (George and Nair (1998)) $\|Tx_\alpha^\delta - y^\delta\| = \frac{\delta}{\sqrt{\alpha}}$.
3. Schock (Schock (1984b,a)) $\|Tx_\alpha^\delta - y^\delta\| = \frac{\delta^p}{\alpha^q}$ $p > 0, q > 0$.
4. Engl (Engl (1987a,b); Engl and Neubauer (1987))
   $\|T^*Tx_\alpha^\delta - T^*y^\delta\| = \frac{\delta^p}{\alpha^q}$ $p > 0, q > 0$.

1.3 FRACTIONAL TIKHONOV REGULARIZATION

Tikhonov regularization method over smoothens the regularized solution x_α^δ (Bianchi et al. (2015), Hochstenbach et al. (2015), Reddy (2018)), i.e., it may not contain all the details of the exact solution $\hat{x}$. Therefore several variations of Tikhonov regularization schemes have been introduced to approximate the exact solution. Klann and Ramlau (Klann and Ramlau (2008)) introduced the Fractional Tikhonov regularization scheme (FTR). Later Hochstenbach and Reichel had studied the FTR with a different approach.

In (Hochstenbach and Reichel (2011)), the regularized solution of FTR is the unique minimizer, say $x^{\delta}_{\alpha,\gamma}$, of the functional

$$J_{\alpha,\gamma}(x) = ||Tx - y^{\delta}||^2_W + \alpha||x||^2, x \in \mathscr{X}, \alpha > 0,$$

where $||y||_W = ||W^{1/2}y||_{\mathscr{Y}}$ with $W = (TT^*)^{(\gamma-1)/2}$ for some parameter $0 < \gamma \leq 1$. Note that, W is the generalized inverse of TT^* when $\gamma < 1$. Also one can see that $x^{\delta}_{\alpha,\gamma}$ is the solution of the well-posed equation defined by

$$((T^*T)^{\frac{\gamma+1}{2}} + \alpha I)x = (T^*T)^{\frac{\gamma-1}{2}} T^* y^{\delta}.$$

Gerth et al. (Gerth et al. (2015)) studied Morozov's parameter choice rule for fractional Tikhonov regularization scheme and obtained the solution at a rate of convergence $O(\delta^{\frac{\nu}{\nu+1}})$ if $\hat{x} \in R((T^*T)^{\frac{\nu}{2}})$ and $0 < \nu \leq 1$. Reddy (2018) has considered the Engl type discrepancy principles for choosing the regularization parameter α and achieved the optimal rate of converegence $O(\delta^{\frac{\nu}{\nu+1}})$ if $\hat{x} \in R((T^*T)^{\frac{\nu}{2}})$ with $\nu = \gamma + 1$.

Kanagaraj et al. (2020), considered the FTR method in the form (with $\beta = \frac{\gamma-1}{2}$)

$$x^{\delta}_{\alpha,\beta} = ((T^*T)^{1+\beta} + \alpha I)^{-1}(T^*T)^{\beta} T^* y^{\delta}, -\frac{1}{2} < \beta \leq 0. \tag{1.3.1}$$

The authors studied Schock type discrepancy principle (Schock (1984b)), for choosing the parameter α and obtained optimal rate $O(\delta^{\frac{2\nu}{2\nu+1}})$ whenever, $\hat{x} \in R((T^*T)^{\nu})$.

We extended the work of Kanagaraj et al. in the finite dimensional settings using the Raus and Gferer type of discrepancy principle (in Chapter 2). More about linear ill-posed problems can be found in(Engl et al. (1996); Groetsch (1984).

1.4 NONLINEAR ILL-POSED PROBLEMS

We are also interested in approximate a solution of the equation

$$\mathscr{H}(u) = y, \tag{1.4.1}$$

where $\mathscr{H} : \mathscr{U} \to \mathscr{U}$ is a nonlinear, monotone, Fréchet differentiable operator defined on a Hilbert space $\mathscr{U}$. First, we introduced some basic definitions:

Definition 1.4.1. *(cf Alber and Ryazantseva (2006))*
Let $\mathscr{E}$ be an open subset of the Hilbert space $\mathscr{U}$ and $\mathscr{H} : \mathscr{E} \to \mathscr{U}$. If there exists a

bounded linear operator $L : \mathscr{U} \to \mathscr{U}$ such that for $u_0 \in \mathscr{E}$

$$\lim_{h \to 0} \frac{||\mathscr{H}(u_0 + h) - \mathscr{H}(u_0) - \mathscr{H}'(u_0)h||}{||h||} = 0,$$

*then $\mathscr{H}$ is said to be **Fréchet differentiable** at u_0 and the bounded linear operator $\mathscr{H}'(u_0) := L$ is called the first Fréchet derivative of $\mathscr{H}$ at u_0.*

Definition 1.4.2. *An Operator $\mathscr{H} : D(\mathscr{H}) \subset \mathscr{U} \to \mathscr{U}$ is said to be a monotone operator if*

$$\langle \mathscr{H}(u) - \mathscr{H}(v), u - v \rangle \geq 0, \forall u, v \in D(\mathscr{H}).$$

Remark 1.4.3. *Kabanikhin (2008), In the Example (1.1.2), if we consider the relationship between $q(t)$ and $f(t)$ is a nonlinear one :i.e.,*

$$\int_0^t K(t, \tau, q) d\tau = f(t),$$

then it is an example of a nonlinear ill-posed problem.

In the nonlinear case, it is always assume that the solution of (1.4.1) exists, however it need not be unique. Therefore, we consider a general u_0 minimum norm solution instead of best approximation solution.

Definition 1.4.4. *(cf: Engl et al. (1996); Tautenhahn and Jin (2003)) A solution $\hat{u}$ is called u_0 **minimum norm solution** if,*

(i) $\mathscr{H}(\hat{u}) = y$ *and*

(ii) $\|\hat{u} - u_0\| = \min\{\|u - u_0\| : \mathscr{H}(u) = y\}$.

The choice of u_0 depends on certain selection criteria (Engl et al. (1989)). For a nonlinear and Fréchet differentiable operator $\mathscr{H}$, the equation (1.4.1) is ill-posed if the inverse of $\mathscr{H}'(.)$ is unbounded.

Lavrentiev regularization is one of the commonly used regularization technique to approximate the solution of nonlinear ill-posed problem when the operator involved in the ill-posed equation is monotone. The Lavrentiev regularized solution $u_\alpha^\delta \in \mathscr{H}$ of (1.4.1) with $y = y^\delta$ is the solution of the equation,

$$\mathscr{H}(u) + \alpha(u - u_0) = y^\delta,$$

where u_0 is an initial guess of the solution $\hat{u}$.

To obtain an error bound for $\|\hat{u} - u_\alpha^\delta\|$, one needs some additional smoothness assumptions on the unknown $\hat{u}$, with respect to the operator $\mathscr{H}'(\hat{u})$ or $\mathscr{H}'(u_0)$. These assumptions are generally called **source conditions.** The best optimal order of the Lavrentiev regularization method is $O\left(\delta^{\frac{\nu}{\nu+1}}\right)$, provided $u_0 - \hat{u} \in R(H'(\hat{u})^\nu)$ or $u_0 - \hat{u} \in R(H'(u_0)^\nu)$ where, $0 < \nu \leq 1$. To achieve this, we can use a posteriori parameter choice strategy. In a posteriori parameter choice strategy, the regularization parameter α (depending on δ and y^δ) is chosen at the time of computation of u_α^δ. Also, it is believed that (Tautenhahn (2002)) a priori parameter-choice strategy is not a good strategy to choose α since the choice depends on the unknown ν. In our work, we have introduced a new parameter choice strategy for the regularization parameter α (depending on δ and y^δ) and independent of ν (in Chapter 3). We have compared the computational results with the adaptive method in (George and Nair (2017)).

In the literature, Pereverzev and Schock have considered an adaptive method to choose the regularization parameter α. The procedure considered by Pereverzev and Schock (2005) begins with a finite number of positive reals, $\alpha_0, \alpha_1, \cdots, \alpha_N$, such that

$$\alpha_0 < \alpha_1 < \cdots < \alpha_N.$$

Assume that there exist increasing functions $\varphi(\alpha)$ and $\phi(\alpha)$ for $\alpha > 0$ such that

$$\lim_{\alpha \to 0} \varphi(\alpha) = 0 = \lim_{\alpha \to 0} \phi(\alpha),$$

and

$$\|\hat{u} - u_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\phi(\alpha)}$$

for all $\alpha > 0, \delta > 0$. Note that the $\varphi(\alpha) + \frac{\delta}{\phi(\alpha)}$ attains its minimum for the choice $\alpha := \alpha_\delta$ such that $\varphi(\alpha) = \frac{\delta}{\phi(\alpha)}$, Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\phi(\alpha_i)}$ and for some $\mu > 1$,

$$\phi(\alpha_i) \leq \mu \phi(\alpha_{i-1}) \quad \forall i \in \{0, 1, 2, \cdots, N\}.$$

We also assume that,

$$max\left\{i : \varphi(\alpha_i) \leq \frac{\delta}{\phi(\alpha_i)}\right\} < N.$$

Let

$$k := max\left\{i : \|u_i^\delta - u_j^\delta\| \leq 4\frac{\delta}{\phi(\alpha_j)}, j = 0, 1, 2, \cdots, i-1\right\}.$$

Then we choose $\alpha = \alpha_k$ as the regularization parameter. We introduced a new source condition in our study to obtain the optimal error bound $O\left(\delta^{\frac{\nu}{\nu+1}}\right), 0<\nu\leq 1$(in Chapter 3).

1.5 REGULARIZED PROJECTION METHODS

We have mentioned that a stable approximate solution to ill-posed problems can be obtained by using regularization methods. One needs regularization methods that can be implementable in finite dimensional spaces for numerical calculations. We consider a sequence

$$V_1 \subset V_2 \subset V_3 \subset \ldots$$

of finite dimensional subspaces of $\mathscr{X}$ whose union is dense in $\mathscr{X}$. We can consider $x_n = T_h^{\dagger} y$, where $T_h = TP_h$ and $P_h\left(h=\frac{1}{n}\right)$, is the orthogonal projection onto V_n, is the stable approximation of $x^{\dagger}$ in least squares projection method. It is proved that (Engl et al. (1996)), $\{x_n\}$ converges to $\hat{x}$ if and only if $\{\|x_n\|\}$ is bounded. In the dual least squares projection method, one can consider a sequence

$$U_1 \subset U_2 \subset U_3 \subset \ldots$$

of finite dimensional subspaces of $clR(T) \subset \mathscr{Y}$ whose union is dense in $\mathscr{Y}$. Then x_n is defined as the best approximate solution of the equation

$$T_h x = y_n, T_h = Q_h T, y_n = Q_h y, \tag{1.5.1}$$

where Q_h is the orthogonal projection onto U_n. Then x_n is a stable approximation of $\hat{x}$.

Theorem 1.5.1. *(Engl et al. (1996)) Let $y \in D(T^{\dagger})$ and let x_n be as in (1.5.1). Then $x_n = P_h \hat{x}$, where P_h is the orthogonal projection onto $V_n = T^* U_n$. Moreover,*

$$x_n \to \hat{x} \quad as \quad n \to \infty.$$

The above projection method is useful to obtain the best possible approximation in V_n in the noise free case. However, one has to combine the projection method with an additional regularization method to approximate the solution of an ill-posed problem with noisy data.

Pereverzev (1995) investigated Tikhonov regularization method combined with projections. Precisely, he considered the regularized solution $x_{\alpha,h}^{\delta}$ is defined as

$$x_{\alpha,h}^{\delta} = (T_h^* T_h + \alpha I)^{-1} T_h^* y^{\delta}, \tag{1.5.2}$$

with

$$T_h = T_m = \sum_{k=1}^{m} (P_{2^k} - P_{2^{k-1}}) T P_{2^{2m-k}} + P_1 T P_{2^{2m}}, \tag{1.5.3}$$

where P_i is the orthogonal projection onto the subspaces V_i. Many studies have carried out in this area, for example; (Engl and Neubauer (1985a); Gfrerer (1987); George and Nair (1998); Nair and Rajan (2001)). In Chapter 2, we have considered the finite dimensional realization of the FTR method using Raus and Gfrerer type discrepancy principle.

1.6 REGULARIZED ITERATIVE METHODS

The iterative regularization method is another approach to approximate the solution $\hat{x}$ of both linear and nonlinear ill-posed problems. There is a vast literature on iteration methods for well-posed problems. Many studies have been carried out on iterative methods, that have self regularizing property, and regularized iterative methods for ill-posed problems (Landweber iteration, Accelerated Landweber methods, The ν methods, Levenberg-Marquardth method, Gauss Newton method) (see Engl et al. (1996); Hanke et al. (1995); Ramlau (1999); Hanke (1997); Bakrushinsky (1992)). Newton type regularized iterative methods are the usual choice for approximating the solution to nonlinear ill-posed problems. However, most of these methods involve the inverse of the Fréchet derivative of the operator.

Ramlau (2003) has considered Tikhonov Gradient method (TIGRA) defined as

$$u_{n+1}^{\delta} = u_n^{\delta} + \beta_n [\mathscr{H}'(u_n^{\delta})^* (y^{\delta} - \mathscr{H}(u_n^{\delta})) + \alpha_n (u_n^{\delta} - u_0)], \forall n = 0, 1, 2, ... \tag{1.6.1}$$

Later, Argyros et al. (2014) modified (1.6.1) as follows,

$$u_{n+1,\alpha}^{\delta} = u_{n,\alpha}^{\delta} - [\mathscr{H}'(u_0^{\delta})^* (\mathscr{H}(u_{n,\alpha}^{\delta}) - y^{\delta}) + \alpha (u_{n,\alpha}^{\delta} - u_0)], \forall n = 0, 1, 2, ... \tag{1.6.2}$$

where $u_{0,\alpha}^{\delta} = u_0$ and α is the regularization parameter was chosen using the adaptive method in (Pereverzev and Schock (2005)). The above method is free of the inverses and the Frechet derivatives need to be computed at the initial point u_0 only.

Semenova (Semenova (2010)) has considered a Lavrentiev regularized iterative method, that is independent of Fréchet derivative of the operator involved, defined as,

$$u_{n+1,\alpha}^{\delta} = u_{n,\alpha}^{\delta} - \gamma [(\mathscr{H}(u_{n,\alpha}^{\delta}) + \alpha (u_{n,\alpha}^{\delta} - u_0) - y^{\delta}], \forall n = 0, 1, 2, ... \tag{1.6.3}$$

for fixed α, δ by assuming that $\mathscr{H}$ is monotone, Lipschitz continuous with Lipschitz constant R and with γ satisfying $0 \leq \gamma \leq min\{\frac{1}{\alpha}, \frac{2\alpha}{\alpha^2} + R^2\}$. Later, George and Nair (2017) have investigated the iterative procedure $(1.6.3)$, but with β independent on the regularization parameter α and the Lipschitz constant R, instead of γ. In Chapter 3, we study finite dimensional realization of the derivative free regularized iterative method considered in (George and Nair (2017)).

This thesis has divided into mainly two sections. The second and third Chapters focus on the numerical realization of regularization methods for ill-posed problems. In Chapter 4 and Chapter 5, we discuss iterative methods for nonlinear problems and their applications in ill-posed problems.

1.7 ITERATIVE METHODS FOR NONLINEAR PROBLEMS

Constructing higher order convergent (see definition 1.7.1) iterative algorithm to solve nonlinear equations in Banach spaces is a vital problem in Computational Mathematics. The importance of higher order convergent iterative methods has grown in recent years because of its applicability in various fields. Numerous applications, including chemical reactions, image processing, economics, population growth, electronic circuits, cryptography, and secure communications, need the simultaneous or sequential solution of integral equations, partial differential equations, and boundary value problems. And those problems can be modeled as a nonlinear equation in Banach space settings. Generally, there is no direct method to solve nonlinear problems. However, the iterative methods use to approximate solutions of such problems numerically.

In the literature, one can see that there are mainly three types of convergence analysis such as local, semilocal, or global convergence. We commonly consider the basic assumption that the solution of the nonlinear equation exists and the iterates are sufficiently close to the solution. The global convergence ensures that the iterative method converges to the solution irrespective of initial iterates. In the case of nonlinear problems, global convergence can not be guaranteed even though it is achievable for a linear case under stringent conditions (Ortega and Rheinboldt (1970)). The local convergence analysis uses information around a solution, whereas the semilocal convergence employs information provided at an initial approximation (Argyros and Magreñán (2016); Argyros (2008)). Mostly, the local convergence analysis uses the expensive Taylor's expansion and assumptions on the higher order Fréchet derivatives of the operator to obtain the order of convergence of the iterative scheme (Cordero et al. (2014, 2012, 2010); Fang et al. (2008); Sharma and Gupta (2014)). The domain of convergence of an iterative method is usually small. The choice of initial approximation and the behavior of nonlinear functions around the solution plays a vital role in the convergence of an iterative method.

In Chapter 4 and Chapter 5, we consider the following nonlinear equation,

$$\mathscr{L}(s) = 0, \tag{1.7.1}$$

where $\mathscr{L} : \Omega \subset \mathscr{U} \to \mathscr{V}$ is an operator from $\mathscr{U}$ to $\mathscr{V}$ and $\mathscr{U}, \mathscr{V}$ are Banach spaces. We also assume that $\mathscr{L}$ is a Fréchet differentiable nonlinear operator and Ω is a nonempty open convex set. Recall Kreyszig (1989), the direct method involves a finite number of arithmetic operations after which an exact solution has attained. Since there is no direct method to solve the nonlinear problem (1.7.1), we reformulate it into a fixed point problem $\mathscr{G}(s) = s$ and use iterative methods to obtain the exact solution Kelley (1995). A suitable initial value, an iterative function, and a stopping rule (numerical criterion for deciding when to stop iteration) are required to perform the iterative method. In Traub (1964), Traub categorized the Iterative methods as follows based on the values used:

(i) **One Point Iterative Function (I.F.) :** If $s_{k+1} = \mathscr{G}(s_k), s_k$ is the kth iterate, we call this one point iterative function. Newton iterative function by equation is an example (see Chapter 4).

(ii) **One Point I.F. with memory :** If $s_{k+1} = \mathscr{G}(s_k; s_{k-1}, ..., s_{k-m})$, then $\mathscr{G}$ is called one-point iteration with memory as it reuses information of past iterations. The semicolon separates the points at which new data are used from the points at which old data are used. Secant method is an example.

$$s_{k+1} = s_k - \mathscr{L}(s_k)\frac{s_k - s_{k-1}}{\mathscr{L}(s_k) - \mathscr{L}(s_{k-1})}, \forall k = 0, 1, 2, ...$$

(iii) **Multi Point I.F. :** If s_{k+1} is determined by new information at $s_k, \omega_1(s_k), ..., \omega_n(s_k)$ and $s_{k+1} = \mathscr{G}(s_k, \omega_1(s_k), ..., \omega_n(s_k))$, the $\mathscr{G}$ is called multi point I.F. The midpoint Newton's method is an example (see Chapter 5).

(iv) **Multi Point I.F. with memory :** Iterative function $\mathscr{G}$ is called multi point I.F. with memory, if p_j denoted $n+1$ quantities $s_k, \omega_1(s_k), ..., \omega_n(s_k)$ and $s_{k+1} = \mathscr{G}(p_k; p_{k-1}, ..., p_{k-m})$.

One needs the concept of order of convergence to analyze the convergence speed of the sequence of iterates $\{s_k\}$ to the exact solution s^* in an iterative scheme.

Definition 1.7.1. *Recall Traub (1964), If for a sequence $\{s_k\}$ converging to solution s^*, there exist some $p \in [0,\infty)$ such that the limit defined as*

$$K_p = \lim_{n\to\infty} \frac{\|s_{k+1} - s^*\|}{\|s_k \; - s^*\|^p}$$

exists. Then, p is called the ***order of convergence*** *of the sequence $\{s_k\}$.*

Ortega and Rheinboldt have discussed a more general concept of R-order and Q-order in Ortega and Rheinboldt (1970). There is an equivalence between the above definition, Q and R orders, with an additional condition $0 < K_p < \infty$, in Petković et al. (2014). We use the following R-order convergence definition in this study (Potra (1989) Ezquerro and Hernandez (2006)).

Definition 1.7.2. *A sequence $\{s_k\}$ converges to s^* with* ***R- order*** *at least p if there are constants C and $\beta \in (0,1)$ such that,*

$$\|s_{k+1} - s^*\| \leq C\beta^{p^k}, n = 0,1,2,... \tag{1.7.2}$$

Note that,

$$\|s_{k+1} - s^*\| \leq \tilde{C}\|s_{k+1} - s^*\|^p, \tilde{C} > 0, n = 0,1,2,..., \tag{1.7.3}$$

for $p \in \mathbb{N}$ implies (1.7.2), provided $\|s_0 - s^*\| < 1$. It can be shown as below,

$$\begin{aligned}
\|s_{k+1} - s^*\| &\leq \tilde{C}\|s_k - s^*\|^p \\
&\leq \tilde{C}\left(\tilde{C}\|s_{k-1} - s^*\|^p\right)^p \\
&= \tilde{C}(\tilde{C})^p\|s_{k-1} - s^*\|^{p^2} \\
&\leq \tilde{C}(\tilde{C})^p(\tilde{C})^{p^2}\|s_{k-2} - s^*\|^{p^3} \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&\leq \tilde{C}(\tilde{C})^p(\tilde{C})^{p^2}...(\tilde{C})^{p^{k-1}}(\tilde{C})^{p^k}\|s_0 - s^*\|^{p^{k+1}} \\
&= (\tilde{C})^{\frac{1-p^k}{1-p}}\|s_0 - s^*\|^{p^{k+1}} \\
&= \left((\tilde{C})^{\frac{1-p^k}{1-p}}\|s_0 - s^*\|\right)\|s_0 - s^*\|^{p^k},
\end{aligned}$$

where $C = (\tilde{C})^{\frac{1-p^k}{1-p}}\|s_0 - s^*\|$ and $\beta = \|s_0 - s^*\|$. We use the relation (1.7.3) as order of convergence in our study.

The Bisection algorithm, Regula falsi, and Secant iterative scheme are some of the simplest methods to solve nonlinear equations. The well known efficient iterative method used to approximate the solution of the nonlinear equation is classical Newton's method of order two (see Chapters 4 and 5).

In the earlier studies, the local analysis has proved by applying the Taylor expansions that need assumptions on derivatives of higher order, which are quite expensive. If the higher order derivatives are unbounded, these schemes have limited applicability. For example, consider the equation $\mathscr{L}(t) = 0$, where $\mathscr{L} : [-\frac{1}{2}, \frac{5}{2}] \to \mathbb{R}$ defined by

$$\mathscr{L}(t) = \begin{cases} t^3 log(t^2) + t^5 - t^4 & t \neq 0 \\ 0 & t = 0. \end{cases}$$

Since the third order derivative of $\mathscr{L}$ is unbounded, the convergence analysis depends on Taylor's expansion cannot be applicable in this example. Our study introduces new higher-order iterative schemes and eliminates Taylor's expansion in the convergence analysis of those methods.

1.8 RESEARCH OBJECTIVES

The overall objectives can be summarize as follows;

(i) Study the finite dimensional realization of Fractional Tikhonov Regularization method.

(ii) Propose a new parameter choice rule for nonlinear ill-posed operator equation.

(iii) Study the finite dimesnional realization of derivative - free iterative method for nonlinear ill-posed equations.

(iv) Study the local convergence analysis of higher order iterative methods in Banach space and use these methods to solve non linear ill - posed problem in Hilbert space settings.

1.9 OUTLINE OF THE THESIS

The rest of the thesis has organized as follows;

Chapter 1, introduces the basic definitions and notations for the rest of the thesis.

Chapter 2, discusses the techniques developed to study the finite dimensional Fractional Tikhonov regularization (FDFTR) method to approximate a solution of linear ill-posed problems. We studied Raus and Gfrerer type discrepancy principle for the FDFTR method and compared the numerical results with other discrepancy principles of the same type.

In Chapter 3, we have considered the derivative-free regularized iterative method in (George and Nair (2017)) in the finite dimensional setting. We have introduced a new parameter choice strategy to compute α before computing the approximate solution u_α^δ. Also, we introduced a new source condition. The optimal order has been obtained using this parameter choice strategy and source condition.

Chapter 4, discusses the Cordero - type iterative methods to solve nonlinear problems in Banach space. We studied Cordero's higher order methods in (Cordero et al. (2009, 2012)) of orders four and six without Taylor's expansion. In this study, we used assumptions only on the first order Fréchet derivative in the convergence analysis. Moreover, we extended the Cordero method to a method having order of convergence eight. We have applied these iterative schemes to ill-posed problems.

In Chapter 5, we have studied Newton's midpoint iterative scheme and introduced two extensions of Newton's midpoint iterative scheme of order five and six. We did the convergence analysis without Taylor's expansion. This study used assumptions on Fréchet derivative of order up to two.

Chapter 6, gives the conclusion of the thesis and future work.

CHAPTER 2

FINITE DIMENSIONAL REALIZATION OF THE FTR METHOD WITH RAUS AND GFRERER TYPE DISCREPANCY PRINCIPLE

2.1 INTRODUCTION

Fractional Tikhonov regularization method is studied in the setting of infinite dimensional Hilbert space (Bianchi et al. (2015); Bianchi and Donatelli (2017); Gerth et al. (2015); Kanagaraj et al. (2020); Hochstenbach and Reichel (2011); Hochstenbach et al. (2015); Huckle and Sedlacek (2012); Klann and Ramlau (2008); Morigi et al. (2017); Reddy (2018)). But, it is not easy to obtain theoretical results in the setting of finite dimentional Hilbert space. In this Chapter, we developed the necessary techniques to study FTR method in the setting of finite dimentional Hilbert space. In this chapter we consider the FDFTR method and Raus and Gfrerer type discrepancy principle. We also compare the numerical results of three similar type discrepancy principle for FDFTR method.

The challenge of approximating the solution $\hat{x}$ of linear ill-posed operator equation is of particular relevance to us. We consider the operator equation of the form

$$Tx = y, \tag{2.1.1}$$

where $T \in L(\mathscr{X}, \mathscr{Y})$, the collection of all bounded linear operators from the Hilbert space $\mathscr{X}$ to the Hilbert space $\mathscr{Y}$ with non closed range $R(T)$ and $y \in R(T)$ (Engl et al. (1996); Groetsch (1977, 1984); Guacaneme (1988); Morozov (1968)). Also, we considered the available data y^δ

$$\|y - y^\delta\| \leq \delta. \tag{2.1.2}$$

We use regularization methods for approximating the exact solution $\hat{x}$ and one of the commonly used regularization method is Tikhonov regularization, (Engl (1987a,b); Engl et al. (1996); Engl and Neubauer (1985a, 1987); Schock (1984b); Engl and Neubauer (1985b); George and Nair (1998, 1994); Groetsch (1984, 1983); Schock (1984a)), in which

$$x_\alpha^\delta = (T^*T + \alpha I)^{-1} T^* y^\delta$$

is taken as an approximation for $\hat{x}$. Here $\alpha > 0$ is a regularization parameter. As we mentioned in the introduction, if $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \leq 1$, then

$$\|\hat{x} - x_\alpha^\delta\| \leq c_1\, \alpha^\nu + c_2 \left(\frac{\delta}{\sqrt{\alpha}} \right).$$

Here and below $c, c_1, c_2, \ldots$ etc denote generic positive constants which may take different values at different places.

It is observed that Tikhonov regularization method over-smoothen the solution $\hat{x}$ of (2.1.1)(Bianchi et al. (2015); Bianchi and Donatelli (2017); Hochstenbach et al. (2015); Krasnoselskii et al. (1966); Louis (1989); Reddy (2018); Tikhonov and Arsenin (1977)). So, many authors examined the fractional or weighted Tikhonov regularization approach to reduce the over smoothening in the Tikhonov regularization method (Bianchi et al. (2015); Gerth et al. (2015); Bianchi and Donatelli (2017); Hochstenbach et al. (2015); Hochstenbach and Reichel (2011); Huckle and Sedlacek (2012); Kanagaraj et al. (2020); Klann and Ramlau (2008); Morigi et al. (2017); Reddy (2018); Reich and Tuyen (2022)). In this method, the approximate solution of $\hat{x}$ is defined as follows,

$$x_{\alpha,\gamma}^\delta = ((T^*T)^{\frac{\gamma+1}{2}} + \alpha I)^{-1} (T^*T)^{\frac{\gamma-1}{2}} T^* y^\delta \tag{2.1.3}$$

where $0 < \gamma \leq 1$. Reddy (2018), studied the following discrepancy principles

$$G(\alpha, y^\delta) := \|\alpha\big((T^*T)^{\frac{\gamma+1}{2}} + \alpha I\big)^{-1} (T^*T)^{\frac{\gamma-1}{2}} T^* y^\delta\|^2 = \tau_1 \frac{\delta^p}{\alpha^q},\ \tau_1 > 0$$

and

$$G_1(\alpha, y^\delta) := \|T^*T x_{\alpha,\gamma}^\delta - T^* y^\delta\|^2 = \frac{\delta^p}{\alpha^q}, \quad p > 0, q > 0,\ \alpha > 0$$

for choosing the regularization parameter α for FTR method .

Later Kanagaraj et al. (2020) considered the FTR method (2.1.3) in the form (with

$\beta = \frac{\gamma-1}{2}$)

$$x^{\delta}_{\alpha,\beta} = ((T^*T)^{1+\beta} + \alpha I)^{-1}(T^*T)^{\beta}T^*y^{\delta}, \tag{2.1.4}$$

with $-\frac{1}{2} < \beta \leq 0$ and studied the Schock type discrepancy principle Schock (1984b);

$$G(\alpha, y^{\delta}) := \|Tx^{\delta}_{\alpha,\beta} - y^{\delta}\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, q > 0, \alpha > 0 \tag{2.1.5}$$

for choosing the parameter α in (2.1.4). In this chapter, we study the finite dimensional realization of the method (2.1.4) and Raus and Gfrerer type discrepancy principle (see Section 2.3). Note that, the above discrepancy principles gives optimal order error estimate

$$\|x^{\delta}_{\alpha,\beta} - \hat{x}\| = O\left(\delta^{\frac{2\nu}{2\nu+1}}\right), \tag{2.1.6}$$

provided p and q are chosen properly (i.e., p and q are chosen depending on the unknown ν and β). Observe that, Raus and Gfrerer type discrepancy principle (2.3.3) (Gfrerer (1987); Raus (1984, 1985)) is independent of parameters like p and q. In the noise free case instead of $x^{\delta}_{\alpha,\beta}$, we define

$$x_{\alpha,\beta} = \left((T^*T)^{1+\beta} + \alpha I\right)^{-1}(T^*T)^{\beta}T^*y \tag{2.1.7}$$

and assume that the solution $\hat{x}$ satisfies the following condition

$$\hat{x} \in R\left((T^*T)^{\nu}\right), \quad 0 < \nu \leq 1 + \beta. \tag{2.1.8}$$

We will be using the following proposition in the sequel.

Proposition 2.1.1. *((Kanagaraj et al., 2020, Proposition 2.1), see also (Klann and Ramlau, 2008, Proposition 3.4.3)) Let $x^{\delta}_{\alpha,\beta}$ and $x_{\alpha,\beta}$ be as in* (2.1.4) *and* (2.1.7)*, respectively. Let $\hat{x}$ satisfies* (2.1.8)*. Then*

(i) $\|x^{\delta}_{\alpha,\beta} - x_{\alpha,\beta}\| \leq c_1 \frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}}$

(ii) $\|\hat{x} - x_{\alpha,\beta}\| \leq c_2 \alpha^{\frac{\nu}{1+\beta}}, \ 0 < \nu \leq 1 + \beta.$

In particular, we have
(iii) $\|\hat{x} - x^{\delta}_{\alpha,\beta}\| \leq c_1 \frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}} + c_2 \alpha^{\frac{\nu}{1+\beta}}.$

The remainder of the Chapter is structured as follows. In Section 2.2, the error analysis of the finite dimensional FTR method is discussed and in Section 2.3 we studied

Raus and Gfrerer type discrepancy principle (cf.Gfrerer (1987); Raus (1984, 1985)). The numerical examples in Section 2.4 compared the error estimates derived by three discrepancy principles. Section 2.5 concludes the Chapter with some remarks.

2.2 FINITE DIMENSIONAL FTR METHOD

Let (V_n) be a sequence of finite dimensional subspace of X with $V_1 \subset V_2 \subset \dots$ such that $\overline{\bigcup_{n\in\mathbb{N}} V_n} = X$ and let $P_h (h = \frac{1}{n})$ denote the orthogonal projection onto $R(P_h) = V_n$. We consider the finite dimensional version of the regularized solution (2.1.4), defined as

$$x_{\alpha,\beta}^{h,\delta} = \left((T_h^* T_h)^{1+\beta} + \alpha I\right)^{-1} (T_h^* T_h)^{\beta} T_h^* y^{\delta}, \tag{2.2.1}$$

where $T_h = TP_h$. Let

$$\varepsilon_h := \|T(I - P_h)\|$$

and assume that $\lim_{h\to 0} \varepsilon_h = 0$. The above assumption is satisfied if, for example, $P_h \to I$ point-wise and if T is a compact operator.

$$x_{\alpha,\beta}^{h} = \left((T_h^* T_h)^{1+\beta} + \alpha I\right)^{-1} (T_h^* T_h)^{\beta} T_h^* y. \tag{2.2.2}$$

Proposition 2.2.1. *Let $x_{\alpha,\beta}^{h,\delta}$ and $x_{\alpha,\beta}^{h}$ be as in* (2.2.1) *and* (2.2.2), *respectively. Then*

$$\|x_{\alpha,\beta}^{h,\delta} - x_{\alpha,\beta}^{h}\| \leq c_1 \frac{\delta}{\alpha^{\frac{1}{2(1+\beta)}}}.$$

Proof. Analogous to the proof of Proposition 2.1 in Kanagaraj et al. (2020) with T_h in place of T. □

Next, we obtain an estimate for $\|x_{\alpha,\beta}^{h} - x_{\alpha,\beta}\|$, we shall make use of the following formula ((Krasnoselskii et al., 1966, Page 287));

$$\begin{aligned} B^z x &= \frac{\sin \pi z}{\pi} \int_0^\infty t^z \left[(B+tI)^{-1}x - \frac{\theta(t)}{t}x + \dots + (-1)^n \frac{\theta(t)}{t^n} B^{n-1}x\right] dt \\ &\quad + \frac{\sin \pi z}{\pi} \left[\frac{x}{z} - \frac{Bx}{z-1} + \dots + (-1)^{n-1} \frac{B^{n-1}x}{z-n+1}\right], \; x \in X, \end{aligned}$$

where

$$\theta(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq 1 \\ 1 & \text{if} \quad 1 < t < \infty \end{cases}$$

for any positive self-adjoint operator B and for any complex number z such that $0 < Re(z) < n$. Taking $z = 1+\beta$, $-\frac{1}{2} \le \beta \le 0$, we have for any $z \in X$,

$$\begin{aligned}
[((T_h^*T_h)^2)^{\frac{1+\beta}{2}} - ((T^*T)^2)^{\frac{1+\beta}{2}}]z &= \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\int_0^\infty t^{\frac{1+\beta}{2}}[((T_h^*T_h)^2+tI)^{-1} \\
&\quad -((T^*T)^2+tI)^{-1}]zdt \\
&= \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\int_0^\infty t^{\frac{1+\beta}{2}}((T_h^*T_h)^2+tI)^{-1} \\
&\quad \times((T^*T)^2-(T_h^*T_h)^2)((T^*T)^2+tI)^{-1}zdt.
\end{aligned} \tag{2.2.3}$$

Proposition 2.2.2. *Suppose $\hat{x}$ satisfies (2.1.8) with $\gamma \le \nu \le 1+\beta$ for some constant $\gamma > 0$, and $\beta \in (-\frac{1}{2}, 0]$. Let $x_{\alpha,\beta}, x_{\alpha,\beta}^h$ be as in (2.1.7), (2.2.2), respectively. Then, the following estimate holds.*

$$\|x_{\alpha,\beta}^h - x_{\alpha,\beta}\| = O\left(\frac{\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}}\right).$$

Proof. From (2.2.2), we have

$$\begin{aligned}
x_{\alpha,\beta}^h &= ((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\beta}T_h^*y \\
&= ((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\beta}T_h^*T\hat{x} \\
&= ((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{1+\beta}\hat{x} + ((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\beta}T_h^*T(I-P_h)\hat{x}, \\
x_{\alpha,\beta} &= ((T^*T)^{1+\beta}+\alpha I)^{-1}(T^*T)^{\beta}T^*y \\
&= ((T^*T)^{1+\beta}+\alpha I)^{-1}(T^*T)^{1+\beta}\hat{x}
\end{aligned}$$

and hence

$$\begin{aligned}
x_{\alpha,\beta}^h - x_{\alpha,\beta} &= [((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{1+\beta} - ((T^*T)^{1+\beta}+\alpha I)^{-1}(T^*T)^{1+\beta}]\hat{x} \\
&\quad +((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\beta}P_hT^*T(I-P_h)\hat{x}.
\end{aligned}$$

So

$$\|x_{\alpha,\beta}^h - x_{\alpha,\beta}\| \le \|\Gamma\| + \|((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\beta}T_h^*T(I-P_h)\hat{x}\|, \tag{2.2.4}$$

where $\Gamma = [((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{1+\beta} - ((T^*T)^{1+\beta}+\alpha I)^{-1}(T^*T)^{1+\beta}]\hat{x}$. Further, we have

$$\begin{aligned}
\|((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\beta}T_h^*T(I-P_h)\hat{x}\| &\leq \|((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\beta+\frac{1}{2}}\| \\
&\quad \times\|T(I-P_h)\hat{x}\| \\
&\leq \sup_{\lambda\in\sigma(T_h^*T_h)}\left|\frac{\lambda^{\beta+\frac{1}{2}}}{\lambda^{\beta+1}+\alpha}\right|\varepsilon_h\|\hat{x}\| \\
&\leq c_3\frac{\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}}, \qquad (2.2.5)
\end{aligned}$$

and (See appendix A)

$$\|\Gamma\| \leq 4\left[\frac{1}{\gamma}c_4+\|T^*T\|c_5\right]\frac{\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}}. \qquad (2.2.6)$$

Thus by (2.2.4), (2.2.5) and (2.2.6), we have

$$\|x_{\alpha,\beta}^h - x_{\alpha,\beta}\| \leq \left(c_3+4\left[\frac{1}{\gamma}c_4+\|T^*T\|c_5\right]\right)\frac{\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}} = O\left(\frac{\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}}\right).$$

□

Combining Proposition 2.2.1 and Proposition 2.2.2, we have the following theorem.

Theorem 2.2.3. *Let $x_{\alpha,\beta}^{h,\delta}$ and $x_{\alpha,\beta}$ be as in (2.2.1) and (2.1.7), respectively. Let $\hat{x}$ satisfies (2.1.8) with $\gamma \leq \nu \leq 1+\beta$ for some constant $\gamma > 0$, and $\beta \in (-\frac{1}{2}, 0]$. Then,*

$$\|x_{\alpha,\beta}^{h,\delta} - x_{\alpha,\beta}\| \leq O\left(\frac{\delta+\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}}\right).$$

In particular, we have

$$\|\hat{x} - x_{\alpha,\beta}^{h,\delta}\| \leq c_6\frac{\delta+\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}} + c_2\alpha^{\frac{\nu}{1+\beta}}. \qquad (2.2.7)$$

Remark 2.2.4. *(cf. (Bianchi and Donatelli, 2017, Proposition 10)) Observe that, $\frac{\delta+\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}}$ is decreasing for $\beta \in (-\frac{1}{2}, 0]$, whereas $\alpha^{\frac{\nu}{1+\beta}}$ is increasing for $\beta \in (-\frac{1}{2}, 0]$. Therefore, one has to choose $\beta \in (-\frac{1}{2}, 0]$, such that $\frac{\delta+\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}} = \alpha^{\frac{\nu}{1+\beta}}$ in order to obtain an optimal order error estimate for $\|\hat{x} - x_{\alpha,\beta}^{h,\delta}\|$. In particular, for $\alpha = (\delta+\varepsilon_h)^{\frac{2(1+\beta)}{2\nu+1}}$ we have by (2.2.7)*

$$\|\hat{x} - x_{\alpha,\beta}^{h,\delta}\| = O((\delta+\varepsilon_h)^{\frac{2\nu}{2\nu+1}}). \qquad (2.2.8)$$

The above order is optimal in the sense that $\frac{\delta+\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}} = \alpha^{\frac{\nu}{1+\beta}} = (\delta+\varepsilon_h)^{\frac{2\nu}{2\nu+1}}$. *But such a choice of* β *and* α *is not possible in practice, because* ν *is unknown. Therefore, in Section 2.3 we study Raus and Gfrerer-type discrepancy principle (cf. Gfrerer (1987); Raus (1984, 1985)) for choosing* α *in (2.2.1), for a fixed* β.

2.3 RAUS AND GFRERER TYPE DISCREPANCY PRINCIPLE (Gfrerer (1987); Raus (1984, 1985))

For $u \in Y$, define

$$\phi(\alpha,u) := \|\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}u\|. \tag{2.3.1}$$

Theorem 2.3.1. *For each* $u \in Y$, *the function* $\alpha \to \phi(\alpha,u)$ *for* $\alpha > 0$, *defined in (2.3.1), is continuous, monotonically increasing and*

$$\lim_{\alpha\to 0}\phi(\alpha,u) = 0, \quad \lim_{\alpha\to\infty}\phi(\alpha,u) = \|u\|.$$

Proof. Note that

$$\left(\frac{\alpha}{\|(T_hT_h^*)^{1+\beta}\|+\alpha}\right)^{\frac{3}{2}}\|u\| \le \alpha^{\frac{3}{2}}\|((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}u\| = \phi(\alpha,u) \le \|u\|.$$

So, $\phi(\alpha,u) \longrightarrow \|u\|$ as $\alpha \to +\infty$ and $\phi(\alpha,u) \longrightarrow 0$ as $\alpha \to 0$. Also $\phi(\alpha,u)$ is strictly increasing and continuous. □

In addition to (2.1.2), we assume that

$$c\delta + d\varepsilon_h \le \|y^\delta\|, c > 0, d > 0. \tag{2.3.2}$$

Then by the intermediate value theorem, the succeeding theorem follows.

Theorem 2.3.2. *If* y^δ *satisfies (2.1.2) and (2.3.2). Then, there exists a unique* α *such that*

$$\phi(\alpha,y^\delta) = c\delta + d\varepsilon_h. \tag{2.3.3}$$

Furthermore, if $y \neq 0$, *then* $\alpha \to 0$ *as* $\delta+\varepsilon_h \to 0$.

Proof. Existence of a unique $\alpha = \alpha(\delta+\varepsilon_h)$ follows from intermediate value theorem. Suppose α does not converges to 0 as $\delta+\varepsilon_h \to 0$. Then there exists a sequence $(\delta_n+\varepsilon_{h_n})$ with $\delta_n+\varepsilon_{h_n} \to 0$ and $\alpha_n = \alpha(\delta_n+\varepsilon_{h_n}) > r > 0$ as $n \longrightarrow \infty$. Using (2.3.3) we have

$$c\delta_n + d\varepsilon_{h_n} = \|\alpha_n^{\frac{3}{2}}((T_{h_n}T_{h_n}^*)^{1+\beta}+\alpha_n I)^{-\frac{3}{2}}y^{\delta_n}\|.$$

In particular, as $\delta_n + \varepsilon_{h_n} \to 0$, we get

$$0 \geq r^{3/2} \left\| \left((TT^*)^{1+\beta} + rI \right)^{-3/2} y \right\|$$

and hence

$$r^{3/2} \left((TT^*)^{1+\beta} + rI \right)^{-3/2} y = 0,$$

implies $y = 0$. This is a contradiction. Thus $\alpha \to 0$ as $\delta + \varepsilon_h \to 0$. □

Remark 2.3.3. *In view of the above Theorem, we assume that α satisfying (2.3.3) is bounded above, say by some α_0.*

Next, we shall show that if $\alpha = \alpha(\delta + \varepsilon_h)$ satisfies (2.3.3), then $\|\hat{x} - x_{\alpha,\beta}\| = O((\delta + \varepsilon_h)^{\frac{2\nu}{2\nu+1}})$. Our proof is based on the following moment inequality

$$\|B^u x\| \leq \|B^v x\|^{\frac{u}{v}} \|x\|^{1-\frac{u}{v}}, \quad 0 \leq u \leq v, \tag{2.3.4}$$

where B is positive self-adjoint operator.

Theorem 2.3.4. *Let $\alpha = \alpha(\delta + \varepsilon_h)$ be the unique solution of (2.3.3) and (2.1.8) holds. Then*

$$\|\hat{x} - x_{\alpha,\beta}\| \leq O((\delta + \varepsilon_h)^{\frac{2\nu}{2\nu+1}}).$$

Proof. Let $u = 2\nu$, $v = 1 + 2\nu$, $B = \alpha^{\frac{1}{2}} ((T^*T)^{1+\beta} + \alpha I)^{-\frac{1}{2}} (T^*T)^{\frac{1}{2}}$ and $x = \alpha^{1-\nu} ((T^*T)^{1+\beta} + \alpha I)^{-(1-\nu)} z$. Then, by (2.3.4), we have

$$\begin{aligned}
\|\hat{x} - x_{\alpha,\beta}\| &= \|\alpha ((T^*T)^{1+\beta} + \alpha I)^{-1} (T^*T)^{\nu} z\| \\
&= \|B^{2\nu} x\| \\
&\leq \|B^{1+2\nu} x\|^{\frac{2\nu}{2\nu+1}} \|x\|^{\frac{1}{2\nu+1}} \\
&= \|\alpha^{\frac{3}{2}} (T^*T)^{\frac{1}{2}} ((T^*T)^{1+\beta} + \alpha I)^{-\frac{3}{2}} \hat{x}\|^{\frac{2\nu}{2\nu+1}} \|z\|^{\frac{1}{2\nu+1}} \\
&\leq \|\alpha^{\frac{3}{2}} ((TT^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}} T\hat{x}\|^{\frac{2\nu}{2\nu+1}} \|z\|^{\frac{1}{2\nu+1}} \\
&= \|\alpha^{\frac{3}{2}} ((TT^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}} y\|^{\frac{2\nu}{2\nu+1}} \|z\|^{\frac{1}{2\nu+1}},
\end{aligned} \tag{2.3.5}$$

where we used the relation $T = U(T^*T)^{\frac{1}{2}}$ with unitary operator U. We have,

$$\begin{aligned}
& \|\alpha^{\frac{3}{2}} ((TT^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}} y\| \\
\leq \; & \|\alpha^{\frac{3}{2}} [((TT^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}} - ((T_h T_h^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}}] y\| \\
& + \|\alpha^{\frac{3}{2}} ((T_h T_h^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}} y\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}} \\
&\quad \times[((T_hT_h^*)^{1+\beta}+\alpha I)^{\frac{3}{2}}-((TT^*)^{1+\beta}+\alpha I)^{\frac{3}{2}}]((TT^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}y\| \\
&\quad +\|\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}y\| \\
&=: G_1+G_2, \qquad (2.3.6)
\end{aligned}
$$

where $G_1=\|\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}[((T_hT_h^*)^{1+\beta}+\alpha I)^{\frac{3}{2}}-((TT^*)^{1+\beta}+\alpha I)^{\frac{3}{2}}]$ $((TT^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}y\|$ and $G_2=\|\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}y\|$. So, first we shall obtain an expression for $((T_hT_h^*)^{1+\beta}+\alpha I)^{\frac{3}{2}}-((TT^*)^{1+\beta}+\alpha I)^{\frac{3}{2}}$. Notice that by (2.2.3), for any $\xi\in Y$, we have

$$
\begin{aligned}
&[((T_hT_h^*)^{1+\beta}+\alpha I)^{\frac{3}{2}}-((TT^*)^{1+\beta}+\alpha I)^{\frac{3}{2}}]\xi \\
=\;& ((T_hT_h^*)^{1+\beta}+\alpha I)[((T_hT_h^*)^{1+\beta}+\alpha I)^{\frac{1}{2}}-((TT^*)^{1+\beta}+\alpha I)^{\frac{1}{2}}]\xi \\
& +[(T_hT_h^*)^{1+\beta}-(TT^*)^{1+\beta}]((TT^*)^{1+\beta}+\alpha I)^{\frac{1}{2}}\xi \\
=\;& ((T_hT_h^*)^{1+\beta}+\alpha I)\left[\frac{1}{\pi}\int_0^\infty u^{\frac{1}{2}}((T_hT_h^*)^{1+\beta}+(\alpha+u)I)^{-1}((TT^*)^{1+\beta}-(T_hT_h^*)^{1+\beta})\right. \\
& \left.\times((TT^*)^{1+\beta}+(\alpha+u)I)^{-1}\xi du\right] \\
& +[(T_hT_h^*)^{1+\beta}-(TT^*)^{1+\beta}]((TT^*)^{1+\beta}+\alpha I)^{\frac{1}{2}}\xi.
\end{aligned}
$$

In particular for $\xi=((TT^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}y$, we have

$$
\begin{aligned}
G_1 \;=\;& \alpha^{\frac{3}{2}}\|((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{1}{2}}\left[\frac{1}{\pi}\int_0^\infty u^{\frac{1}{2}}((T_hT_h^*)^{1+\beta}+(\alpha+u)I)^{-1}((TT^*)^{1+\beta}\right. \\
& \left.-(T_hT_h^*)^{1+\beta})Z_1du\right] \\
& +\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}[(T_hT_h^*)^{1+\beta}-(TT^*)^{1+\beta}]Z_2\| \\
\leq:\;& \|\Gamma_1\|+\|\Gamma_2\|, \qquad (2.3.7)
\end{aligned}
$$

where, $Z_1=((TT^*)^{1+\beta}+(\alpha+u)I)^{-1}((TT^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}y, Z_2=((TT^*)^{1+\beta}+\alpha I)^{-1}y$,

$$
\begin{aligned}
\|\Gamma_1\| \;=\;& \alpha^{\frac{3}{2}}\left\|((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{1}{2}}\left[\frac{1}{\pi}\int_0^\infty u^{\frac{1}{2}}((T_hT_h^*)^{1+\beta}+(\alpha+u)I)^{-1}\right.\right. \\
& \left.\left.\times((TT^*)^{1+\beta}-(T_hT_h^*)^{1+\beta})Z_1du\right]\right\| \\
& \text{and} \\
\|\Gamma_2\| \;=\;& \alpha^{\frac{3}{2}}\|((T_hT_h^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}[(T_hT_h^*)^{1+\beta}-(TT^*)^{1+\beta}]Z_2\|
\end{aligned}
$$

One can prove (see appendix B)

$$\|\Gamma_1\| \leq \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}(4\varepsilon_h + 2\varepsilon_h\|TT^*\|^{\frac{3}{2}})c_8$$

and

$$\|\Gamma_2\| \leq \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}(4\varepsilon_h + 2\varepsilon_h\|TT^*\|^{\frac{3}{2}})c_9.$$

Therefore, we have

$$G_1 \leq \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}(4\varepsilon_h + 2\varepsilon_h\|TT^*\|^{\frac{3}{2}})(c_8 + c_9) := c_{10}\varepsilon_h \tag{2.3.8}$$

and

$$\begin{aligned} G_2 &\leq \|\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}}(y - y^\delta)\| \\ &\quad + \|\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}}y^\delta\| \\ &\leq \delta + c\delta + d\varepsilon_h. \end{aligned} \tag{2.3.9}$$

Thus by (2.3.5)- (2.3.9), we have

$$\|\hat{x} - x_{\alpha,\beta}\| \leq [\max\{c_{10} + d, 1 + c\}(\delta + \varepsilon_h)]^{\frac{2\nu}{2\nu+1}}\|z\|^{\frac{1}{2\nu+1}}.$$

□

Hereafter, we assume that $c > 1$ and $d > c_{10}$.

Theorem 2.3.5. *Suppose, $\alpha = \alpha(\delta + \varepsilon_h)$ is chosen as a solution of (2.3.3). Then* $\frac{\delta+\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}} = O\left((\delta + \varepsilon_h)^{\frac{2\nu}{2\nu+1}}\right)$.

Proof. From our Discrepancy principle, we have

$$\begin{aligned} c\delta + d\varepsilon_h &= \|\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}}y^\delta\| \\ &\leq \|\alpha^{\frac{3}{2}}((T_hT_h^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}}(y^\delta - y)\| \\ &\quad + \|\alpha^{\frac{3}{2}}[((T_hT_h^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}} - ((TT^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}}]y\| \\ &\quad + \|\alpha^{\frac{3}{2}}((TT^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}}y\| \\ &\leq \delta + G_1 + \|\alpha^{\frac{3}{2}}((TT^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}}y\| \\ &\leq \delta + c_{10}\varepsilon_h + \|\alpha^{\frac{3}{2}}((TT^*)^{1+\beta} + \alpha I)^{-\frac{3}{2}}y\|, \end{aligned} \tag{2.3.10}$$

and hence

$$\begin{aligned}(c-1)\delta+(d-c_{10})\varepsilon_h &\leq \|\alpha^{\frac{3}{2}}((TT^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}y\| \\ &= \|\alpha^{\frac{3}{2}}T((T^*T)^{1+\beta}+\alpha I)^{-\frac{3}{2}}\hat{x}\| \\ &= \|\alpha^{\frac{3}{2}}U(T^*T)^{1/2}((T^*T)^{1+\beta}+\alpha I)^{-\frac{3}{2}}\hat{x}\| \\ &\leq \|\alpha^{\frac{3}{2}}((T^*T)^{1+\beta}+\alpha I)^{-\frac{3}{2}}(T^*T)^{1/2+\nu}z\| \\ &\leq c_{11}\alpha^{\frac{2\nu+1}{2(1+\beta)}}. \end{aligned} \tag{2.3.11}$$

Thus, by (2.3.11),

$$\delta+\varepsilon_h \leq \frac{c_{11}}{\min\{c-1,d-c_{10}\}}\alpha^{\frac{2\nu+1}{2(1+\beta)}},$$

so,

$$\frac{\delta+\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}} = (\delta+\varepsilon_h)^{\frac{2\nu}{2\nu+1}}\left(\frac{\delta+\varepsilon_h}{\alpha^{\frac{2\nu+1}{2(1+\beta)}}}\right)^{\frac{1}{2\nu+1}} \leq (\delta+\varepsilon_h)^{\frac{2\nu}{2\nu+1}}\left(\frac{c_{11}}{\min\{c-1,d-c_{10}\}}\right)^{\frac{1}{2\nu+1}}. \tag{2.3.12}$$

□

By combining Theorem 2.3.2, Theorem 2.3.4 and Theorem 2.3.5, we obtain the following Theorem.

Theorem 2.3.6. *Suppose Assumptions in Theorem 2.3.2,Theorem 2.3.4 and Theorem 2.3.5 hold and if $\alpha = \alpha(\delta+\varepsilon_h)$ is chosen as a solution of (2.3.3). Then*

$$\|x_{\alpha,\beta}^{h,\delta}-\hat{x}\| = O\left((\delta+\varepsilon_h)^{\frac{2\nu}{2\nu+1}}\right).$$

□

Remark 2.3.7. *The above theorem shows that, we obtain the optimal order $O\left((\delta+\varepsilon_h)^{\frac{2\nu}{2\nu+1}}\right)$ for $\gamma \leq \nu \leq 1+\beta$, for some $\gamma > 0$.*

2.4 NUMERICAL EXAMPLES

In this section, the performance of FTFDR method is compared when the regularization parameter α is determined using the Raus and Gfrerer type discrepancy principles (2.3.3) Gfrerer (1987); Raus (1984, 1985), Schock type discrepancy principle Schock (1984b)

$$\|T_h x_{\alpha,\beta}^{h,\delta}-y^\delta\| = \frac{(c\delta+d\varepsilon_h)^p}{\alpha^q}, p>0, q>0 \tag{2.4.1}$$

and Morozov's discrepancy principle Morozov (1968)

$$\|T_h x_{\alpha,\beta}^{h,\delta} - y^\delta\| = c\delta + d\varepsilon_h, \tag{2.4.2}$$

for some positive constants c, d. For the comparison, we pick up two examples whose discrete version is taken from Regularization Toolbox by Hansen Hansen (2007). We take the singular value decomposition (SVD)

$$T = U\Sigma V^T, \tag{2.4.3}$$

where $V = [v_1,\ v_2,\ \ldots,\ v_n] \in \mathbb{R}^{n\times n}$ and $U = [u_1,\ u_2,\ \ldots,\ u_n] \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and

$$\Sigma = diag[\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n] \in \mathbb{R}^{n\times n},$$

whose singular values are ordered according to

$$\lambda_1 \geq \lambda_2 \geq \ldots\lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0.$$

Substituting the SVD (2.4.3) into (2.2.1), (2.3.3), (2.4.1) and ((2.4.2)) yield

$$x_{\alpha,\beta}^\delta = V(\Sigma^{2(\beta+1)} + \alpha I)^{-1}\Sigma^{2\beta+1}U^T y^\delta, \tag{2.4.4}$$

$$\varphi(\alpha, y^\delta) : \|\alpha^{3/2}U(\Sigma^{2(\beta+1)} + \alpha I)^{-3/2}U^T y^\delta\| = c\delta + d\varepsilon_h, \tag{2.4.5}$$

$$\|\alpha U(\Sigma^{2(\beta+1)} + \alpha I)^{-1}U^T y^\delta\| = \frac{(c\delta + d\varepsilon_h)^p}{\alpha^q}. \tag{2.4.6}$$

and

$$\|\alpha U(\Sigma^{2(\beta+1)} + \alpha I)^{-1}U^T y^\delta\| = c\delta + d\varepsilon_h. \tag{2.4.7}$$

In order to solve above nonlinear equations for α with different values β, δ and q with $q = p - 1$, we used Newton's approach. Relative errors $e_{\alpha,\beta} := \left(\frac{\|x_{\alpha,\beta}^\delta - \hat{x}\|}{\|\hat{x}\|}\right)$, and α are presented in the tables for different values of β, $n = 1000$ (size of the mesh) and noise level δ. In each figure, plot (a) contains the exact data and noise data; plot (b) contains the computed solution (C.S) and exact solution (exact sol.) when α is chosen according to (2.4.2); plots (c) contains the computed solution (C.S) and exact solution (exact sol.) when α is chosen according to (2.4.1) and plots (d) contains the computed solution (C.S) and exact solution (exact sol.) when α is chosen according to (2.3.3).

Example 2.4.1. *Consider Foxgood example from the Regularization Toolbox by Hansen Hansen (2007) with n points. It is defined as follows:*

$$[Tx](s) := \int_0^1 \sqrt{s^2+t^2}x(t)dt = y(s),\ 0 \le s \le 1 \tag{2.4.8}$$

with noise free data $y(s) = \frac{1}{3}((1+s^2)^{3/2} - s^3)$ *and solution* $\hat{x}(t) = t$. *We have introduced the random noise level* $\delta = 0.1, 0.01$ *and* 0.001 *in the exact data and* $\varepsilon_h = \varepsilon_{\frac{1}{n}} = \frac{1}{n^2}$. *The exact data has been tainted by the addition of random noise level* $\delta = 0.1, \delta = 0.01$ *and* $\delta = 0.001$. *In Table 2.1, we present the relative errors as well as* α *values using discrepancy principle (2.4.2), (2.4.1) and (2.3.3) for different values of* β *and* δ. *Plots of Foxgood example for different values of* δ, *and* β *are given in Figures 2.1–2.12, with captions.*

Example 2.4.2. *We choose Shaw example from the Regularization Toolbox by Hansen Hansen (2007) with n points. It is defined as follows:*

$$[Tx](s) := \int_{-\pi}^{\pi} k(s,t)x(t)dt = y(s),\ -\pi \le s \le \pi, \tag{2.4.9}$$

where $k(s,t) = (\cos(s)+\cos(t))^2(\frac{\sin(u)}{u})^2$, $u = \pi(\sin(s)+\sin(t))$. *The solution* $\hat{x}$ *is given by* $\hat{x}(t) = 2\exp(-6(t-0.5)^2) + 2\exp(-2(t-0.8)^2)$. *We have introduced the random noise level* $\delta = 0.1, 0.01$ *and* 0.001 *in the exact data and* $\varepsilon_h = \varepsilon_{\frac{1}{n}} = \frac{1}{n^2}$. *The exact data has been tainted by the addition of random noise level* $\delta = 0.1, \delta = 0.01$ *and* $\delta = 0.001$. *In Table (2.2), we present the relative errors as well as* α *values using discrepancy principle (2.4.2), (2.4.1) and (2.3.3) for different values of* β *and* δ. *Plots of Shaw example for different values of* δ, *and* β *are given in Figures 2.13–2.24, with captions.*

2.5 CONCLUSION

In this Chapter, we studied, finite dimensional realization of FTR method with Raus and Gfrerer type discrepancy principle for choosing the regularization parameter α in FDFTR method. As mentioned earlier, it is difficult to choose $\beta \in (-\frac{1}{2}, 0]$ to obtain a better error estimate, but we observe (see Table 2.1 and Table 2.2) that the relative errors $e_{\alpha,\beta} = \left(\frac{\|x^\delta_{\alpha,\beta} - \hat{x}\|}{\|\hat{x}\|}\right) < e_{\alpha,0} := \left(\frac{\|x^\delta_\alpha - \hat{x}\|}{\|\hat{x}\|}\right)$ holds, when α is chosen according to the Raus and Gfrerer type discrepancy principle, Schock-type discrepancy principle and Morozov's type discrepancy principle for $\beta \in (-\frac{1}{2}, 0]$. This demonstrates that the FTR approach provides a better estimate of error than the standard Tikhonov regularization method. The best feasible choice of β is still an open problem (Morigi et al. (2017)).

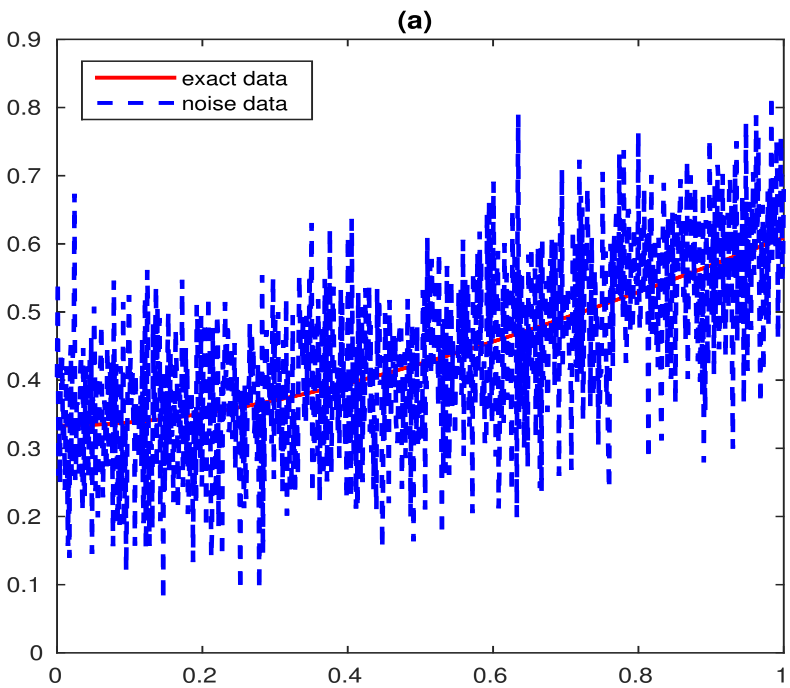

Figure 2.1 Data of *Foxgood* example with $\delta = 0.1$.

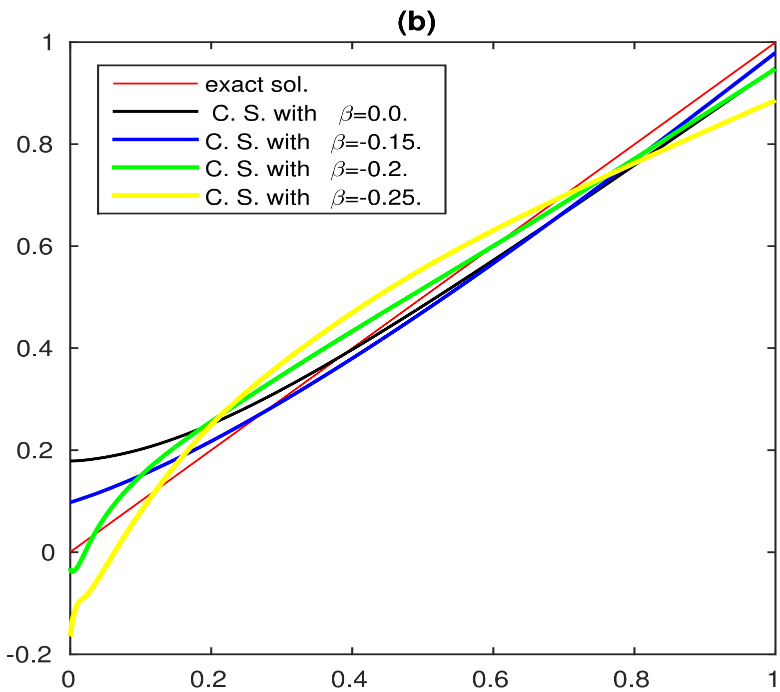

Figure 2.2 Solutions using Morozov's type discrepancy principle with $\delta = 0.1$.

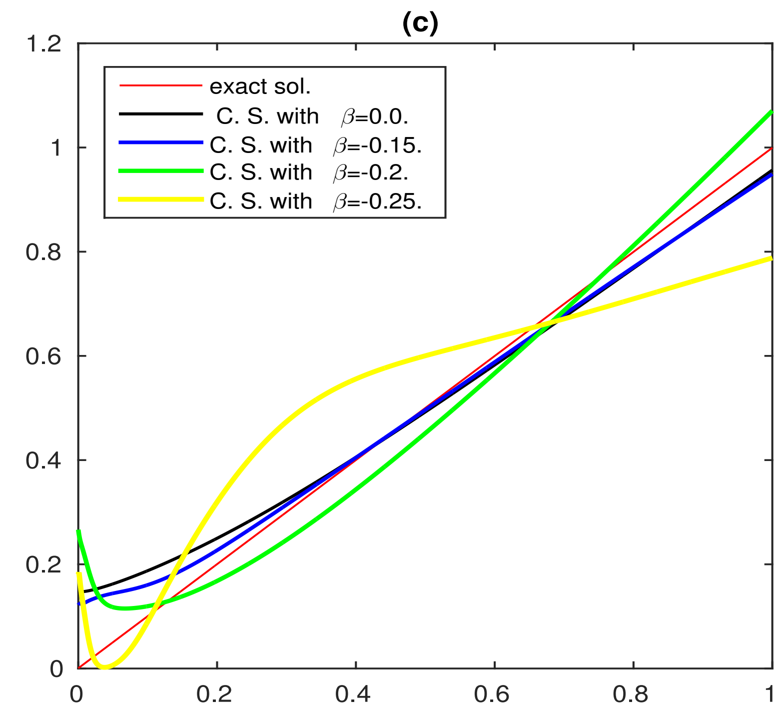

Figure 2.3 Solutions using Schock-type discrepancy principle with $\delta = 0.1$.

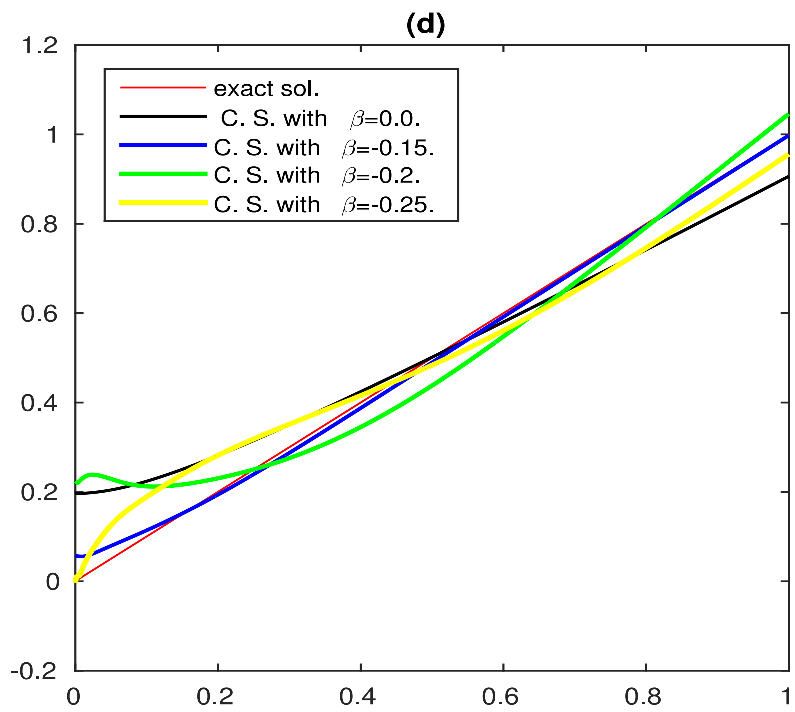

Figure 2.4 Solutions using Raus and Gfrerer type discrepancy principle with $\delta = 0.1$.

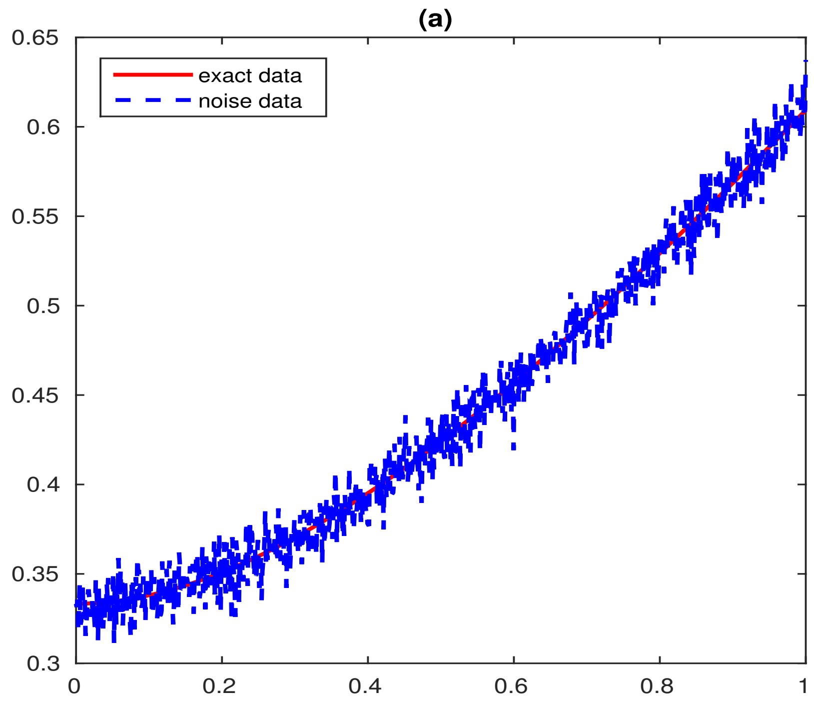

Figure 2.5 Data of *Foxgood* example with $\delta = 0.01$.

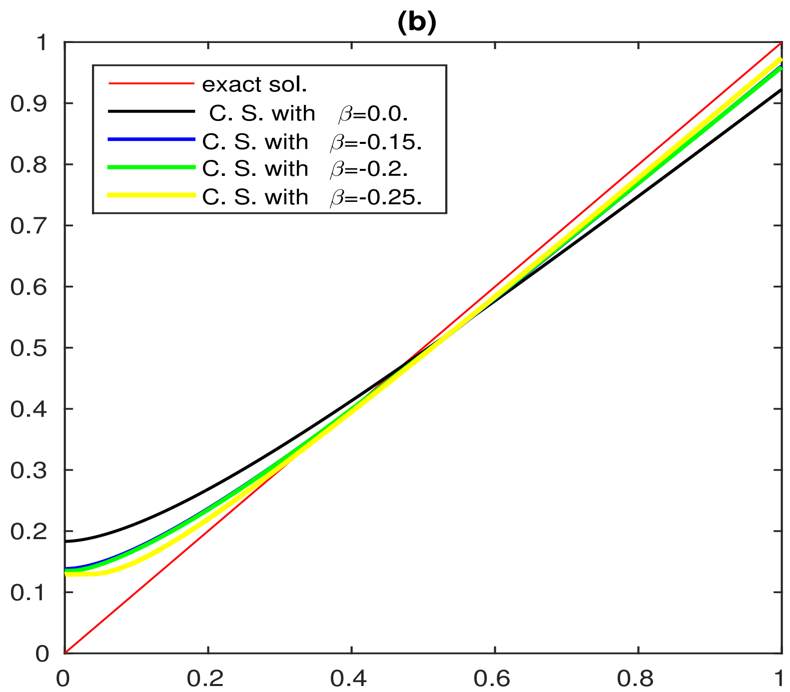

Figure 2.6 Solutions using Morozov's type discrepancy principle with $\delta = 0.01$.

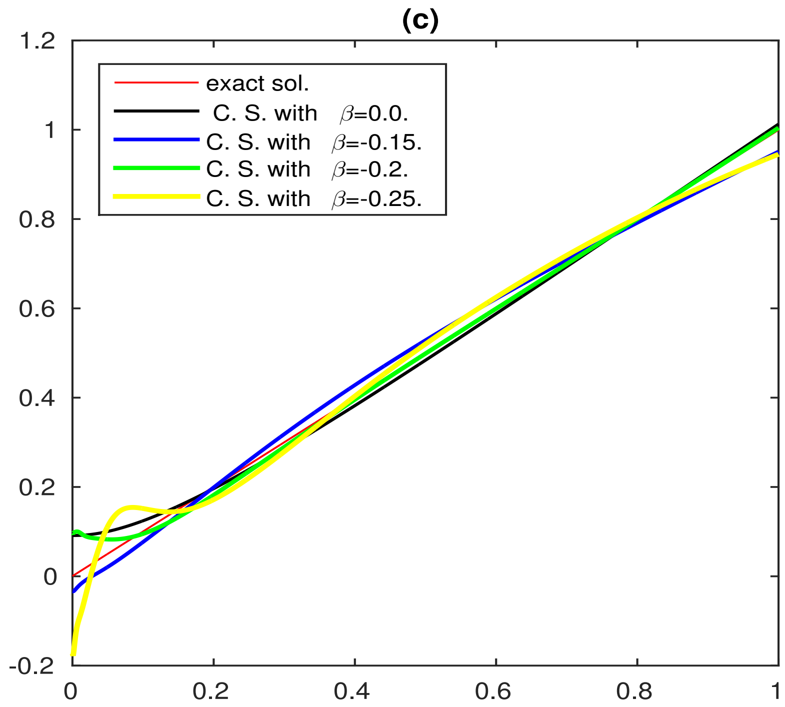

Figure 2.7 Solutions using Schock-type discrepancy principle with $\delta = 0.01$.

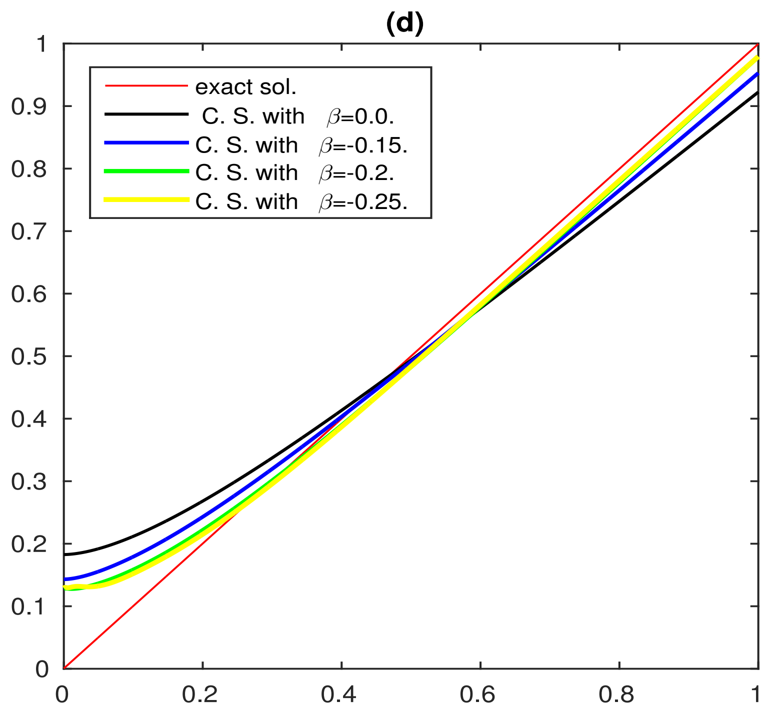

Figure 2.8 Solutions using Raus and Gfrerer type discrepancy principle with $\delta = 0.01$.

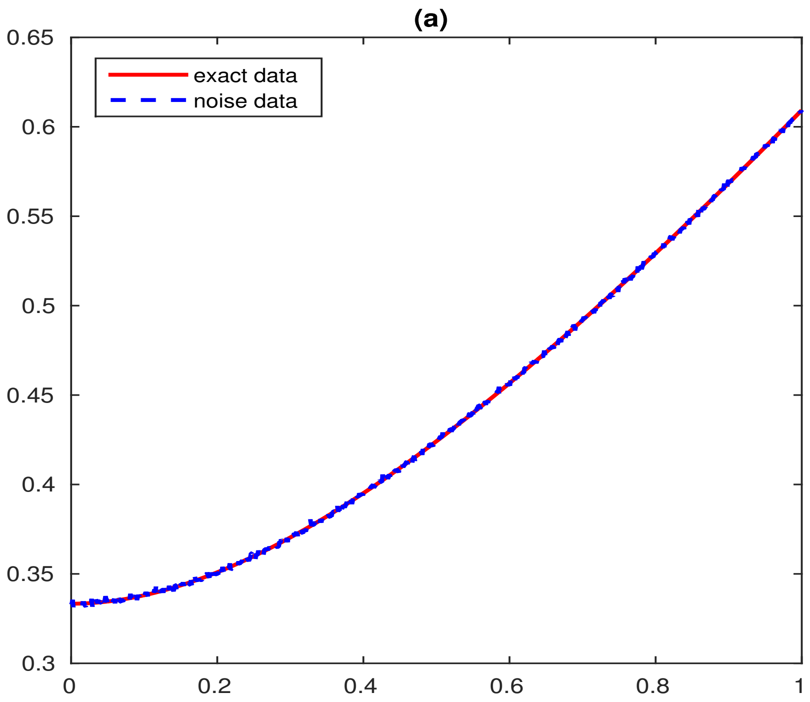

Figure 2.9 Data of *Foxgood* example with $\delta = 0.001$.

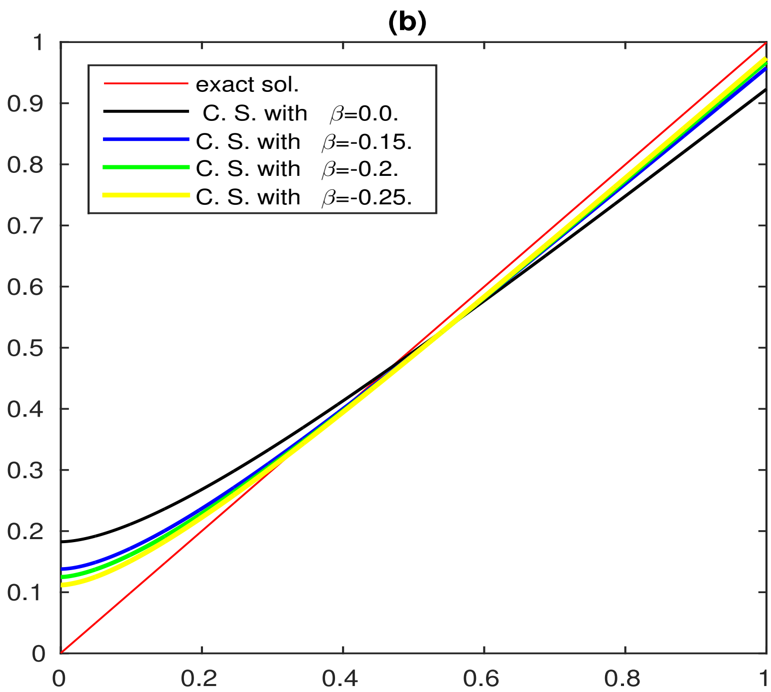

Figure 2.10 Solutions using Morozov's type discrepancy principle with $\delta = 0.001$.

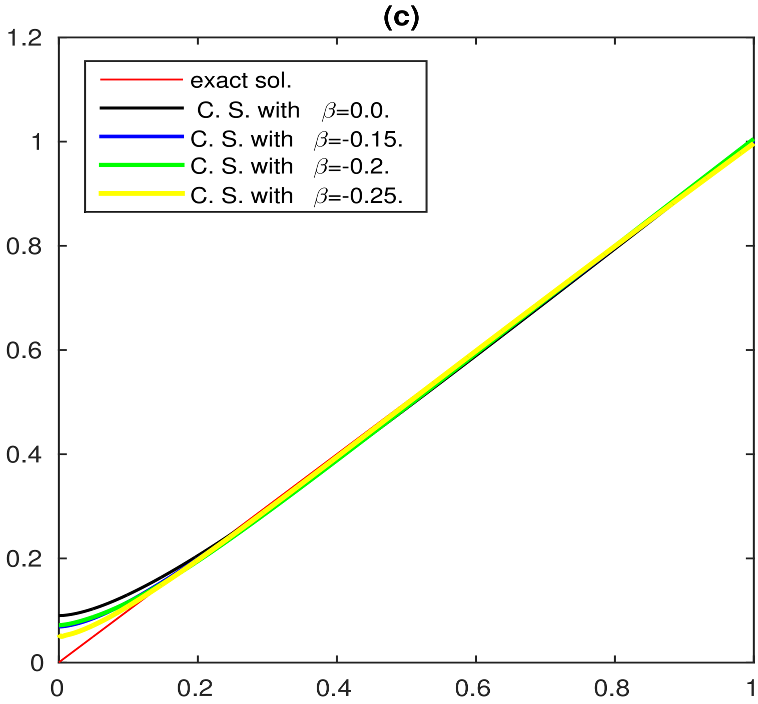

Figure 2.11 Solutions using Schock-type discrepancy principle with $\delta = 0.001$.

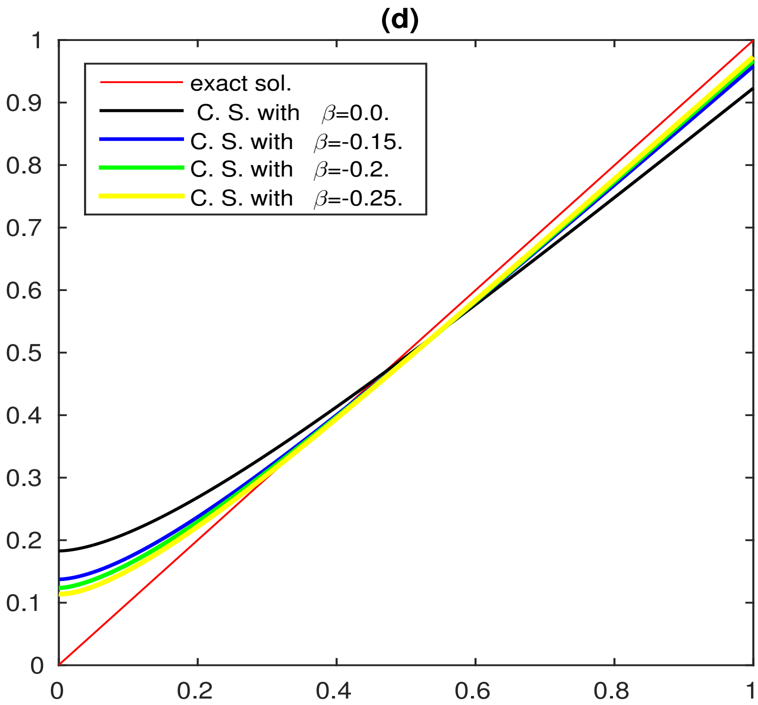

Figure 2.12 Solutions using Raus and Gfrerer type discrepancy principle with $\delta = 0.001$.

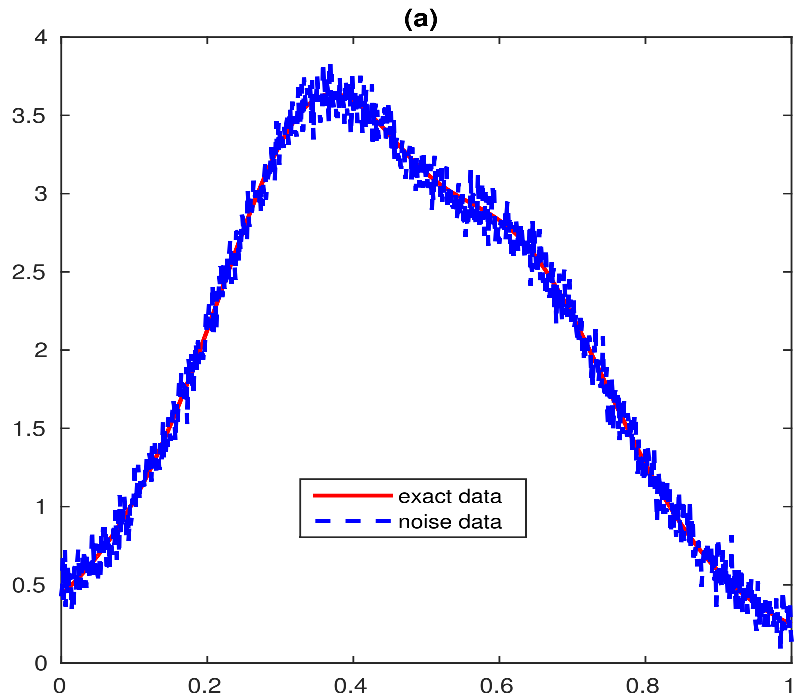

Figure 2.13 Data of *Shaw* example with $\delta = 0.1$.

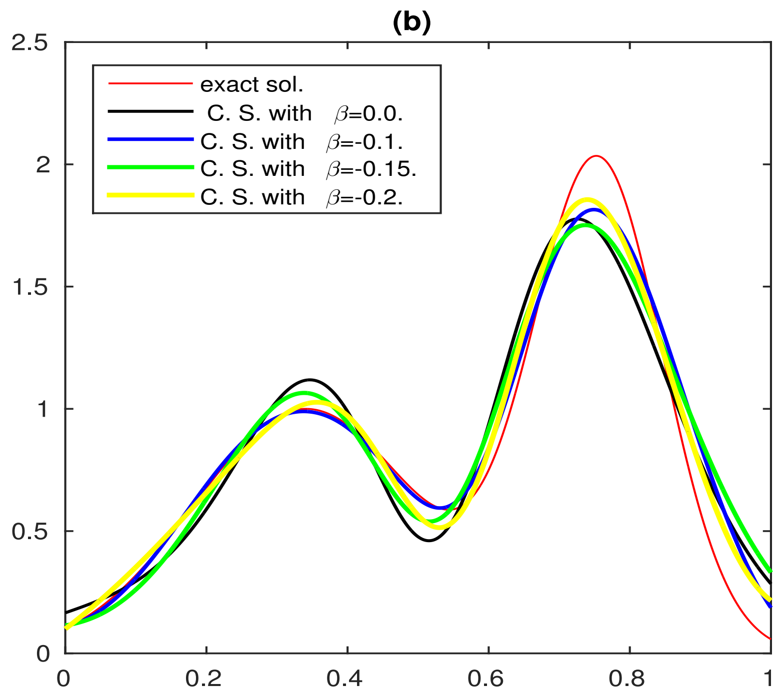

Figure 2.14 Solutions using Morozov's type discrepancy principle with $\delta = 0.1$.

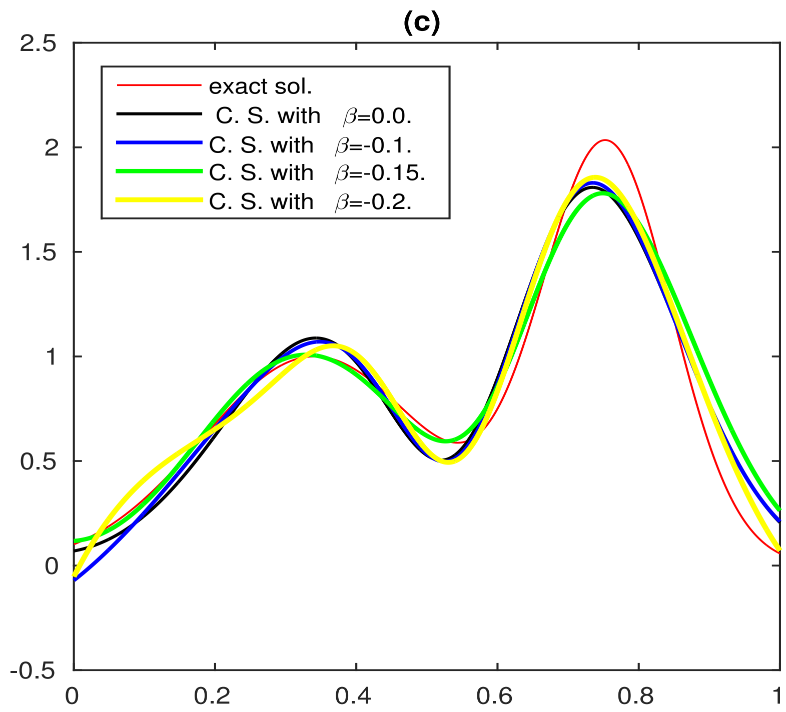

Figure 2.15 Solutions using Schock-type discrepancy principle with $\delta = 0.1$.

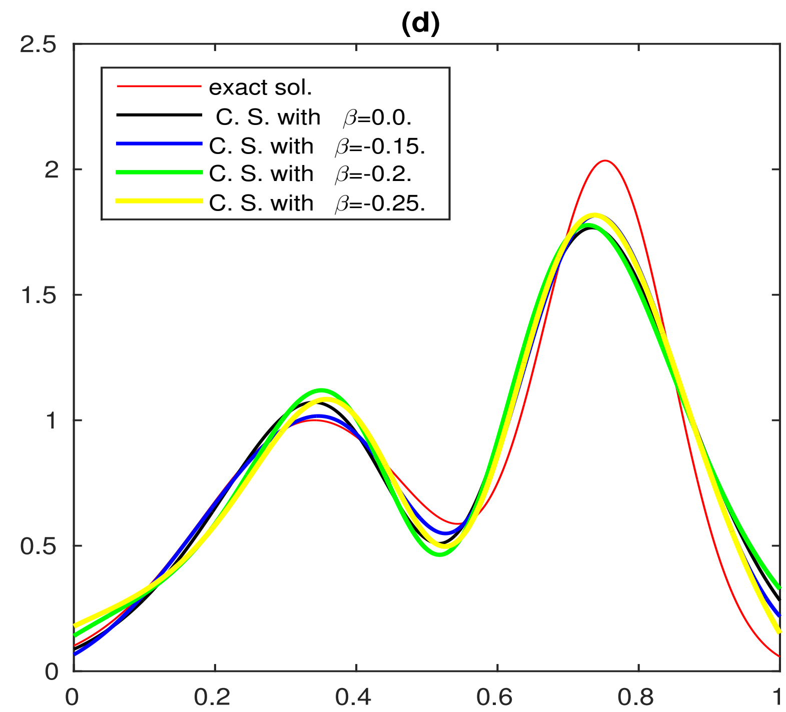

Figure 2.16 Solutions using Raus and Gfrerer type discrepancy principle with $\delta = 0.1$.

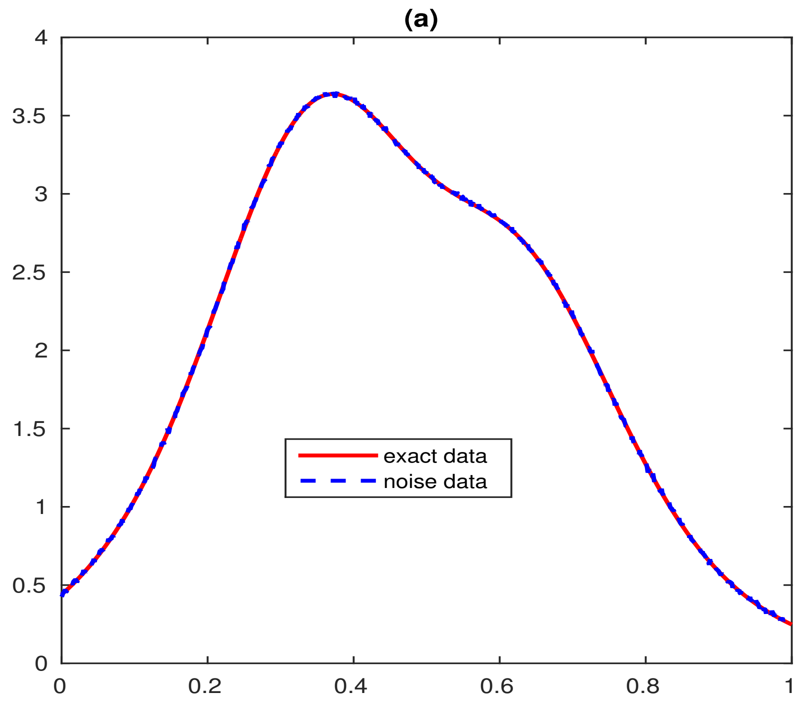

Figure 2.17 Data of *Shaw* example with $\delta = 0.01$.

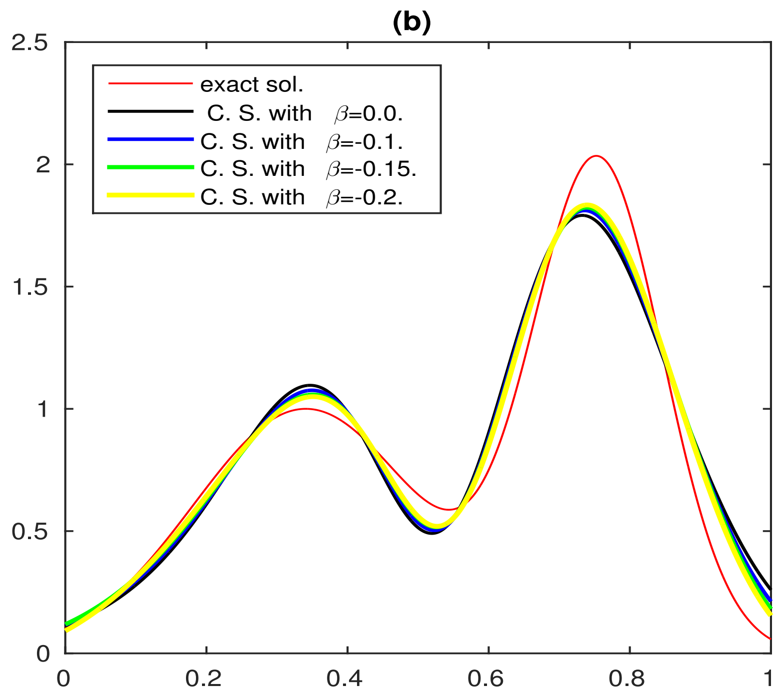

Figure 2.18 Solutions using Morozov's type discrepancy principle with $\delta = 0.01$.

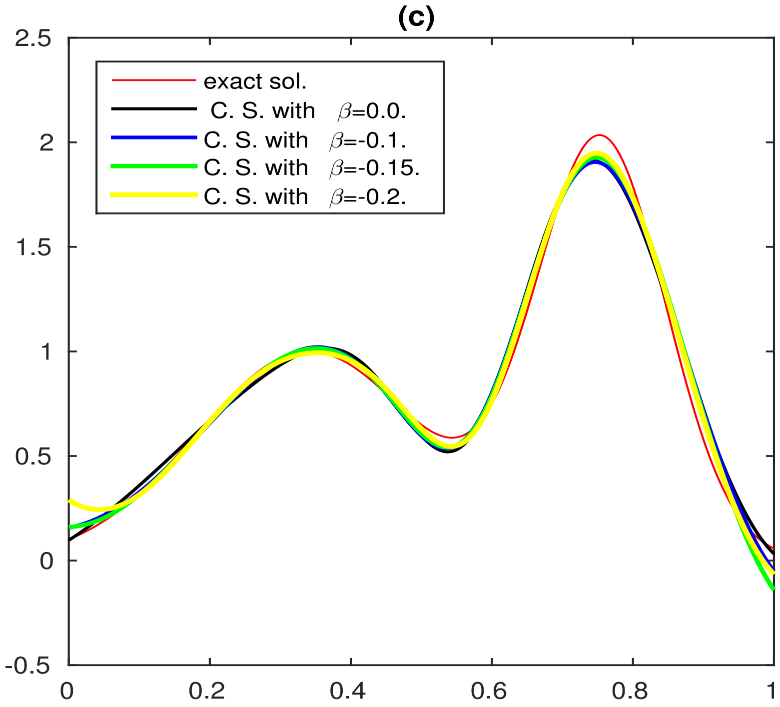

Figure 2.19 Solutions using Schock-type discrepancy principle with $\delta = 0.01$.

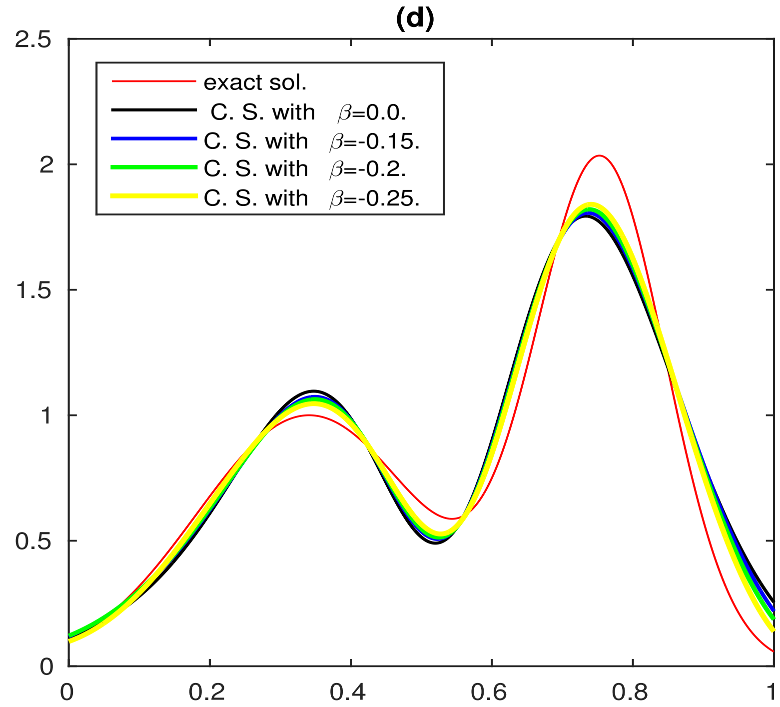

Figure 2.20 Solutions using Raus and Gfrerer type discrepancy principle with $\delta = 0.01$.

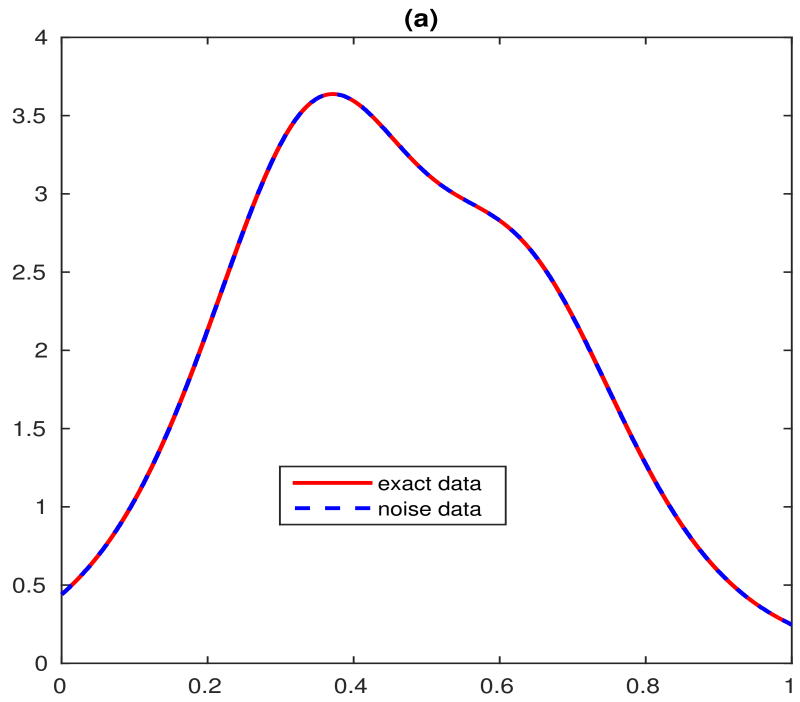

Figure 2.21 Data of *Shaw* example with $\delta = 0.001$.

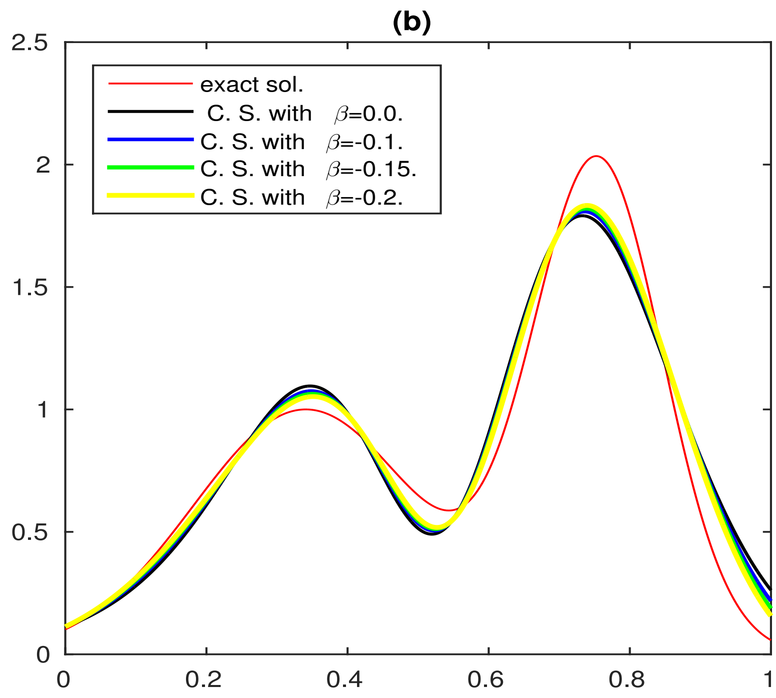

Figure 2.22 Solutions using Morozov's type discrepancy principle with $\delta = 0.001$.

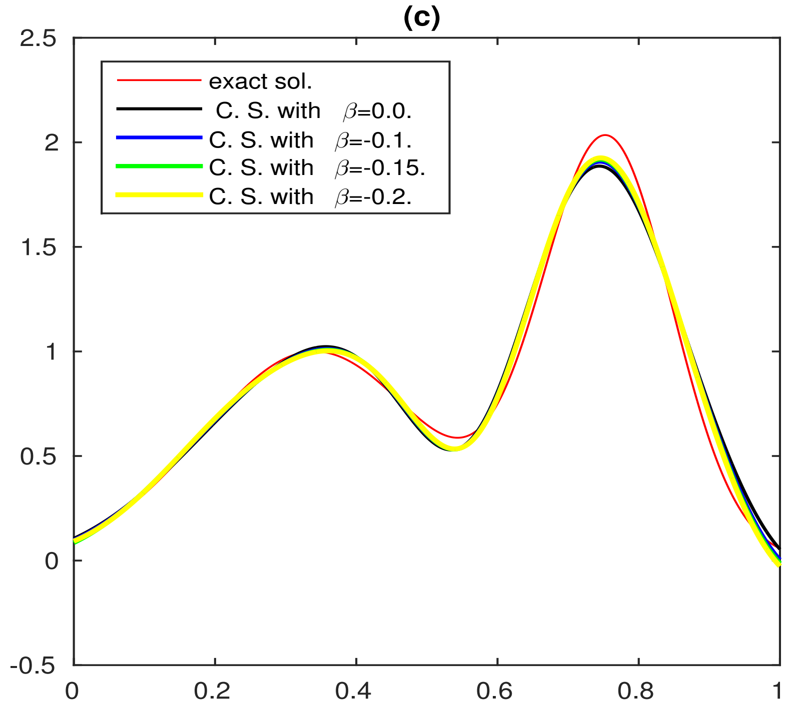

Figure 2.23 Solutions using Schock-type discrepancy principle with $\delta = 0.001$.

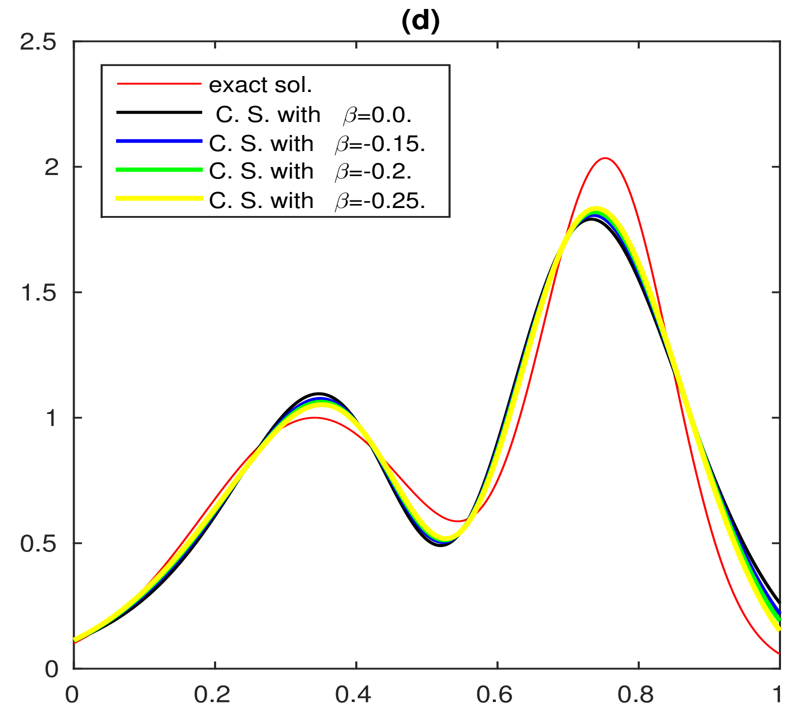

Figure 2.24 Solutions using Raus and Gfrerer type discrepancy principle with $\delta = 0.001$.

Table 2.1 Relative errors employing different discrepancy rules in the Foxgood example.

		Morozov's type method			Schock's type method			Raus and Gfrerer type method		
	δ/β	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001
0	α	$4.773800e-03$	$5.497968e-03$	$5.499283e-03$	$3.416657e-03$	$3.224783e-04$	$7.138374e-04$	$5.151365e-03$	$5.499240e-03$	$5.499896e-03$
	$e_{\alpha,\beta}$	$1.006404e-01$	$1.154653e-01$	$1.149971e-01$	$8.429121e-02$	$3.634466e-02$	$3.570536e-02$	$1.299482e-01$	$1.151339e-01$	$1.151811e-01$
-0.15	α	$4.792023e-03$	$5.498288e-03$	$5.499627e-03$	$3.416159e-03$	$3.389354e-04$	$7.252900e-04$	$5.187231e-03$	$5.499252e-03$	$5.499970e-03$
	$e_{\alpha,\beta}$	$6.018369e-02$	$7.400377e-02$	$7.453973e-02$	$6.777801e-02$	$3.995668e-02$	$2.434411e-02$	$2.225650e-02$	$8.053778e-02$	$7.441076e-02$
-0.2	α	$4.826014e-03$	$5.498266e-03$	$5.499693e-03$	$3.447664e-03$	$3.261034e-04$	$7.024664e-04$	$5.134205e-03$	$5.499292e-03$	$5.499980e-03$
	$e_{\alpha,\beta}$	$6.298842e-02$	$7.183043e-02$	$6.413678e-02$	$9.036714e-02$	$2.985094e-02$	$2.634077e-02$	$1.225282e-01$	$6.064762e-02$	$6.394360e-02$
-0.25	α	$4.765742e-03$	$5.498398e-03$	$5.499743e-03$	$3.387352e-03$	$3.337148e-04$	$6.763532e-04$	$5.128512e-03$	$5.499343e-03$	$5.499985e-03$
	$e_{\alpha,\beta}$	$1.042475e-01$	$5.779342e-02$	$5.484780e-02$	$2.011399e-01$	$5.512364e-02$	$1.581466e-02$	$9.344107e-02$	$5.772382e-02$	$5.501765e-02$

Table 2.2 Relative errors employing different discrepancy rules in the Shaw example.

		Morozov's type method			Schock's type method			Raus and Gfrerer type method		
	δ/β	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001
0	α	$5.494892e-03$	$5.499930e-03$	$5.499978e-03$	$3.342907e-03$	$3.485044e-04$	$4.940953e-04$	$5.497493e-03$	$5.499969e-03$	$5.499995e-03$
	$e_{\alpha,\beta}$	$1.566804e-01$	$1.390961e-01$	$1.393482e-01$	$1.258053e-01$	$6.274838e-02$	$7.164926e-02$	$1.444955e-01$	$1.375854e-01$	$1.390921e-01$
-0.1	α	$5.495304e-03$	$5.499930e-03$	$5.499982e-03$	$3.435283e-03$	$3.304696e-04$	$5.909074e-04$	$5.497594e-03$	$5.499972e-03$	$5.499997e-03$
	$e_{\alpha,\beta}$	$1.118636e-01$	$1.213782e-01$	$1.229959e-01$	$1.211811e-01$	$6.149359e-02$	$6.032795e-02$	$1.100015e-01$	$1.239868e-01$	$1.232654e-01$
-0.15	α	$5.495425e-03$	$5.499934e-03$	$5.499984e-03$	$3.241767e-03$	$3.239556e-04$	$6.348895e-04$	$5.497589e-03$	$5.499974e-03$	$5.499998e-03$
	$e_{\alpha,\beta}$	$1.565930e-01$	$1.094870e-01$	$1.123341e-01$	$1.336406e-01$	$5.658549e-02$	$5.545215e-02$	$1.598642e-01$	$1.121039e-01$	$1.130612e-01$
-0.2	α	$5.495176e-03$	$5.499936e-03$	$5.499985e-03$	$3.252457e-03$	$3.438961e-04$	$6.750084e-04$	$5.497466e-03$	$5.499972e-03$	$5.499999e-03$
	$e_{\alpha,\beta}$	$9.718942e-02$	$1.012614e-01$	$1.025344e-01$	$1.002055e-01$	$5.198294e-02$	$5.195900e-02$	$1.195230e-01$	$9.552077e-02$	$1.015265e-01$

CHAPTER 3

A NEW PARAMETER CHOICE STRATEGY FOR LAVRENTIEV REGULARIZATION METHOD FOR NONLINEAR ILL-POSED EQUATIONS

3.1 INTRODUCTION

Let $\mathscr{H} : D(\mathscr{H}) \subseteq \mathscr{U} \to \mathscr{U}$ be a nonlinear monotone operator (see Definition 1.4.2) defined on the real Hilbert space $\mathscr{U}$. We are concerned with finite dimensional approximation of the solution of the ill-posed equation

$$\mathscr{H}(u) = y, \tag{3.1.1}$$

which has a solution $\hat{u}$ for exact data y. However, we have $y^\delta \in \mathscr{U}$ for some $\delta > 0$, are the available data, such that

$$\|y - y^\delta\| \leq \delta. \tag{3.1.2}$$

Due to the ill-posedness of (3.1.1), one has to apply regularization method to obtain an approximation for $\hat{u}$. For (3.1.1) with monotone $\mathscr{H}$, Lavrentiev regularization (LR) method is widely used (George and Nair (2008); Hofmann et al. (2016); Janno and Tautenhahn (2003); Mahale and Nair (2013); Tautenhahn (2002); Vasin and George (2014)). In (LR) method the solution u_α^δ of the equation

$$\mathscr{H}(u) + \alpha(u - u_0) = y^\delta, \tag{3.1.3}$$

is used as an approximation for $\hat{u}$. Here (and below) u_0 is an initial approximation of $\hat{u}$ with $\|u_0 - \hat{u}\| \leq r_0$ for some $r_0 > 0$. The solution of (3.1.3), with y in place of y^δ is denoted by u_α, i.e., (cf. Tautenhahn (2002))

$$\mathscr{H}(u_\alpha)+\alpha(u_\alpha-u_0)=y. \tag{3.1.4}$$

Let u_α^δ and u_α be as in (3.1.3) and (3.1.4) , respectively. Then, we have the following inequalities (cf. Tautenhahn (2002)).

$$\begin{aligned} \|u_\alpha-\hat{u}\|^2 &\le \langle u_0-\hat{u}, u_\alpha-\hat{u}\rangle, \\ \|u_\alpha^\delta-u_\alpha\| &\le \frac{\delta}{\alpha}, \end{aligned} \tag{3.1.5}$$

and hence,

$$\|\hat{u}-u_\alpha^\delta\| \le \|\hat{u}-u_\alpha\|+\frac{\delta}{\alpha} \tag{3.1.6}$$

and

$$\|\hat{u}-u_\alpha\| \le \|\hat{u}-u_0\|. \tag{3.1.7}$$

For proving our result, we assume that, either $\mathscr{H}'(u)$ is self-adjoint or $\mathscr{H}'(u)$ is positive type, i.e.,

$$\sigma(\mathscr{H}'(u))\subseteq[0,\infty)$$

and

$$\|(\mathscr{H}'(u)+sI)^{-1}\|\le\frac{c}{s},\ s>0,\ \text{for some constant } c>0,\ u\in\overline{B(u_0,r)}$$

(see Nair and Ravishankar (2008)). Here and below $\mathscr{H}'(u)$ is the Fréchet derivative of $\mathscr{H}(u)$ (if $\mathscr{H}'(u)$ is self-adjoint, then $c=1$).

Remark 3.1.1. *If $\mathscr{H}'(u)$ is positive type, then*

$$\|(\mathscr{H}'(u)+sI)^{-1}\mathscr{H}'(u)\|=\|I-s(\mathscr{H}'(u)+sI)^{-1}\|\le 1+c.$$

Further as in George and Sabari (2018) (Lemma 2.2) one can prove

$$\|(\mathscr{H}'(u)+sI)^{-1}\mathscr{H}'(u)^\mu\|=O(s^\mu),\ 0\le\mu<1.$$

So, the results in this paper hold for positive type operator $\mathscr{H}'(u)$ up to a constant. Therefore, for convenience, hereafter we assume $\mathscr{H}'(.)$ is positive self-adjoint.

In earlier studies such as (Mahale and Nair (2013); Tautenhahn (2002); Vasin and

George (2014); George (2010); Semenova (2010)), the following source condition:

$$u_0 - \hat{u} = \mathscr{H}'(\hat{u})^{\mu_1} z, \ \|z\| \leq \rho, \quad 0 < \mu_1 \leq 1. \tag{3.1.8}$$

or

$$u_0 - \hat{u} = \mathscr{H}'(u_0)^{\mu_2} z, \ \|z\| \leq \rho, \quad 0 < \mu_2 \leq 1 \tag{3.1.9}$$

was used to obtain an estimate for $\|\hat{u} - u_\alpha\|$. In fact, if the source condition (3.1.8) is satisfied, then, we have Tautenhahn (2002)

$$\|\hat{u} - u_\alpha\| = O(\alpha^{\mu_1})$$

and if (3.1.9) is satisfied, then, we have Hofmann et al. (2016)

$$\|\hat{u} - u_\alpha\| = O(\alpha^{\mu_2}).$$

In this study, we introduce a new source condition,

$$u_0 - \hat{u} = A^\nu z, \ \|z\| \leq \rho, \quad 0 < \nu \leq 1, \tag{3.1.10}$$

where $\rho > 0$ and $A = \int_0^1 \mathscr{H}'(\hat{u} + t(u_0 - \hat{u}))dt$. We shall use this source condition (3.1.10) to obtain a convergence rate for $\|\hat{u} - u_\alpha\|$ and to introduce a new parameter-choice strategy.

Remark 3.1.2. *(a) Note that in a posteriori parameter-choice strategy, the regularization parameter α (depending on δ and y^δ) is chosen at the time of computing u_α^δ (see de Hoog (1980)). The new source condition (3.1.10) is used to choose the parameter α (depending on δ and y^δ) and independent of ν, before computing u_α^δ (see Section 3.2) and also it gives the best known convergence order (see Remark 3.2.3). This is the innovation of our approach.*
(b) Notice that, the operator A and A^ν are used to obtain an estimate for $\|\hat{u} - u_\alpha\|$. In actual computation of the approximation $u_{n+1,\alpha}^{h,\delta}$ (see Equation (3.3.7)) and α (see Section 3.4) we do not require the operator A or A^ν.

Let $z = \nu$, and $B = \mathscr{H}'(.)$ in (2.2.3). Then, we have

$$\mathscr{H}'(.)^\nu x = \frac{\sin \pi(\nu)}{\pi} \left[\frac{x}{\nu} + \int_0^\infty \tau^\nu (\mathscr{H}'(.) + \tau I)^{-1} x d\tau - \int_1^\infty \frac{x}{\tau^{1-\nu}} d\tau \right]. \tag{3.1.11}$$

Note that, if $\mathscr{H}'(.)$ is positive self-adjoint, then, A is self-adjoint. Further, suppose

$\mathscr{H}'(.)$ is positive type, then we have

$$\begin{aligned}
\|(A+sI)^{-1}\| &= \left\| \left(\int_0^1 \mathscr{H}'(\hat{u}+t(u_0-\hat{u}))dt + sI)^{-1} \right) \right\| \\
&= \left\| \left(\int_0^1 (\mathscr{H}'(\hat{u}+t(u_0-\hat{u})) + sI)dt \right)^{-1} \right\| \\
&\leq \int_0^1 \|(\mathscr{H}'(\hat{u}+t(u_0-\hat{u})) + sI)^{-1}\| dt \\
&\leq \frac{c}{s},
\end{aligned}$$

i.e., A is positive type. Next, we shall prove that (3.1.10) implies

$$u_0 - \hat{u} = \begin{cases} \mathscr{H}'(u_0)^{\nu_1}\xi_z, \ \|\xi_z\| \leq \rho_0 \text{ for } & 0 < \nu_1 < \nu < 1 \\ \mathscr{H}'(u_0)\xi_{z_1}, \ \|\xi_{z_1}\| \leq \rho_1 \text{ for } & \nu = 1, \end{cases} \tag{3.1.12}$$

for some constants ρ_0 and ρ_1. For this, we use the standard non-linear assumptions in the literature (cf. Mahale and Nair (2013, 2009)).

Assumption 3.1.3. For every $u, v \in \overline{B(u_0, r)}$ and $w \in \mathscr{U}$, there exists $k_0 > 0$ and an element $\Phi(u, v, w) \in \mathscr{U}$ with

$$[\mathscr{H}'(u) - \mathscr{H}'(v)]w = \mathscr{H}'(v)\Phi(u, v, w)$$

and

$$\|\Phi(u, v, w)\| \leq k_0 \|w\| \|u - v\|.$$

Suppose (3.1.10) holds for $\nu < 1$, then

$$\begin{aligned}
u_0 - \hat{u} &= A^\nu z \\
&= [A^\nu - \mathscr{H}'(u_0)^\nu]z + \mathscr{H}'(u_0)^\nu z \\
&= -\frac{\sin \pi(\nu)}{\pi} \int_0^\infty \tau^\nu (\mathscr{H}'(u_0) + \tau I)^{-1}(A - \mathscr{H}'(u_0))(A + \tau I)^{-1} z d\tau \\
&\quad + \mathscr{H}'(u_0)^\nu z,
\end{aligned}$$

so by the definition of A and Assumption 3.1.3, we have

$$\begin{aligned}
u_0 - \hat{u} &= [A^\nu - \mathscr{H}'(u_0)^\nu]z + \mathscr{H}'(u_0)^\nu z \\
&= -\frac{\sin \pi(\nu)}{\pi} \int_0^\infty \tau^\nu (\mathscr{H}'(u_0) + \tau I)^{-1} \\
&\quad \times \int_0^1 (\mathscr{H}'(\hat{u}+t(u_0-\hat{u})) - \mathscr{H}'(u_0))dt (A + \tau I)^{-1} z d\tau
\end{aligned}$$

$$
\begin{aligned}
& +\mathscr{H}'(u_0)^{\nu} z \\
= & -\frac{\sin \pi(\nu)}{\pi} \int_0^{\infty} \tau^{\nu} (\mathscr{H}'(u_0) + \tau I)^{-1} \mathscr{H}'(u_0) \\
& \times \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, (A + \tau I)^{-1} z) dt d\tau + \mathscr{H}'(u_0)^{\nu} z \\
= & \mathscr{H}'(u_0) \left[-\frac{\sin \pi(\nu)}{\pi} \int_0^{\infty} \tau^{\nu} (\mathscr{H}'(u_0) + \tau I)^{-1} \right. \\
& \left. \times \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, (A + \tau I)^{-1} z) dt d\tau \right] + \mathscr{H}'(u_0)^{\nu} z \\
= & \mathscr{H}'(u_0)^{\nu_1} \xi_z, \ \nu_1 < \nu,
\end{aligned}
$$

where $\xi_z = \mathscr{H}'(u_0)^{1-\nu_1}(-\frac{\sin \pi(\nu)}{\pi} \int_0^{\infty} \tau^{\nu} (\mathscr{H}'(u_0) + \tau I)^{-1} \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, (A + \tau I)^{-1} z) dt) d\tau + \mathscr{H}'(u_0)^{\nu - \nu_1} z$. Further note that

$$
\begin{aligned}
\|\xi_z\| \leq & \frac{1}{\pi} \Big\| \left(\int_0^{\infty} \mathscr{H}'(u_0)^{1-\nu_1} \tau^{\nu} (\mathscr{H}'(u_0) + \tau I)^{-1} \right. \\
& \left. \times \int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, (A + \tau I)^{-1} z) dt \right) d\tau \Big\| + \|\mathscr{H}'(u_0)^{\nu - \nu_1} z\| \\
\leq & \frac{1}{\pi} \left(\int_0^1 \tau^{\nu} \|\mathscr{H}'(u_0)^{1-\nu_1} (\mathscr{H}'(u_0) + \tau I)^{-1}\| k_0 \frac{\|u_0 - \hat{u}\|}{2} \|(A + \tau I)^{-1} z\| d\tau \right. \\
& \left. + \int_1^{\infty} \tau^{\nu} \|\mathscr{H}'(u_0)^{1-\nu_1} (\mathscr{H}'(u_0) + \tau I)^{-1}\| k_0 \frac{\|u_0 - \hat{u}\|}{2} \|(A + \tau I)^{-1} z\| d\tau \right) \\
& + \|\mathscr{H}'(u_0)^{\nu - \nu_1}\| \rho \\
\leq & \frac{1}{\pi} \left(\int_0^1 \tau^{\nu - \nu_1 - 1} d\tau k_0 \frac{\|u_0 - \hat{u}\|}{2} \|z\| \right. \\
& \left. + \|\mathscr{H}'(u_0)^{1-\nu_1}\| \int_1^{\infty} \tau^{\nu - 2} d\tau k_0 \frac{\|u_0 - \hat{u}\|}{2} \|z\| \right) + \|\mathscr{H}'(u_0)^{\nu - \nu_1}\| \rho \\
\leq & \frac{1}{\pi} \left[\frac{1}{\nu - \nu_1} + \frac{\|\mathscr{H}'(u_0)^{1-\nu_1}\|}{1 - \nu} \right] k_0 \frac{r_0}{2} \rho + \|\mathscr{H}'(u_0)^{\nu - \nu_1}\| \rho := \rho_0.
\end{aligned}
$$

Suppose,

$$
\begin{aligned}
u_0 - \hat{u} = & Az \\
= & (A - \mathscr{H}'(u_0) + \mathscr{H}'(u_0)) z \\
= & \left(\int_0^1 (\mathscr{H}'(\hat{u} + t(u_0 - \hat{u})) - \mathscr{H}'(u_0)) dt + \mathscr{H}'(u_0) \right) z \\
= & \mathscr{H}'(u_0) \left(\int_0^1 \Phi(\hat{u} + t(u_0 - \hat{u}), u_0, z) dt + z \right) \\
= & \mathscr{H}'(u_0) \xi_{z_1},
\end{aligned}
$$

where $\xi_{z_1} = \int_0^1 \Phi(\hat{u}+t(u_0-\hat{u}), u_0, z)dt + z$. Observe that

$$\|\xi_{z_1}\| \leq \left(k_0 \frac{\|\hat{u}-u_0\|}{2} + 1\right)\|z\| \leq \left(\frac{k_0 r_0}{2} + 1\right)\rho = \rho_1.$$

So $u_0 - \hat{u} = Az$ implies $u_0 - \hat{u} = \mathscr{H}'(u_0)\xi_{z_1}$, $\|\xi_{z_1}\| \leq \rho_1$ i.e., (3.1.10) implies (3.1.12). Similarly one can show that (3.1.10) implies

$$u_0 - \hat{u} = \begin{cases} \mathscr{H}'(\hat{u})^{\nu_1}\xi_z, \ \|\xi_z\| \leq \rho_2 \text{ for} & 0 < \nu_1 < \nu < 1 \\ \mathscr{H}'(\hat{u})\xi_{z_1}, \ \|\xi_{z_1}\| \leq \rho_1 \text{ for} & \nu = 1, \end{cases}$$

for some constant ρ_2. Throughout the paper, we use the relation (Fundamental Theorem of Integration),

$$\mathscr{H}(u) - \mathscr{H}(x) = \left[\int_0^1 \mathscr{H}'(x+t(u-x))\,dt\right](u-x)$$

for all x and u in a ball contained in $D(\mathscr{H})$.

Remark 3.1.4. *In general, it is believed that (see Tautenhahn (2002)) a priori parameter-choice strategy is not a good strategy to choose α since the choice is depending on the unknown ν. In this study, we introduce a new parameter-choice strategy which is not depending on unknown ν and gives the best known convergence order $O(\delta^{\frac{\nu}{\nu+1}})$.*

In some recent papers, the first author and his collaborators considered iterative methods for obtaining stable approximate solutions for (3.1.3) (see George and Sabari (2018); George and Nair (2017)). In most of the iterative methods Fréchet derivative of the operator involved is used. Semenova (2010) considered the iterative method defined for fixed α, δ, by

$$u^{\delta}_{n+1,\alpha} = u^{\delta}_{n,\alpha} - \gamma[\mathscr{H}(u^{\delta}_{n,\alpha}) + \alpha(u^{\delta}_{n,\alpha} - u_0) - y^{\delta}]. \tag{3.1.13}$$

Note that, the above iterative method is derivative-free. Convergence analysis in (Semenova (2010)) is based on the assumption that $\mathscr{H}$ is Lipschitz continuous and the Lipschitz constant R satisfies

$$0 < \gamma < \min\left\{\frac{1}{\alpha}, \frac{2\alpha}{\alpha^2 + R^2}\right\}, \tag{3.1.14}$$

where γ is a constant. Contraction mapping arguments are used to prove the convergence in (Semenova (2010)). George and Nair (2017) considered the method (3.1.13), but with β independent on the regularization parameter α and the Lipschitz constant R, instead of γ. The source condition on $u_0 - \hat{u}$ in (George and Nair (2017)) depends on the known u_0 and the analysis in (George and Nair (2017)) is not based on the contraction mapping arguments as in (Semenova (2010)).

The purpose of this Chapter is threefold: (1) introduce a new source condition, (2)introduce a new parameter-choice strategy, and (3) apply the parameter-choice strategy to the (finite–dimensional setting of the) method in (George and Nair (2017)).

The remainder of the Chapter is organized as follows. In Section 3.2, we present the error bounds under the source condition (3.1.10) and a new parameter-choice strategy. In Section 3.3, we present the finite dimensional realization of method (3.1.13). In Section 3.4, we present the finite dimensional realization of (3.1.10). Section 3.5 contains the numerical example and the conclusion is given in Section 3.6.

3.2 ERROR BOUNDS UNDER (3.1.10) AND A NEW PARAMETER CHOICE STRATEGY

First we obtain an estimate for $\|\hat{u} - u_\alpha\|$ using (3.1.10).

Theorem 3.2.1. *Let $\frac{3}{2}k_0 r_0 < 1$, Assumption 3.1.3 and (3.1.10) be satisfied. Then,*

$$\|\hat{u} - u_\alpha\| \leq \frac{2 + k_0 r_0}{3 - 2k_0 r_0} \alpha^\nu \|z\|.$$

Proof. Since $\mathscr{H}(\hat{u}) = y$ and $\mathscr{H}(u_\alpha) + \alpha(u_\alpha - u_0) = y$, we have

$$\mathscr{H}(u_\alpha) - \mathscr{H}(\hat{u}) + \alpha(u_\alpha - u_0) = 0,$$

i.e.,

$$\mathscr{H}(u_\alpha) - \mathscr{H}(\hat{u}) + \alpha(u_\alpha - \hat{u}) = \alpha(u_0 - \hat{u}), \tag{3.2.1}$$

or

$$(M_\alpha + \alpha I)(u_\alpha - \hat{u}) = \alpha(u_0 - \hat{u}), \tag{3.2.2}$$

where

$$M_\alpha = \int_0^1 \mathscr{H}'(\hat{u} + t(u_\alpha - \hat{u}))\,dt.$$

Again (3.2.2) can be written as

$$(A_0 + \alpha I)(u_\alpha - \hat{u}) = (A_0 - M_\alpha)(u_\alpha - \hat{u}) + \alpha(u_0 - \hat{u}),$$

where $A_0 = \mathscr{H}'(u_0)$. Thus, we have

$$\begin{aligned}
u_\alpha - \hat{u} &= -(A_0+\alpha I)^{-1}(M_\alpha - A_0)(u_\alpha - \hat{u}) + \alpha(A_0+\alpha I)^{-1}(u_0-\hat{u}) \\
&= -(A_0+\alpha I)^{-1}\left[\int_0^1 [\mathscr{H}'(\hat{u}+t(u_\alpha-\hat{u})) - \mathscr{H}'(u_0)]\right](u_\alpha-\hat{u})\,dt \\
&\quad +\alpha(A_0+\alpha I)^{-1}(u_0-\hat{u}) \\
&= -(A_0+\alpha I)^{-1}A_0\int_0^1 \Phi(\hat{u}+t(u_\alpha-\hat{u}), u_0, u_\alpha-\hat{u})\,dt \\
&\quad +\alpha(A_0+\alpha I)^{-1}(u_0-\hat{u})
\end{aligned}$$

and hence

$$\begin{aligned}
\|u_\alpha-\hat{u}\| &\le k_0\left[\frac{\|u_\alpha-\hat{u}\|}{2} + \|\hat{u}-u_0\|\right]\|u_\alpha-\hat{u}\| \\
&\quad +\|\alpha(A_0+\alpha I)^{-1}(u_0-\hat{u})\| \\
&\le \frac{3}{2}k_0r_0\|u_\alpha-\hat{u}\| + \alpha\|(A+\alpha I)^{-1}(u_0-\hat{u})\| \text{ by (3.1.7)} \\
&\quad +\alpha\|[(A_0+\alpha I)^{-1}-(A+\alpha I)^{-1}](u_0-\hat{u})\| \\
&\le \frac{3}{2}k_0r_0\|u_\alpha-\hat{u}\| + \alpha\|(A+\alpha I)^{-1}(u_0-\hat{u})\| \\
&\quad +\|(A_0+\alpha I)^{-1}(A-A_0)\alpha(A+\alpha I)^{-1}(u_0-\hat{u})\| \\
&\le \frac{3}{2}k_0r_0\|u_\alpha-\hat{u}\| + \alpha\|(A+\alpha I)^{-1}(u_0-\hat{u})\| \\
&\quad +\left\|A_0(A_0+\alpha I)^{-1}\int_0^1 \Phi(\hat{u}+t(u_0-\hat{u}), u_0, \alpha(A+\alpha I)^{-1}(u_0-\hat{u}))dt\right\| \\
&\le \frac{3}{2}k_0r_0\|u_\alpha-\hat{u}\| + \alpha\|(A+\alpha I)^{-1}(u_0-\hat{u})\| \\
&\quad +\frac{k_0r_0}{2}\|\alpha(A+\alpha I)^{-1}(u_0-\hat{u})\|
\end{aligned}$$

i.e.,

$$\begin{aligned}
\left(1-\frac{3}{2}k_0r_0\right)\|u_\alpha-\hat{u}\| &\le \left(1+\frac{k_0r_0}{2}\right)\|\alpha(A+\alpha I)^{-1}(u_0-\hat{u})\| \\
\frac{2-3k_0r_0}{2+k_0r_0}\|\hat{u}-u_\alpha\| &\le \|\alpha(A+\alpha I)^{-1}A^\nu z\| \text{ by (3.1.10)} \qquad (3.2.3) \\
&\le \sup_{\lambda\in\sigma(A)}\left|\frac{\alpha\lambda^\nu}{\lambda+\alpha}\right|\|z\| \\
&\le \alpha^\nu\|z\|.
\end{aligned}$$

□

Theorem 3.2.2. *Suppose Assumption 3.1.3 and (3.1.10) hold. Then,*

$$\|u_\alpha^\delta - \hat{u}\| \leq \max\left\{1, \frac{2+k_0r_0}{3-2k_0r_0}\|z\|\right\}\left(\frac{\delta}{\alpha}+\alpha^\nu\right).$$

In particular, if $\alpha = \delta^{\frac{1}{\nu+1}}$, *then*

$$\|u_\alpha^\delta - \hat{u}\| = O\left(\delta^{\frac{\nu}{\nu+1}}\right).$$

Proof. Follows from (3.1.6) and Theorem 3.2.1. □

Remark 3.2.3. *Note that the best value for* $\frac{\delta}{\alpha}+\alpha^\nu$ *is attained when* $\frac{\delta}{\alpha} = \alpha^\nu$, *i.e.,* $\alpha = \delta^{\frac{1}{\nu+1}}$, *and in this case the optimal order is* $O\left(\delta^{\frac{\nu}{\nu+1}}\right)$. *However, the above choice of* α *is depending on the unknown* ν. *In view of this, our aim is to choose* α *(not depending on* ν*), so that we obtain* $\|u_\alpha^\delta - \hat{u}\| = O\left(\delta^{\frac{\nu}{\nu+1}}\right)$.

3.2.1 A NEW PARAMETER CHOICE STRATEGY

For $u \in \mathscr{U}$, define

$$\phi(\alpha,u) := \|\alpha^2(A_0+\alpha I)^{-2}(\mathscr{H}(u_0)-u)\|, \tag{3.2.4}$$

where $A_0 = \mathscr{H}'(u_0)$.

Theorem 3.2.4. *For each* $u \in \mathscr{U}$, *and* $\alpha > 0$ *the function* $\alpha \to \phi(\alpha,u)$ *is continuous, monotonically increasing and*

$$\lim_{\alpha\to 0}\phi(\alpha,u) = 0 \quad \text{and} \lim_{\alpha\to\infty}\phi(\alpha,u) = \|\mathscr{H}(u_0)-u\|.$$

Proof. Note that

$$\phi(\alpha,u)^2 = \int_0^{\|A_0\|}\left(\frac{\alpha}{\lambda+\alpha}\right)^4 d\|E_\lambda(\mathscr{H}(u_0)-u)\|^2,$$

where E_λ is the spectral family of A_0. Note that for each $\lambda > 0$,

$$\alpha \longrightarrow \left(\frac{\alpha}{\lambda+\alpha}\right)^4$$

is strictly increasing and satisfies $\lim_{\alpha\longrightarrow 0}\left(\frac{\alpha}{\lambda+\alpha}\right)^4 = 0$ and $\lim_{\alpha\longrightarrow\infty}\left(\frac{\alpha}{\lambda+\alpha}\right)^4 = 1$. Hence, by Dominated Convergence Theorem $\phi(\alpha,u)$ is strictly increasing, continuous,

$\lim_{\alpha\to 0}\phi(\alpha,u)$
$= 0$ and $\lim_{\alpha\to\infty}\phi(\alpha,u) = \|\mathscr{H}(u_0)-u\|$. □

In addition to (3.1.2), we assume that

$$c\delta \le \|\mathscr{H}(u_0)-y^\delta\|, \tag{3.2.5}$$

for some $c > 1$. The following theorem is a consequence of the intermediate value theorem.

Theorem 3.2.5. *Let y^δ satisfies (3.1.2) and (3.2.5). Then,*

$$\phi(\alpha,y^\delta) = c\delta \tag{3.2.6}$$

has a unique solution α.

Next, we shall show that if $\alpha = \alpha(\delta,u_0)$ satisfies (3.1.10) and (3.2.6) hold, then $\|\hat{u}-u_\alpha\|$
$= O(\delta^{\frac{\nu}{\nu+1}})$. Our proof is based on the following moment inequality for positive type operator B (see Krasnoselskii et al. (1966), p. 290)

$$\|B^u x\| \le \|B^v x\|^{\frac{u}{v}}\,\|x\|^{1-\frac{u}{v}}, \quad 0\le u\le v. \tag{3.2.7}$$

Theorem 3.2.6. *Let $\frac{3}{2}k_0r_0 < 1$, Assumption 3.1.3 and (3.1.10) be satisfied. Let $\alpha = \alpha(\delta,u_0)$ be the solution of (3.2.6). Then,*

$$\|\hat{u}-u_\alpha\| \le O(\delta^{\frac{\nu}{\nu+1}}).$$

Proof. By taking $B = \alpha(A+\alpha I)^{-1}A$ and $x = \alpha^{1-\nu}(A+\alpha I)^{-(1-\nu)}z$ in (3.2.3) and then using (3.2.7) with $u = \nu$, $v = 1+\nu$, we have

$$\begin{aligned}
\frac{2-3k_0r_0}{2+k_0r_0}\|\hat{u}-u_\alpha\| &\le \|B^\nu x\| \\
&\le \|B^{1+\nu}x\|^{\frac{\nu}{1+\nu}}\,\|x\|^{\frac{1}{1+\nu}} \\
&= \|\alpha^2(A+\alpha I)^{-2}A^{1+\nu}z\|^{\frac{\nu}{1+\nu}}\,\|z\|^{\frac{1}{1+\nu}} \\
&= \|\alpha^2(A+\alpha I)^{-2}A(u_0-\hat{u})\|^{\frac{\nu}{1+\nu}}\,\|z\|^{\frac{1}{1+\nu}} \\
&= \|\alpha^2(A+\alpha I)^{-2}(\mathscr{H}(u_0)-y)\|^{\frac{\nu}{1+\nu}}\,\|z\|^{\frac{1}{1+\nu}} \\
&\le \Big(\|\alpha^2(A+\alpha I)^{-2}(\mathscr{H}(u_0)-y^\delta)\| \\
&\quad + \|\alpha^2(A+\alpha I)^{-2}(y^\delta-y)\|\Big)^{\frac{\nu}{1+\nu}}\,\|z\|^{\frac{1}{1+\nu}}
\end{aligned} \tag{3.2.8}$$

$$= \quad (\mathscr{B}_1+\delta)^{\frac{\nu}{1+\nu}}\|z\|^{\frac{1}{1+\nu}},$$

where $\mathscr{B}_1 = \|\alpha^2(A+\alpha I)^{-2}(\mathscr{H}(u_0)-y^\delta)\|$ and we used the inequality,

$$\|\alpha^2(A+\alpha I)^{-2}(y^\delta - y)\| \le \delta.$$

We have,

$$\begin{aligned}
\mathscr{B}_1 &= \|\alpha^2(A+\alpha I)^{-2}(\mathscr{H}(u_0)-y^\delta)\| \\
&= \|\alpha^2[(A+\alpha I)^{-2}-(A_0+\alpha I)^{-2}](\mathscr{H}(u_0)-y^\delta) \\
&\quad +\alpha^2(A_0+\alpha I)^{-2}(\mathscr{H}(u_0)-y^\delta)\| \\
&\le \|\alpha^2[(A+\alpha I)^{-2}-(A_0+\alpha I)^{-2}](\mathscr{H}(u_0)-y^\delta)\| \\
&\quad +\|\alpha^2(A_0+\alpha I)^{-2}(\mathscr{H}(u_0)-y^\delta)\| \\
&=: \mathscr{D}_1+\phi(\alpha,y^\delta), \qquad (3.2.9)
\end{aligned}$$

where $\mathscr{D}_1 = \|\alpha^2[(A+\alpha I)^{-2}-(A_0+\alpha I)^{-2}](\mathscr{H}(u_0)-y^\delta)\|$. Let $w = \alpha^2(A_0+\alpha I)^{-2}(\mathscr{H}(u_0)$ $-y^\delta)$. Note that,

$$\begin{aligned}
\mathscr{D}_1 &= \|\alpha^2[(A+\alpha I)^{-2}-(A_0+\alpha I)^{-2}](\mathscr{H}(u_0)-y^\delta)\| \\
&= \|(A+\alpha I)^{-2}[A_0^2-A^2+2\alpha(A_0-A)]w\| \\
&= \|(A+\alpha I)^{-2}[(A+A_0)+2\alpha I](A_0-A)w\| \\
&= \|(A+\alpha I)^{-2}[A_0-A+2A+2\alpha I](A_0-A)w\| \\
&= \|[(A+\alpha I)^{-1}(A_0-A)]^2w+2(A+\alpha I)^{-1}(A_0-A)w\| \\
&\le (\|\Gamma\|^2+2\|\Gamma\|)\|w\| = (\|\Gamma\|^2+2\|\Gamma\|)\phi(\alpha,y^\delta), \qquad (3.2.10)
\end{aligned}$$

where $\Gamma = (A+\alpha I)^{-1}(A_0-A)$. By Assumption 3.1.3, we obtain

$$\begin{aligned}
\|\Gamma x\| &\le \|[(A+\alpha I)^{-1}-(A_0+\alpha I)^{-1}](A_0-A)x\| \\
&\quad +\|(A_0+\alpha I)^{-1}(A_0-A)x\| \\
&= \|(A_0+\alpha I)^{-1}[A_0-A](A+\alpha I)^{-1}(A_0-A)x\| \\
&\quad +\|(A_0+\alpha I)^{-1}(A_0-A)x\| \\
&\le \Big\|(A_0+\alpha I)^{-1}A_0 \\
&\quad \times\int_0^1 \Phi(\hat{u}+t(u_0-\hat{u}),u_0,(A+\alpha I)^{-1}(A_0-A)x)dt\Big\| \\
&\quad +\Big\|(A_0+\alpha I)^{-1}A_0\int_0^1 \Phi((\hat{u}+t(u_0-\hat{u}),u_0,x)dt\Big\|
\end{aligned}$$

$$\leq \frac{k_0 r_0}{2}\|\Gamma x\| + \frac{k_0 r_0}{2}\|x\|,$$

i.e.,

$$\left(1 - \frac{k_0 r_0}{2}\right)\|\Gamma x\| \leq k_0 r_0 \|x\|, \tag{3.2.11}$$

and hence

$$\mathscr{B}_1 \leq \left[\frac{2k_0 r_0}{2 - k_0 r_0}\left(\frac{2k_0 r_0}{2 - k_0 r_0} + 2\right) + 1\right]\phi(\alpha, y^\delta) = O(\delta). \tag{3.2.12}$$

The result now follows from (3.2.9)–(3.2.12). □

Theorem 3.2.7. *Suppose Assumption 3.1.3 and (3.1.10) hold and if* $\alpha = \alpha(\delta, u_0)$ *is chosen as a solution of (3.2.6). Then,*

$$\frac{\delta}{\alpha} = O\left(\delta^{\frac{\nu}{\nu+1}}\right).$$

Proof. By (3.2.6), we have

$$\begin{aligned} c\delta &= \|\alpha^2 (A_0 + \alpha I)^{-2}(\mathscr{H}(u_0) - y^\delta)\| \\ &\leq \|\alpha^2 (A_0 + \alpha I)^{-2}(\mathscr{H}(u_0) - y)\| \\ &\quad + \|\alpha^2 (A_0 + \alpha I)^{-2}(y - y^\delta))\| \\ &\leq \|\alpha^2 (A_0 + \alpha I)^{-2}(\mathscr{H}(u_0) - y)\| + \delta, \end{aligned}$$

so

$$\begin{aligned} (c-1)\delta &\leq \|\alpha^2 \left[(A_0 + \alpha I)^{-2} - (A + \alpha I)^{-2}\right](\mathscr{H}(u_0) - y)\| \\ &\quad + \|\alpha^2 (A + \alpha I)^{-2}(\mathscr{H}(u_0) - y)\| \\ &= \|(A_0 + \alpha I)^{-2}\left[(A + \alpha I)^2 - (A_0 + \alpha I)^2\right] \\ &\quad \times \alpha^2 (A + \alpha I)^{-2}(\mathscr{H}(u_0) - y)\| \\ &\quad + \|\alpha^2 (A + \alpha I)^{-2}(\mathscr{H}(u_0) - y)\|. \end{aligned} \tag{3.2.13}$$

Let $w_1 = \alpha^2 (A + \alpha I)^{-2}(\mathscr{H}(u_0) - y)$. Then, similar to (3.2.10), we have

$$(c_1 - 1)\delta \leq (\|\Gamma_1\|^2 + 2\|\Gamma_1\| + 1)\|w_1\|, \tag{3.2.14}$$

where $\Gamma_1 = (A_0 + \alpha I)^{-1}(A - A_0)$. Note that,

$$\|\Gamma_1 x\| = \|(A_0 + \alpha I)^{-1}(A - A_0)x\|$$

$$
\begin{aligned}
&= \left\|(A_0+\alpha I)^{-1}A_0\int_0^1 \Phi(\hat{u}+t(u_0-\hat{u}),u_0,x)dt\right\| \\
&\leq \frac{k_0}{2}\|u_0-\hat{u}\|\|x\| \\
&\leq \frac{k_0 r_0}{2}\|x\|,
\end{aligned}
$$

so

$$\|\Gamma_1\| \leq \frac{k_0 r_0}{2}. \tag{3.2.15}$$

Therefore, by (3.1.10), (3.2.14) and (3.2.15), we have

$$
\begin{aligned}
(c-1)\delta &\leq \left[\left(\frac{k_0 r_0}{2}\right)^2 + k_0 r_0 + 1\right] \|w_1\| \\
&= \left[\left(\frac{k_0 r_0}{2}\right)^2 + k_0 r_0 + 1\right] \|\alpha^2 (A+\alpha I)^{-2} A (u_0-\hat{u})\| \\
&= \left[\left(\frac{k_0 r_0}{2}\right)^2 + k_0 r_0 + 1\right] \|\alpha^2 (A+\alpha I)^{-2} A^{1+\nu} z\| \\
&\leq \left[\left(\frac{k_0 r_0}{2}\right)^2 + k_0 r_0 + 1\right] \|\alpha^2 (A+\alpha I)^{-1} A^{\nu} z\| \\
&\leq \left[\left(\frac{k_0 r_0}{2}\right)^2 + k_0 r_0 + 1\right] \alpha^{1+\nu} \|z\|,
\end{aligned}
$$

or

$$\alpha^{1+\nu} \geq \frac{c-1}{\left[(\frac{k_0 r_0}{2})^2 + k_0 r_0 + 1\right] \|z\|}\delta. \tag{3.2.16}$$

Thus,

$$\frac{\delta}{\alpha} = \delta^{\frac{\nu}{\nu+1}} \left(\frac{\delta}{\alpha^{\nu+1}}\right)^{\frac{1}{\nu+1}} = O(\delta^{\frac{\nu}{\nu+1}}).$$

□

Combining Theorem 3.2.6 and Theorem 3.2.7, we obtain:

Theorem 3.2.8. *Let Assumption 3.1.3 and (3.1.10) be satisfied and let $\alpha = \alpha(\delta, u_0)$ be the solution of (3.2.6). Then,*

$$\|u_\alpha^\delta - \hat{u}\| = O(\delta^{\frac{\nu}{\nu+1}}).$$

In George and Nair (2017), the following estimates was given (see George and Nair

(2017), Theorem 2.3)

$$\|u_\alpha^\delta - u_{n,\alpha}^\delta\| \leq kq_\alpha^n, \tag{3.2.17}$$

where $q_\alpha = 1-\beta\alpha$ and $k \geq r_0+1$ with $\beta = \frac{1}{\beta_0+\alpha}$, $\beta_0 \geq \|\mathscr{H}'(u)\|$, $\forall u \in \overline{B(u,2(r_0+1))}$. Suppose

$$n_{\alpha,\delta} := \min\{n \in \mathbb{N} : \alpha q_\alpha^n \leq \delta\}.$$

Theorem 3.2.9. *Let Assumption 3.1.3 and (3.1.10) be satisfied and let $\alpha = \alpha(\delta, u_0)$ be the solution of (3.2.6). Then,*

$$\|u_{n_{\alpha,\delta},\alpha}^\delta - \hat{u}\| = O(\delta^{\frac{\nu}{\nu+1}}).$$

Proof. Follows from the inequality

$$\|u_{n_{\alpha,\delta},\alpha}^\delta - \hat{u}\| \leq \|u_{n_{\alpha,\delta},\alpha}^\delta - u_\alpha^\delta\| + \|u_\alpha^\delta - \hat{u}\|,$$

Equation (3.2.17), Theorem 3.2.7, and Theorem 3.2.8. □

3.3 FINITE DIMENSIONAL REALIZATION OF (3.1.13)

Consider a family $\{P_h\}_{h>0}$ of orthogonal projections of $\mathscr{U}$ onto the range $R(P_h)$ of P_h. Let there exists $b_0 > 0$ such that

$$\|(I-P_h)\hat{u}\| := b_h \leq b_0,$$

and let

$$r \geq 2(2r_0 + \max\{\|\hat{u}\|, 1\} + b_h) \quad \text{with} \quad r_0 := \|\hat{u} - u_0\|.$$

We assume that;

(i) $\overline{B(P_h u_0, r)} \subseteq D(\mathscr{H})$,

(ii) there exists $\beta_0 > 0$ such that

$$\|P_h \mathscr{H}'(u) P_h\| \leq \beta_0 \; \forall \, u \in \overline{B(P_h u_0, r)}. \tag{3.3.1}$$

(iii) there exists $\varepsilon_0 > 0$ such that

$$\|\mathscr{H}'(u)(I-P_h)\| := \varepsilon_h(u) \leq \varepsilon_h \leq \varepsilon_0 \; \forall \, u \in \overline{B(P_h u_0, r)}. \tag{3.3.2}$$

Remark 3.3.1. *(a) Suppose $\mathscr{H}'(u)$ is self-adjoint for $u \in \overline{B(P_h u_0, r)}$. Then, $\|\mathscr{H}'(u)(I-P_h)\| = \|(I-P_h)\mathscr{H}'(u)\|$, and by Assumption 3.1.3, we have $\mathscr{H}'(u)v = \mathscr{H}'(P_h u_0)(v+\varphi(u,P_h u_0,v))$. Hence,*

$$\begin{aligned}\|\mathscr{H}'(u)(I-P_h)v\| &= \|(I-P_h)\mathscr{H}'(P_h u_0)(v+\varphi(u,P_h u_0,v))\| \\ &\le \|(I-P_h)\mathscr{H}'(P_h u_0)\|[\|v\|+k_0\|u-P_h u_0\|\|v\|] \\ &\le (1+k_0 r)\|(I-P_h)\mathscr{H}'(P_h u_0)\|\|v\|,\end{aligned}$$

so, $\|\mathscr{H}'(u)(I-P_h)\| \le (1+k_0 r)\|(I-P_h)\mathscr{H}'(P_h u_0)\|$.

Therefore, in this case, we can take, $\varepsilon_h = (1+k_0 r)\|(I-P_h)\mathscr{H}'(P_h u_0)\|$.

(b) Suppose, $\mathscr{H}'(u)$ is not self-adjoint for $u \in \overline{B(P_h u_0, r)}$. In this case, under the additional assumption (see Kaltenbacher (1997))

$$\mathscr{H}'(u) = R_u \mathscr{H}'(P_h u_0),\ u \in \overline{B(P_h u_0, r)}$$

with $\|I-R_u\| \le C_R\|u-P_h u_0\|$, we have

$$\begin{aligned}\|\mathscr{H}'(u)(I-P_h)\| &= \|R_u\mathscr{H}'(P_h u_0)(I-P_h)\| \\ &\le \|R_u\|\|\mathscr{H}'(P_h u_0)(I-P_h)\| \\ &\le (1+C_R r)\|\mathscr{H}'(P_h u_0)(I-P_h)\|.\end{aligned}$$

Therefore, in this case, we can take, $\varepsilon_h = (1+C_R r)\|\mathscr{H}'(P_h u_0)(I-P_h)\|$.

From now on, we assume $\delta \in (0,d]$ and $\alpha \in [\delta+\varepsilon_h, a)$ with $a > d+\varepsilon_0$. First we shall prove that

$$(P_h\mathscr{H}P_h)(u) + \alpha P_h(u-u_0) = P_h y^\delta \tag{3.3.3}$$

has a unique solution $u_\alpha^{h,\delta} \in R(P_h)$, under the assumption

$$R(P_h) \subseteq D(\mathscr{H}). \tag{3.3.4}$$

Proposition 3.3.2. *Suppose (3.3.4) holds. Then (3.3.3) has a unique solution $u_\alpha^{h,\delta}$ in $B(P_h u_0, r)$ for all $u_0 \in \mathscr{U}$ and $y^\delta \in \mathscr{U}$.*

Proof. Since $\mathscr{H}$ is monotone, we have

$$\langle (P_h\mathscr{H}P_h)(u) - (P_h\mathscr{H}P_h)(v), u-v\rangle = \langle \mathscr{H}(P_h(u)) - \mathscr{H}(P_h(v)), P_h(u)-P_h(v)\rangle \ge 0,$$

so $P_h\mathscr{H}P_h$ is monotone and $D(P_h\mathscr{H}P_h) = \mathscr{U}$. Hence by Minty–Browder Theorem (see Alber and Ryazantseva (2006); Deimling (1985)), Equation (3.3.3) has a unique solution $u_\alpha^{h,\delta}$ for all $u_0 \in \mathscr{U}$ and $y^\delta \in \mathscr{U}$. Next, we shall prove that $u_\alpha^{h,\delta} \in B(P_h u_0, r)$. Note that by (3.3.3), we have

$$P_h \mathscr{H}(P_h u_\alpha^{h,\delta}) + \alpha P_h(u_\alpha^{h,\delta} - \hat{u}) - P_h \mathscr{H}(\hat{u}) = P_h(y^\delta - y) + \alpha P_h(u_0 - \hat{u}). \tag{3.3.5}$$

Let $M = \int_0^1 \mathscr{H}'(\hat{u} + t(P_h u_\alpha^{h,\delta} - \hat{u}))dt$. Then by (3.3.5), we have

$$P_h M(P_h u_\alpha^{h,\delta} - \hat{u}) + \alpha P_h(u_\alpha^{h,\delta} - \hat{u}) = P_h(y^\delta - y) + \alpha P_h(u_0 - \hat{u}).$$

or

$$(P_h M P_h + \alpha I)(u_\alpha^{h,\delta} - P_h \hat{u}) = P_h(y^\delta - y) + \alpha P_h(u_0 - \hat{u}) + P_h M(I - P_h)\hat{u}.$$

So, we have

$$\begin{aligned}
\|u_\alpha^{h,\delta} - P_h \hat{u}\| &= \|(P_h M P_h + \alpha I)^{-1} \\
&\quad \times [\alpha P_h(u_0 - \hat{u}) + P_h(y^\delta - y) + (P_h M(I - P_h))(\hat{u})]\| \\
&\leq \|P_h(u_0 - \hat{u})\| + \frac{\|P_h(y^\delta - y)\|}{\alpha} + \frac{\|P_h M(I - P_h)\|\|\hat{u}\|}{\alpha} \\
&\leq r_0 + \frac{\delta}{\alpha} + \frac{\varepsilon_h \|\hat{u}\|}{\alpha}
\end{aligned}$$

and hence

$$\begin{aligned}
\|u_\alpha^{h,\delta} - P_h u_0\| &\leq \|u_\alpha^{h,\delta} - P_h \hat{u}\| + \|P_h(\hat{u} - u_0)\| \\
&\leq 2r_0 + \max\{\|\hat{u}\|, 1\} \frac{\delta + \varepsilon_h}{\alpha} \\
&\leq 2r_0 + \max\{\|\hat{u}\|, 1\} < r,
\end{aligned} \tag{3.3.6}$$

i.e., $u_\alpha^{h,\delta} \in B(P_h u_0, r)$. □

The method: The rest of this section, $\mathscr{H}'(u)$, $u \in \overline{B(P_h u_0, r)}$ is assumed to be positive self-adjoint operator. We consider the sequence $\{u_{n,\alpha}^{h,\delta}\}$ defined iteratively by

$$u_{n+1,\alpha}^{h,\delta} = u_{n,\alpha}^{h,\delta} - \beta P_h[F P_h(u_{n,\alpha}^{h,\delta}) + \alpha(u_{n,\alpha}^{h,\delta} - u_0) - y^\delta], \tag{3.3.7}$$

where

$$u_{0,\alpha}^{h,\delta} = P_h u_0 \quad \text{and} \quad \beta := \frac{1}{\beta_0 + a}.$$

Note that if $\lim_{n \longrightarrow \infty}\{u_{n,\alpha}^{h,\delta}\}$ exists, then the limit is the solution $u_\alpha^{h,\delta}$ of (3.3.3).

Theorem 3.3.3. *Let $\delta \in (0, d]$, $\alpha \in [\delta + \varepsilon_h, a)$, $u_\alpha^{h,\delta}$ and u_α^δ are solutions of (3.1.3) and*

(3.3.3), respectively. Then

$$\|u_\alpha^{h,\delta} - u_\alpha^\delta\| \le \|\hat{u}\|\frac{\varepsilon_h}{\alpha} + b_h + 2\|u_\alpha^\delta - \hat{u}\|.$$

Proof. Note that by (3.1.3), we have

$$P_h\mathscr{H}(u_\alpha^\delta) + \alpha P_h(u_\alpha^\delta - u_0) = P_h y^\delta. \tag{3.3.8}$$

Therefore, by (3.3.3) and (3.3.8), we have

$$P_h(\mathscr{H}(u_\alpha^{h,\delta}) - \mathscr{H}(u_\alpha^\delta)) + \alpha P_h(u_\alpha^{h,\delta} - u_\alpha^\delta) = 0. \tag{3.3.9}$$

Let $T_h := \int_0^1 \mathscr{H}'(u_\alpha^\delta + t(u_\alpha^{h,\delta} - u_\alpha^\delta))dt$. Then by (3.3.9), we have

$$P_hT_h(u_\alpha^{h,\delta} - u_\alpha^\delta) + \alpha P_h(u_\alpha^{h,\delta} - u_\alpha^\delta) = 0$$

or

$$P_hT_hP_h(u_\alpha^{h,\delta} - u_\alpha^\delta) + \alpha P_h(u_\alpha^{h,\delta} - u_\alpha^\delta) = P_hT_h(I - P_h)u_\alpha^\delta. \tag{3.3.10}$$

Notice that

$$\begin{aligned}
\|u_\alpha^\delta + t(u_\alpha^{h,\delta} - u_\alpha^\delta) - P_hu_0\| &= \|(1-t)(u_\alpha^\delta - \hat{u} + \hat{u} - P_hu_0) + t(u_\alpha^{h,\delta} - P_hu_0)\| \\
&= \|(1-t)[(u_\alpha^\delta - \hat{u}) + (I - P_h)\hat{u} + P_h(\hat{u} - u_0)] \\
&\quad + t(u_\alpha^{h,\delta} - P_hu_{0)}\| \\
&\le (1-t)[\|u_\alpha^\delta - \hat{u}\| + \|(I - P_h)\hat{u}\| + \|P_h(\hat{u} - u_0)\|] \\
&\quad + \|t(u_\alpha^{h,\delta} - P_hu_0)\| \\
&\le (1-t)[\|u_\alpha^\delta - \hat{u}\| + b_h + r_0] + t\|u_\alpha^{h,\delta} - P_hu_0\| \\
&\le (1-t)[\left(\frac{\delta}{\alpha} + 2r_0\right) + b_h] + t(2r_0 + \max\{1, \|\hat{u}\|\}) \\
&\quad \text{by (3.1.7) and (3.3.6)} \\
&\le r,
\end{aligned}$$

that is $u_\alpha^\delta + t(u_\alpha^{h,\delta} - u_\alpha^\delta) \in B(P_hu_0, r)$. So, $P_hT_hP_h$ is self-adjoint and hence by (3.3.10),

$$\begin{aligned}
\|u_\alpha^{h,\delta} - P_hu_\alpha^\delta\| &= \|(P_hT_hP_h + \alpha I)^{-1}P_hT_h(I - P_h)u_\alpha^\delta\| \\
&\le \frac{\|P_hT_h(I - P_h)u_\alpha^\delta\|}{\alpha} \\
&\le \frac{\varepsilon_h}{\alpha}\|u_\alpha^\delta\|
\end{aligned}$$

$$\leq \frac{\varepsilon_h}{\alpha}(\|\hat{u}\| + \|\hat{u} - u_\alpha^\delta\|) \tag{3.3.11}$$

and

$$\|(I-P_h)u_\alpha^\delta\| \leq \|(I-P_h)\hat{u}\| + \|u_\alpha^\delta - \hat{u}\|. \tag{3.3.12}$$

Since $\frac{\varepsilon_h}{\alpha} \leq 1$, by (3.3.11) and (3.3.12), we have

$$\begin{aligned} \|u_\alpha^{h,\delta} - u_\alpha^\delta\| &\leq \|u_\alpha^{h,\delta} - P_h u_\alpha^\delta\| + \|(I-P_h)u_\alpha^\delta\| \\ &\leq \|\hat{u}\|\frac{\varepsilon_h}{\alpha} + b_h + 2\|u_\alpha^\delta - \hat{u}\|. \end{aligned}$$

□

Remark 3.3.4. *If $\alpha b_h \leq \delta + \varepsilon_h$ and $\alpha = (\delta + \varepsilon_h)^{\frac{1}{\nu+1}}$, then by Theorems 3.2.2 and 3.3.3, we have*

$$\|u_\alpha^{h,\delta} - \hat{u}\| = O((\delta + \varepsilon_h)^{\frac{\nu}{\nu+1}}).$$

Theorem 3.3.5. *Let $\delta \in (0,d]$ and $\alpha \in [\delta + \varepsilon_h, a)$. Then, $\{u_{n,\alpha}^{h,\delta}\} \in \overline{B(P_h u_0, r)}$ and $\lim_{n\longrightarrow\infty} u_{n,\alpha}^{h,\delta}$ $= u_\alpha^{h,\delta}$. Further*

$$\|u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \leq \kappa q_{\alpha,h}^n,$$

where $q_{\alpha,h} := 1 - \beta\alpha$, $\kappa \geq 2r_0 + \max\{1, \|\hat{u}\|\}$ and $\beta := 1/(\beta_0 + a)$.

Proof. We shall show the following using induction;

(a) $u_{n,\alpha}^{h,\delta} \in \overline{B(P_h u_0, r)}$,

(b) the operator

$$A_n^h := \int_0^1 \mathscr{H}'(u_\alpha^{h,\delta} + t(u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta}))dt$$

is positive self-adjoint, well defined and

(c) $\|u_{n+1,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \leq (1-\beta\alpha)\|u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \; \forall n = 0,1,2,\ldots$

Clearly, $u_{0,\alpha}^{h,\delta} = P_h u_0 \in \overline{B(P_h u_0, r)}$. Furthermore, we have by Proposition 3.3.2, $u_\alpha^{h,\delta} \in \overline{B(P_h u_0, r)}$, so by (3.3.1), A_0^h is a well defined and positive self-adjoint operator with $\|P_h A_0^h P_h\| \leq \beta_0$. So (a) and (b) hold for $n = 0$. Note that

$$u_{1,\alpha}^{h,\delta} - u_\alpha^{h,\delta} = u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta} - \beta P_h[\mathscr{H}(u_{0,\alpha}^{h,\delta}) - \mathscr{H}(u_\alpha^{h,\delta}) + \alpha(u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta})].$$

Since,

$$\mathscr{H}(u_{0,\alpha}^{h,\delta}) - \mathscr{H}(u_\alpha^{h,\delta}) = \int_0^1 \mathscr{H}'(u_\alpha^{h,\delta} + t(u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta}))(u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta})dt = A_0^h(u_{0,\alpha}^{h,\delta} - u_\alpha^{h,\delta})$$

we have

$$u_{1,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta} = [I - \beta(P_h A_0^h P_h + \alpha I)](u_{0,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta})]. \tag{3.3.13}$$

Since $P_h A_0^h P_h$ is a positive self-adjoint operator (cf. Nair (2002)),

$$\begin{aligned} \|I - \beta(P_h A_0^h P_h + \alpha I)\| &= \sup_{\|u\|=1} \left| \left\langle [(1-\beta\alpha)I - \beta P_h A_0^h P_h]u, u \right\rangle \right| \\ &= \sup_{\|u\|=1} \left|(1-\beta\alpha) - \beta \left\langle P_h A_0^h P_h u, u \right\rangle \right| \end{aligned}$$

and since $\|P_h A_0^h P_h\| \leq \beta_0$ and $\beta = 1/(\beta_0 + a)$, we have

$$0 \leq \beta \left\langle P_h A_0^h P_h u, u \right\rangle \leq \beta \|P_h A_0^h P_h\| \leq \beta\beta_0 < 1 - \beta\alpha \; \forall \alpha \in (0, a).$$

Therefore,

$$\|I - \beta(P_h A_0^h P_h + \alpha I)\| \leq 1 - \beta\alpha.$$

Thus, by (3.3.13), we have

$$\|u_{1,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}\| \leq (1-\beta\alpha)\|u_{0,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}\| \leq q_{\alpha,h}\|P_h u_0 - u_{\alpha}^{h,\delta}\|.$$

Therefore, we have

$$\|u_{1,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}\| \leq q_{\alpha,h}(2r_0 + \max\{1, \|\hat{u}\|\}), \text{ by (3.3.6)} = \kappa q_{\alpha,h}.$$

and

$$\begin{aligned} \|u_{1,\alpha}^{h,\delta} - P_h u_0\| &\leq \|u_{1,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}\| + \|u_{\alpha}^{h,\delta} - P_h u_0\| \\ &\leq 2\|P_h u_0 - u_{\alpha}^{h,\delta}\| \leq 2(2r_0 + \max\{1, \|\hat{u}\|\}) \leq r. \end{aligned}$$

Thus, $u_{1,\alpha}^{h,\delta} \in \overline{B(P_h u_0, r)}$. So, for $n = 0$, (a)–(c) hold. The induction for (a)–(c) is completed, if we simply replace $u_{1,\alpha}^{h,\delta}, u_{0,\alpha}^{h,\delta}$ in the preceding arguments with $u_{n+1,\alpha}^{h,\delta}, u_{n,\alpha}^{h,\delta}$, respectively. The result now follows from (c). □

Theorem 3.3.6. *Let $\delta \in (0, d]$, $\alpha \in (\delta + \varepsilon_h, a]$ with $d + \varepsilon_0 < a$. Let u_{α}^{δ} and u_{α} be solutions of (3.1.3) and (3.1.4), respectively. For $\delta \in (0, d]$ and $\alpha \in [\delta + \varepsilon_h, a)$, let $\{u_{n,\alpha}^{h,\delta}\}$ be as in (3.3.7). Let*

$$n_{\alpha,\delta} := \min\{m \in \mathfrak{N} : \alpha q_{\alpha,h}^m \leq \delta + \varepsilon_h\} \tag{3.3.14}$$

and

$$\alpha b_h \leq \delta + \varepsilon_h.$$

Then,

$$\|u^{h,\delta}_{n_{\alpha,\delta},\alpha} - \hat{u}\| = (\kappa + 1 + \max\{\|\hat{u}\|, 3\})\Big(\|\hat{u} - u_\alpha\| + \frac{\delta + \varepsilon_h}{\alpha}\Big). \tag{3.3.15}$$

Proof. By Theorem 3.3.3 and Theorem 3.3.5, we have

$$\begin{aligned}
\|u^{h,\delta}_{n_{\alpha,\delta},\alpha} - \hat{u}\| &\leq \|u^{h,\delta}_{n_{\alpha,\delta},\alpha} - u^{h,\delta}_\alpha\| + \|u^{h,\delta}_\alpha - u^\delta_\alpha\| + \|u^\delta_\alpha - \hat{u}\| \\
&\leq q^n_{\alpha,h} + \frac{\varepsilon_h}{\alpha}\|\hat{u}\| + b_h + 3\|u^\delta_\alpha - \hat{u}\| \qquad (3.3.16) \\
&\leq q^n_{\alpha,h} + \frac{\varepsilon_h}{\alpha}\|\hat{u}\| + b_h + 3(\frac{\delta}{\alpha} + \|u_\alpha - \hat{u}\|) \\
&\leq (\kappa + 1 + \max\{3, \|\hat{u}\|\})\Big(\|\hat{u} - u_\alpha\| + \frac{\delta + \varepsilon_h}{\alpha}\Big). \qquad (3.3.17)
\end{aligned}$$

Here, we used the fact that $q^n_{\alpha,h} \leq \frac{\delta+\varepsilon_h}{\alpha}$ for $n = n_{\alpha,\delta}$ and $b_h \leq \frac{\delta+\varepsilon_h}{\alpha}$. Thus, we obtain the required estimate in the theorem. □

Finite dimensional realization of (3.2.6) is considered next.

3.4 FINITE DIMENSIONAL REALIZATION OF THE NEW PARAMETER CHOICE STRATEGY (3.2.6)

For $u \in \mathscr{U}$, define

$$\phi^h(\alpha, u) := \|\alpha^2 (P_h A_0 P_h + \alpha I)^{-2} P_h(\mathscr{H}(u_0) - u)\|. \tag{3.4.1}$$

The proof of the next theorem is similar to that of Theorem 3.2.4, so the proof is omitted.

Theorem 3.4.1. *For each $u \in \mathscr{U}$, the function $\alpha \to \phi^h(\alpha, u)$ for $\alpha > 0$, defined in (3.4.1), is continuous, monotonically increasing and*

$$\lim_{\alpha\to 0} \phi^h(\alpha, u) = 0, \quad \lim_{\alpha\to\infty} \phi^h(\alpha, u) = \|P_h(\mathscr{H}(u_0) - u)\|.$$

In addition to (3.1.2), we assume that

$$c_1\delta + d_1\varepsilon_h \leq \|P_h(\mathscr{H}(u_0) - y^\delta)\|, \tag{3.4.2}$$

for some $c_1 > 1$ and $d_1 > \frac{k_0 r_0^2}{2} + r_0$. The proof of the following theorem follows from the intermediate value theorem.

Theorem 3.4.2. *If y^δ satisfies (3.1.2) and (3.4.2). Then,*

$$\phi^h(\alpha, y^\delta) = c_1\delta + d_1\varepsilon_h \tag{3.4.3}$$

has a unique solution $\alpha = \alpha(\delta, h, u_0)$.

Next, we shall show that if $\alpha = \alpha(\delta, h, u_0)$ satisfies (3.4.3), then $\|\hat{u} - u_\alpha\| = O((\delta + \varepsilon_h)^{\frac{\nu}{\nu+1}})$. Our proof is based on the moment inequality (3.2.7).

Theorem 3.4.3. *Let Assumption 3.1.3 and (3.1.10) be satisfied and let $\alpha = \alpha(\delta, h, u_0)$ satisfies (3.4.3). Then,*

$$\|\hat{u} - u_\alpha\| \leq O((\delta + \varepsilon_h)^{\frac{\nu}{\nu+1}}).$$

Proof. By (3.2.10), the result follows once we prove $\|w\| = O(\delta + \varepsilon_h)$. This can be seen as follows,

$$\begin{aligned}
\|w\| &= \|\alpha^2(A_0+\alpha I)^{-2}(\mathscr{H}(u_0) - y^\delta)\| \\
&\leq \|\alpha^2[(A_0+\alpha I)^{-2} - (P_hA_0P_h + \alpha P_h)^{-2}](\mathscr{H}(u_0) - y^\delta)\| \\
&\quad + \|\alpha^2(P_hA_0P_h + \alpha I)^{-2}P_h(\mathscr{H}(u_0) - y^\delta)\| \\
&\leq \|\alpha^2(A_0+\alpha I)^{-2}[(P_hA_0P_h + \alpha P_h)^2 - (A_0+\alpha I)^2] \\
&\quad \times (P_hA_0P_h + \alpha P_h)^{-2}(\mathscr{H}(u_0) - y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\alpha^2(A_0+\alpha I)^{-2}[(P_hA_0P_h)^2 + 2\alpha P_hA_0P_h - (A_0^2 + 2\alpha A_0) + \alpha^2(P_h - I)] \\
&\quad \times (P_hA_0P_h + \alpha P_h)^{-2}(\mathscr{H}(u_0) - y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\alpha^2(A_0+\alpha I)^{-2}[(P_hA_0P_h + A_0)(P_hA_0P_h - A_0) + 2\alpha(P_hA_0P_h - A_0)] \\
&\quad \times (P_hA_0P_h + \alpha P_h)^{-2}(\mathscr{H}(u_0) - y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\alpha^2(A_0+\alpha I)^{-2}[(P_hA_0P_h - A_0) + 2(A_0+\alpha I)] \\
&\quad \times (P_hA_0P_h - A_0)(P_hA_0P_h + \alpha P_h)^{-2}(\mathscr{H}(u_0) - y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\,[[(A_0+\alpha I)^{-1}(P_hA_0P_h - A_0)]^2 + 2(A_0+\alpha I)^{-1}(P_hA_0P_h - A_0)] \\
&\quad \times \alpha^2(P_hA_0P_h + \alpha P_h)^{-2}(\mathscr{H}(u_0) - y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\,[[(A_0+\alpha I)^{-1}(P_hA_0P_h - A_0P_h + A_0P_h - A_0)]^2 \\
&\quad + 2(A_0+\alpha I)^{-1}(P_hA_0P_h - A_0P_h + A_0P_h - A_0)] \\
&\quad \times \alpha^2(P_hA_0P_h + \alpha P_h)^{-2}(\mathscr{H}(u_0) - y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&= \|\,[[(A_0+\alpha I)^{-1}(P_hA_0P_h - A_0P_h)]^2 + 2(A_0+\alpha I)^{-1}(P_hA_0P_h - A_0P_h)] \\
&\quad \times \alpha^2(P_hA_0P_h + \alpha P_h)^{-2}(\mathscr{H}(u_0) - y^\delta)\| + c_1\delta + d_1\varepsilon_h
\end{aligned}$$

$$
\begin{aligned}
&\leq \| \left[[(A_0+\alpha I)^{-1}(P_h-I)A_0P_h]^2 + 2(A_0+\alpha I)^{-1}((P_h-I)A_0P_h) \right] \\
&\quad \times \alpha^2 (P_hA_0P_h+\alpha P_h)^{-2}(\mathscr{H}(u_0)-y^\delta)\| + c_1\delta + d_1\varepsilon_h \\
&\leq \left(\left[\frac{\varepsilon_h}{\alpha} + 2 \right] \frac{\varepsilon_h}{\alpha} + 1 \right)(c_1\delta + d_1\varepsilon_h),
\end{aligned} \tag{3.4.4}
$$

where, we used $(P_h-I)P_h = 0$. Next, we shall show that $\frac{\varepsilon_h}{\alpha}$ is bounded. Note that,

$$
\begin{aligned}
c_1\delta + d_1\varepsilon_h &= \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(\mathscr{H}(u_0)-y^\delta)\| \\
&\leq \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(y-y^\delta)\| \\
&\quad + \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(\mathscr{H}(u_0)-y)\| \\
&\leq \delta + \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_hA(u_0-\hat{u})\| \\
&\leq \delta + \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(A-A_0)(u_0-\hat{u})\| \\
&\quad + \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_hA_0(u_0-\hat{u})\| \\
&= \delta + \left\| \alpha^2(P_hA_0P_h+\alpha I)^{-2}P_hA_0 \int_0^1 \Phi(\hat{u}+t(u_0-\hat{u}),u_0,u_0-\hat{u}) \right\| \\
&\quad + \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_hA_0[P_h+I-P_h])(u_0-\hat{u})\| \\
&\leq \delta + \left\| \alpha^2(P_hA_0P_h+\alpha)^{-2}P_hA_0[P_h+I-P_h] \int_0^1 \Phi(\hat{u}+t(u_0-\hat{u}),u_0,u_0-\hat{u}) \right\| \\
&\quad + \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_hA_0[P_h+I-P_h])(u_0-\hat{u})\| \\
&\leq \delta + (\alpha + \|A_0(I-P_h)\|)\left\| \int_0^1 \Phi(\hat{u}+t(u_0-\hat{u}),u_0,u_0-\hat{u})dt \right\| \\
&\quad + (\alpha + \|A_0(I-P_h)\|)\|u_0-\hat{u}\| \\
&\leq \delta + (\alpha+\varepsilon_h)\frac{k_0}{2}\|u_0-\hat{u}\|^2 \\
&\quad + (\alpha+\varepsilon_h)\|u_0-\hat{u}\| \\
&\leq \delta + \left(\frac{k_0r_0^2}{2} + r_0 \right)\varepsilon_h + \left(\frac{k_0r_0^2}{2} + r_0 \right)\alpha
\end{aligned}
$$

so, we have

$$
\left(d_1 - \left(\frac{k_0r_0^2}{2} + r_0 \right) \right)\varepsilon_h < (c_1-1)\delta + \left(d_1 - \left(\frac{k_0r_0^2}{2} + r_0 \right) \right)\varepsilon_h \leq \left(\frac{k_0r_0^2}{2} + r_0 \right)\alpha
$$

and hence

$$
\frac{\varepsilon_h}{\alpha} \leq \frac{\frac{k_0r_0^2}{2} + r_0}{d_1 - \left(\frac{k_0r_0^2}{2} + r_0 \right)} := C_{r_0}. \tag{3.4.5}
$$

Now, the result follows from (3.4.4) and (3.4.5). □

Theorem 3.4.4. *Suppose Assumption 3.1.3 and (3.1.10) hold and if* $\alpha = \alpha(\delta, h, u_0)$ *is* *chosen as a solution of (3.4.3). Then,*

$$\frac{\delta+\varepsilon_h}{\alpha}=O\left((\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}}\right).$$

Proof. By (3.4.3), we have

$$\begin{aligned}
c_1\delta+d_1\varepsilon_h &= \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(\mathscr{H}(u_0)-y^\delta)\| \\
&\leq \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(\mathscr{H}(u_0)-y)\| \\
&\quad +\|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(y-y^\delta))\| \\
&\leq \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(\mathscr{H}(u_0)-y)\|+\delta,
\end{aligned}$$

so

$$\begin{aligned}
(c_1-1)\delta+d_1\varepsilon_h &\leq \|\alpha^2(P_hA_0P_h+\alpha I)^{-2}P_h(\mathscr{H}(u_0)-y)\| \\
&\leq \|\alpha^2\left[(P_hA_0P_h+\alpha P_h)^{-2}-(A+\alpha I)^{-2}\right](\mathscr{H}(u_0)-y)\| \\
&\quad +\|\alpha^2(A+\alpha I)^{-2}(\mathscr{H}(u_0)-y)\| \\
&= \|(P_hA_0P_h+\alpha P_h)^{-2}\left[P_h(A+\alpha I)^2-(P_hA_0P_h+\alpha I)^2\right] \\
&\quad \times\alpha^2(A+\alpha I)^{-2}(\mathscr{H}(u_0)-y)\| \\
&\quad +\|\alpha^2(A+\alpha I)^{-2}(\mathscr{H}(u_0)-y)\|. \qquad (3.4.6)
\end{aligned}$$

Let $w_1=\alpha^2(A+\alpha I)^{-2}(\mathscr{H}(u_0)-y)$. Then, similar to (3.2.10), we have

$$(c_1-1)\delta+d_1\varepsilon_h\leq(\|\Gamma_2\|^2+2\|\Gamma_2\|+1)\|w_1\|, \qquad (3.4.7)$$

where $\Gamma_2=(P_hA_0P_h+\alpha I)^{-1}(P_hA-P_hA_0P_h)$. Note that,

$$\begin{aligned}
\|\Gamma_2x\| &= \|(P_hA_0P_h+\alpha I)^{-1}[P_h(A-A_0)+P_hA_0(I-P_h)]x\| \\
&\leq \|(P_hA_0P_h+\alpha I)^{-1}P_h(A-A_0)x\|+\|(P_hA_0P_h+\alpha I)^{-1}P_hA_0(I-P_h)x\| \\
&= \left\|(P_hA_0P_h+\alpha I)^{-1}P_hA_0\int_0^1\Phi(\hat{u}+t(u_0-\hat{u}),u_0,x)dt\right\| \\
&\quad +\|(P_hA_0P_h+\alpha I)^{-1}P_hA_0(I-P_h)x\| \\
&= \left\|(P_hA_0P_h+\alpha I)^{-1}P_hA_0[P_h+I-P_h)]\int_0^1\Phi(\hat{u}+t(u_0-\hat{u}),u_0,x)dt\right\| \\
&\quad +\|(P_hA_0P_h+\alpha I)^{-1}P_hA_0(I-P_h)x\| \\
&\leq \left[\left(1+\frac{\varepsilon_h}{\alpha}\right)\frac{k_0}{2}\|u_0-\hat{u}\|+\frac{\varepsilon_h}{\alpha}\right]\|x\| \\
&\leq \left[(1+C_{r_0})\frac{k_0}{2}r_0+C_{r_0}\right]\|x\|,
\end{aligned}$$

so

$$\|\Gamma_2\| \leq \left[(1+C_{r_0})\frac{k_0}{2}r_0 + C_{r_0}\right] := C_{\Gamma_2}. \tag{3.4.8}$$

Therefore, by (3.1.10), (3.4.7) and (3.4.8), we have

$$\begin{aligned}(c_1-1)\delta + d_1\varepsilon_h &< \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right] \|w_1\| \\ &= \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right] \|\alpha^2 (A+\alpha I)^{-2} A (u_0 - \hat{u})\| \\ &= \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right] \|\alpha^2 (A+\alpha I)^{-2} A^{1+\nu} z\| \\ &\leq \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right] \|\alpha^2 (A+\alpha I)^{-1} A^{\nu} z\| \\ &\leq \left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right] \alpha^{1+\nu} \|z\|,\end{aligned}$$

or

$$\alpha^{1+\nu} \geq \frac{\min\{c_1-1, d_1\}}{\left[C_{\Gamma_2}^2 + 2C_{\Gamma_2} + 1\right] \|z\|} (\delta + \varepsilon_h). \tag{3.4.9}$$

Thus

$$\frac{\delta+\varepsilon_h}{\alpha} = (\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}} \left(\frac{\delta+\varepsilon_h}{\alpha^{\nu+1}}\right)^{\frac{1}{\nu+1}} = O((\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}}).$$

□

By combining Theorem 3.3.6, Theorem 3.4.3 and Theorem 3.4.4, we have the following Theorem.

Theorem 3.4.5. *Suppose Assumption 3.1.3 and (3.1.10) hold and if $\alpha = \alpha(\delta, h, u_0)$ is chosen as a solution of (3.4.3). Then*

$$\|u_{n_{\alpha,\delta},\alpha}^{h,\delta} - \hat{u}\| = O\left((\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}}\right).$$

Remark 3.4.6. *Note that in the proposed method a system of equation is solved to obtain the parameter α and used it for computing $u_{n_{\alpha,\delta},\alpha}^{h,\delta}$. Whereas in the classical discrepancy principle one has to compute α and $u_{n_{\alpha,\delta},\alpha}^{h,\delta}$ in each iteration step. This is an advantage of our proposed approach.*

3.5 NUMERICAL EXAMPLES

The following steps are involved in the computation of $u_{n_{\alpha,\delta},\alpha}^{h,\delta}$.
Step I Compute $\alpha = \alpha(\delta, h, u_0) =: \alpha(\delta, \varepsilon_h)$ satisfying (3.4.3).
Step II Choose n such that $q_{\alpha,h}^n = (1-\beta\alpha(\delta,\varepsilon_h))^n \leq \frac{\delta+\varepsilon_h}{\alpha(\delta,\varepsilon_h)}$.
Step III Compute $u_{n_{\alpha,\delta},\alpha}^{h,\delta}$ using (3.3.7).

To compute $u_{n,\alpha}^{h,\delta}$, consider a sequence (V_m), of finite dimensional subspaces, where $V_m = span\{v_1, v_2, \ldots, v_{m+1}\}$ with $v_i, i = 1, 2, \ldots, m+1$ as the linear splines (in a uniform grid of $m+1$ points in $[0,1]$), so that dimension $V_m = m+1$. Since $u_{n,\alpha}^{h,\delta} \in V_m$, $u_{n,\alpha}^{h,\delta} = \sum_{i=1}^{m+1} \lambda_i^{(n)} v_i$, $\lambda_i, i = 1, 2, \ldots m+1$ are some scalars. Then, from (3.3.7), we have

$$\sum_{i=1}^{m+1} \lambda_i^{(n+1)} v_i = \sum_{i=1}^{m+1} \lambda_i^{(n)} v_i - \beta P_m[\mathscr{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i) + \alpha \sum_{i=1}^{m+1} (\lambda_i^{(n)} - \lambda_i^{(0)}) v_i - y^\delta], \quad (3.5.1)$$

where $P_m := P_{h_m}$ is the projection on to V_m with $h_m = \frac{1}{m}$. In this case one can prove as in Groetsch et al. (1982) that $\|\mathscr{H}'(u)(I - P_m)\| = O(\frac{1}{m^2})$. So we have taken $\varepsilon_{h_m} = \frac{1}{m^2}$ in our computation. Since $P_m \mathscr{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i) \in V_m$, $P_m y^\delta \in V_m$, we approximate

$$P_m \mathscr{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i) = \sum_{i=1}^{m+1} \mathscr{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i)(t_i) v_i, \; P_m y^\delta = \sum_{i=1}^{m+1} y^\delta(t_i) v_i,$$

where $t_i, i = 1, 2, \ldots, m+1$ are grid points. So $\lambda^{(n+1)} = (\lambda_1^{(n+1)}, \lambda_2^{(n+1)}, \ldots, \lambda_{m+1}^{(n+1)})^T$ satisfies (3.5.1), if $\lambda^{(n+1)}$ satisfies the equation

$$Q[\lambda^{(n+1)} - \lambda^{(n)}] = Q\beta[Y^\delta - (\mathscr{H}_h + \alpha(\lambda^{(n)} - \lambda^{(0)})],$$

where

$$Q = (\langle v_i, v_j \rangle)_{i,j},\ i, j = 1, 2, \ldots m+1,$$

$$Y^\delta = (y^\delta(t_1), y^\delta(t_2), \ldots y^\delta(t_{m+1}))^T,$$

and

$$\mathscr{H}_h = (\mathscr{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i)(t_1), \mathscr{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i)(t_2), \ldots, \mathscr{H}(\sum_{i=1}^{m+1} \lambda_i^{(n)} v_i)(t_{m+1}))^T.$$

To compute the α satisfying (3.4.3), we follow the following steps:

Let $z = (P_m A_0 P_m + \alpha I)^{-2} P_m(\mathscr{H}(u_0) - y^\delta)$, Then $z \in V_m$, so $z = \sum_{i=1}^{m+1} \xi_i v_i$ for some scalars $\xi_i, i = 1, 2, \ldots m+1$. Note that $(P_m A_0 P_m + \alpha I)^2 z = P_m(\mathscr{H}(u_0) - y^\delta)$ or $(P_m A_0 P_m + \alpha I) Z = P_m(\mathscr{H}(u_0) - y^\delta)$, where $Z = (P_m A_0 P_m + \alpha I) z$.

Since $Z \in V_m$, we have $Z = \sum_{i=1}^{m+1} \varsigma_i v_i$. Further $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_{m+1})^T$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_{m+1})^T$ satisfies the equations

$$(M + \alpha Q)\varsigma = QB,$$

and

$$(M + \alpha Q)\xi = Q\varsigma,$$

respectively, where

$$M = (\langle A_0 v_i, v_j\rangle)_{i,j}, i,j = 1,2,\dots m+1$$

and

$$B = ((\mathscr{H}(u_0) - y^\delta)(t_1), (\mathscr{H}(u_0) - y^\delta)(t_2), \dots, (\mathscr{H}(u_0) - y^\delta)(t_{m+1}))^T.$$

We compute α in (3.4.3), using Newton's method as follows. Let $f(\alpha) = \alpha^4\|z\|^2 - (c_1\delta + d_1\varepsilon_h)^2$. Then

$$f'(\alpha) = 4\alpha^3\|z\|^2 + 4\alpha^4\langle z, ZZ\rangle,$$

where $ZZ = (P_m A_0 P_m + \alpha I)^{-3} P_m(\mathscr{H}(u_0) - y^\delta)$. Let $ZZ = \sum_{i=1}^{m+1} \Theta_i v_i$.
The $\Theta = (\Theta_1, \Theta_2, \dots, \Theta_{m+1})^T$ satisfies the equation

$$(M + \alpha Q)\Theta = Q\xi.$$

So,

$$f(\alpha) = \alpha^4 \xi^T Q\xi - (c_1\delta + d_1\varepsilon_h)^2$$

and

$$f'(\alpha) = 4\alpha^3 \xi^T Q\xi + 4\alpha^4 \xi^T Q\Theta.$$

Then, using Newton's iteration we compute the $(k+1)^{th}$ iterate as; $\alpha_{k+1} = \alpha_k - \frac{f(\alpha)}{f'(\alpha)}$. In our computation, we stop the iterate when $\alpha_{k+1} - \alpha_k \leq 10^{-5}$. We consider a simple one dimensional example studied in (Tautenhahn (2002); Nair and Ravishankar (2008); Hofmann and Scherzer (1994); Groetsch (1993)) to illustrate our results in the previous sections. We also compare our computational results with that adaptive method considered in (George and Nair (2017); Pereverzev and Schock (2005)). Let us briefly explain the adaptive method considered in (George and Nair (2017)). Choose $\alpha_0 = \delta + \varepsilon_h$, $\alpha_j = \rho^j \alpha_0$. For each j find n_j such that $n_j = \min\{i : q^i_{\alpha,h} \leq \frac{1}{\rho^j}\}$. Then, find k such that

$$k := \max\{i : \|u^{h,\delta}_{n_i,\delta,\alpha_i} - u^{h,\delta}_{n_j,\delta,\alpha_j}\| \leq 4\frac{1}{\rho^j}, j = 0,1,\dots,i-1\}.$$

Choose $\alpha = \alpha_k$ as the regularization parameter.

Example 3.5.1. *Let $c > 0$ be a constant. Consider the inverse problem of identifying the distributed growth law $u(t), t \in (0,1)$, in the initial value problem*

$$\frac{dy}{dt} = u(t)y(t),\ y(0) = c,\ t \in (0,1) \tag{3.5.2}$$

from the noisy data $y^{\delta}(t) \in L^2(0,1)$. One can reformulate the above problem as an (ill-posed) operator equation $\mathscr{H}(u) = y$ with

$$[\mathscr{H}(u)](t) = ce^{\int_0^t u(\theta)d\theta},\ u \in L^2(0,1),\ t \in (0,1). \tag{3.5.3}$$

Then $\mathscr{H}'$ is given by

$$[\mathscr{H}'(u)h](t) = [\mathscr{H}(u)](t)\int_0^t h(\theta)d\theta. \tag{3.5.4}$$

It is proved in (Nair and Ravishankar (2008)), that $\mathscr{H}'$ is positive type (sectorial) and spectrum of $\mathscr{H}'(u)$ is the singleton set $\{0\}$. Further it is proved in (Tautenhahn (2002)) that $\mathscr{H}'$ satisfies Assumption 3.1.3 and that $\hat{u} - u_0 \in R(\mathscr{H}'(\hat{u}))$ provided $u^* := \hat{u} - u_0 \in H^1(0,1)$ and $u^*(0) = 0$. Now since $\hat{u} - u_0 = \mathscr{H}'(\hat{u})w$, we have

$$\begin{aligned}
[\hat{u} - u_0](t) &= [\mathscr{H}(\hat{u})](t)\int_0^t w(\theta)d\theta \\
&= ce^{\int_0^t \hat{u}(\theta)d\theta}\int_0^t w(\theta)d\theta \\
&= \frac{\int_0^1 ce^{\int_0^t [\hat{u}+\tau(u_0-\hat{u})](\theta)d\theta}d\tau \int_0^t w(\theta)d\theta}{\int_0^1 e^{\int_0^t [\tau(u_0-\hat{u})](\theta)d\theta}d\tau} \\
&= [A\bar{w}](t),
\end{aligned}$$

where $\bar{w} = \frac{w}{\int_0^1 e^{\int_0^t [\tau(u_0-\hat{u})](\theta)d\theta}d\tau}$. This shows the source condition (3.1.10) is satisfied. For our computation we have taken $\hat{u}(t) = t, u_0(t) = 0$ and $y(t) = e^{\frac{t^2}{2}}$. In Table 3.1, we present the relative error $E_\alpha = \frac{\|u^{h,\delta}_{n\alpha,\delta,\alpha} - \hat{u}\|}{\|u^{h,\delta}_{n\alpha,\delta,\alpha}\|}$, and α values using a new method (3.4.3) and adaptive method considered in (George and Nair (2017)) for different values of δ and n. Furthermore, we provide computational time (CT) for both the methods mentioned above. The relative error obtained for our a new method (3.4.3) is lesser than that the adaptive method in (George and Nair (2017)). As the relative error decreases the accuracy of reconstruction increases. The solutions obtained for different δ values ($\delta = 0.01, 0.001, 0.0001$) for $n = 500$ are shown in Figures 3.1–3.6, and for $n = 1000$ are shown in Figures 3.7–3.12, respectively. The exact and noisy data are shown in subfigure (a) of these figures and the computed solution is shown in subfigure(b) (C.S-A priori denotes the figure corresponding to the method (3.4.3)).The computed solution for the new method is closer to the actual solution.

3.6 CONCLUSION

We introduced a new source condition and a new parameter-choice strategy. The proposed a new parameter-choice strategy is independent of the unknown parameter ν and it provides the optimal order $O(\delta^{\frac{\nu}{\nu+1}})$, for $0 \leq \nu \leq 1$.

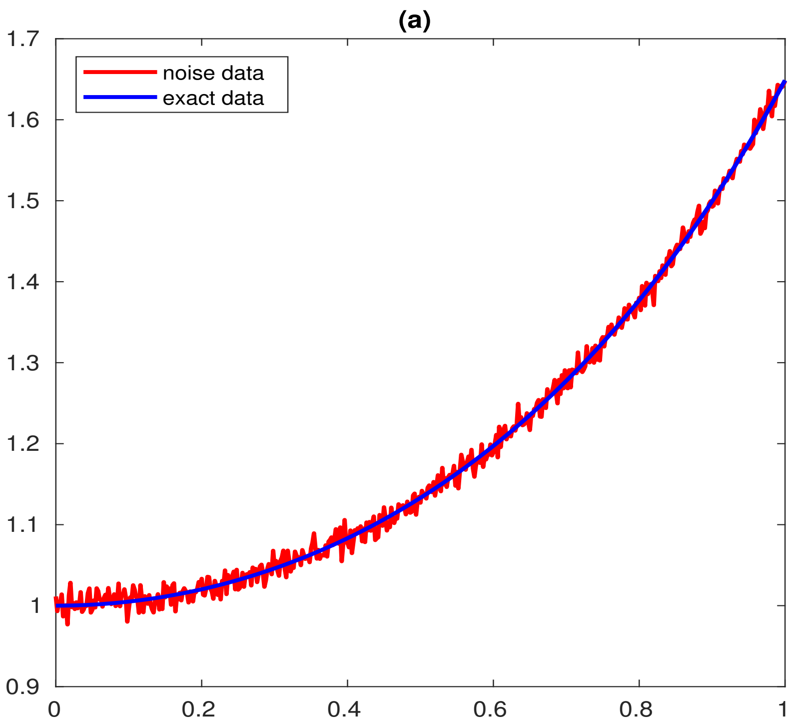

Figure 3.1 (**a**) Data with $\delta = 0.01$ and $n = 500$.

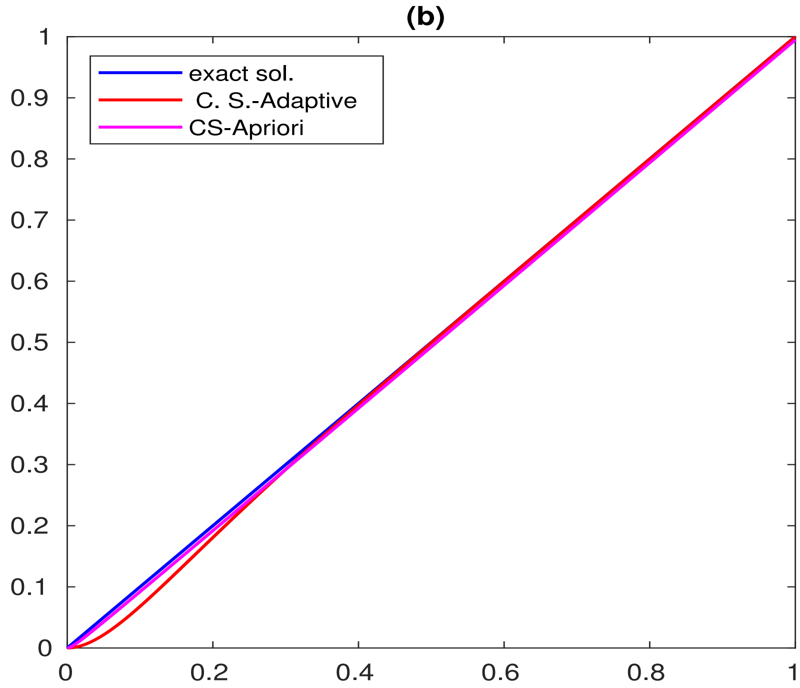

Figure 3.2 (**b**) Solution with $\delta = 0.01$ and $n = 500$.

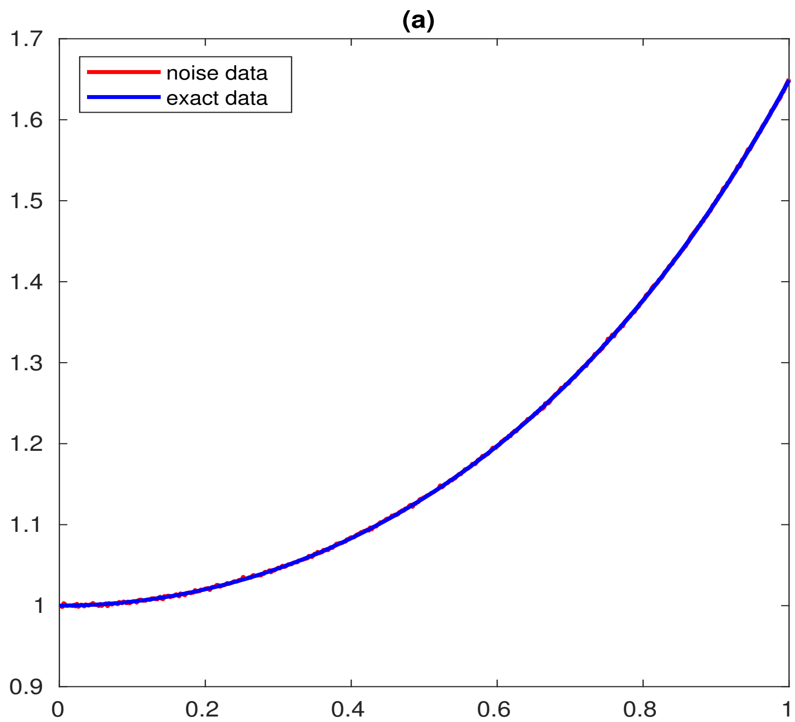

Figure 3.3 (**a**) Data with $\delta = 0.001$ and $n = 500$.

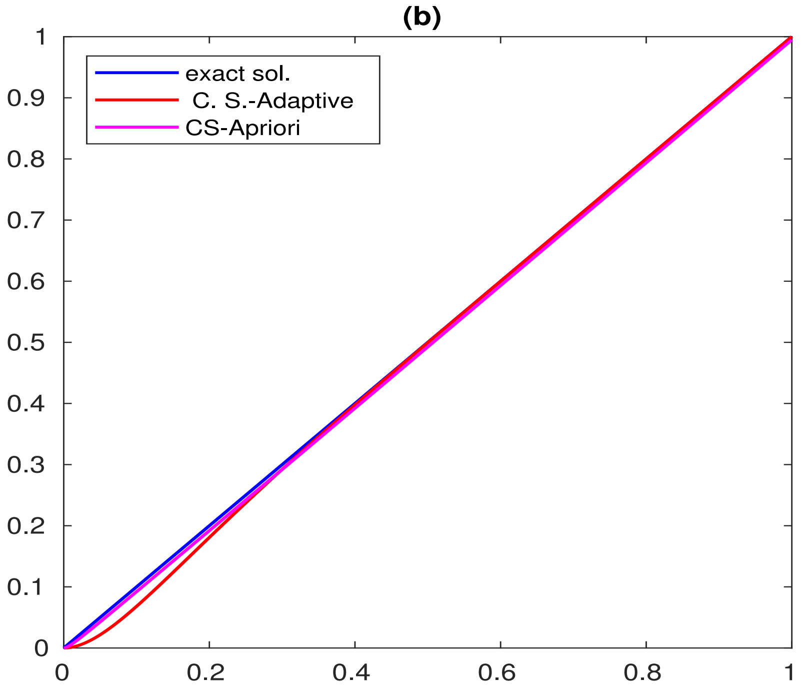

Figure 3.4 (**b**) Solution with $\delta = 0.001$ and $n = 500$.

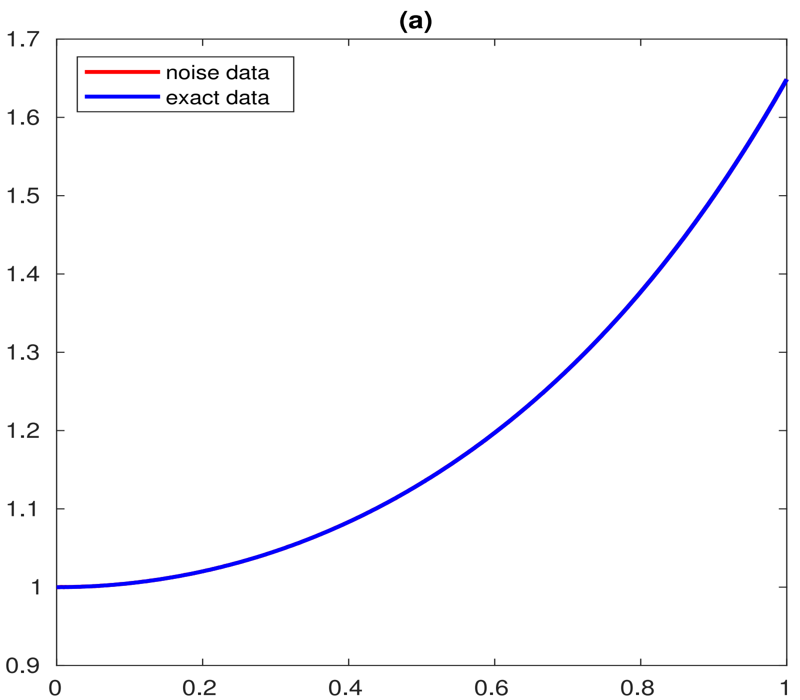

Figure 3.5 (**a**) Data with $\delta = 0.0001$ and $n = 500$.

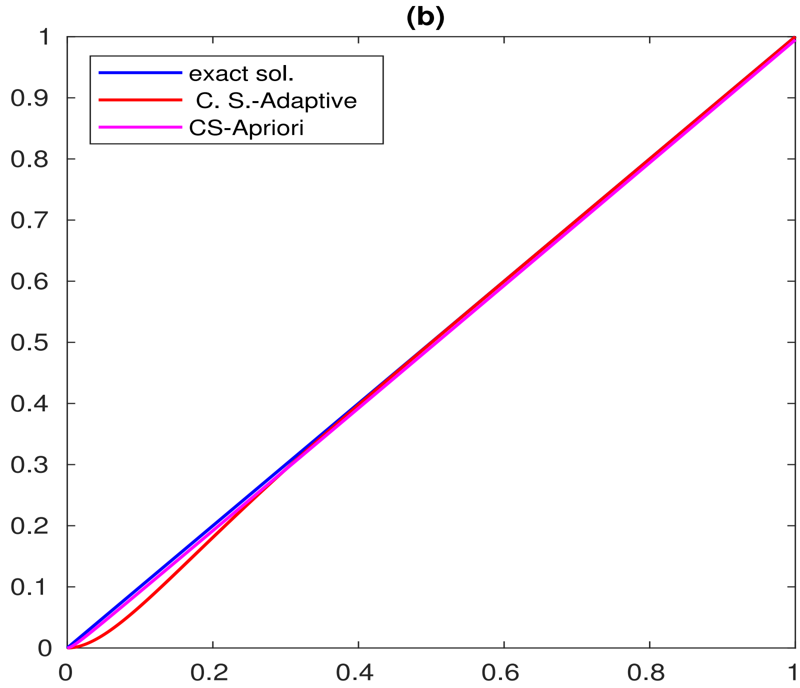

Figure 3.6 (**b**) Solution with $\delta = 0.0001$ and $n = 500$.

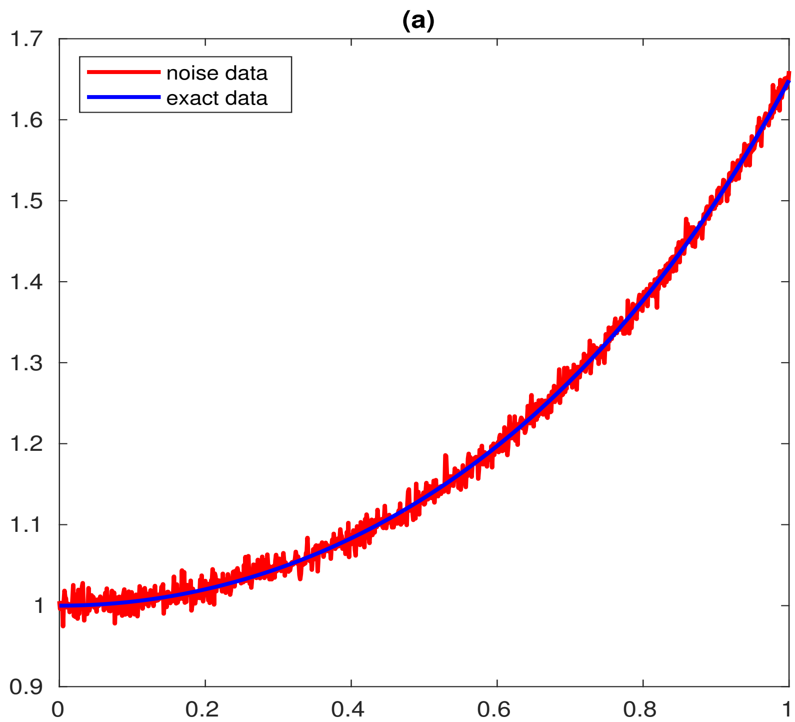

Figure 3.7 (**a**) Data with $\delta = 0.01$ and $n = 1000$.

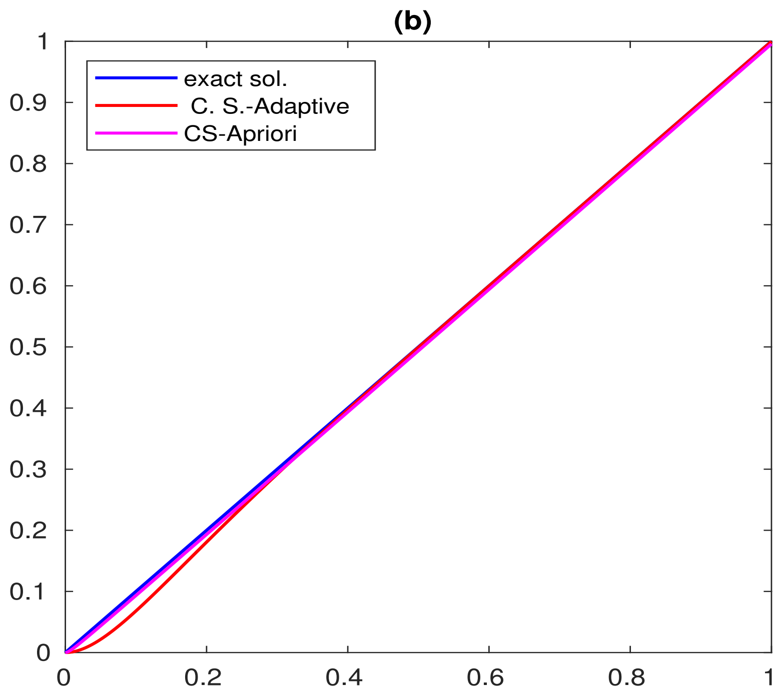

Figure 3.8 (**b**) Solution with $\delta = 0.01$ and $n = 1000$.

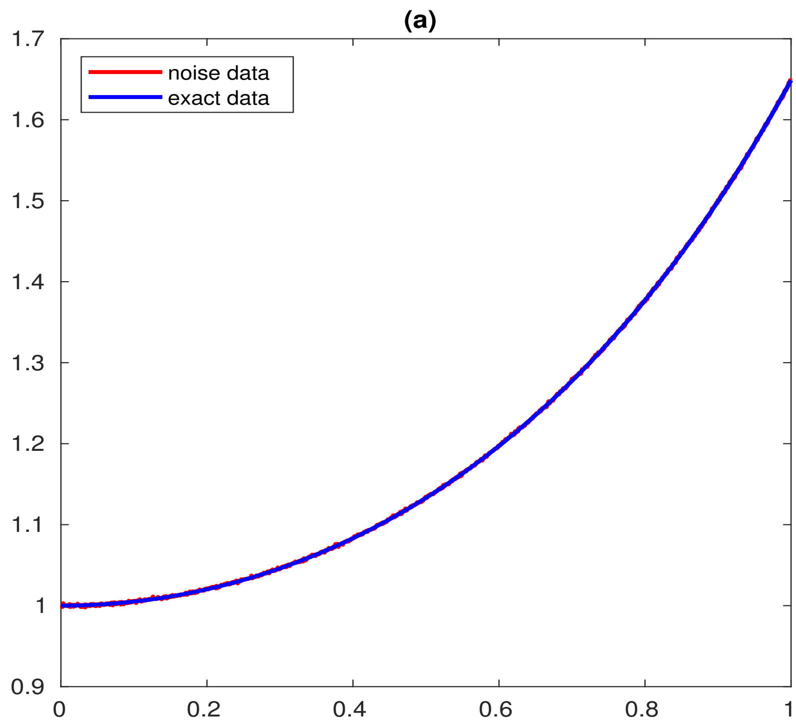

Figure 3.9 (**a**) Data with $\delta = 0.001$ and $n = 1000$.

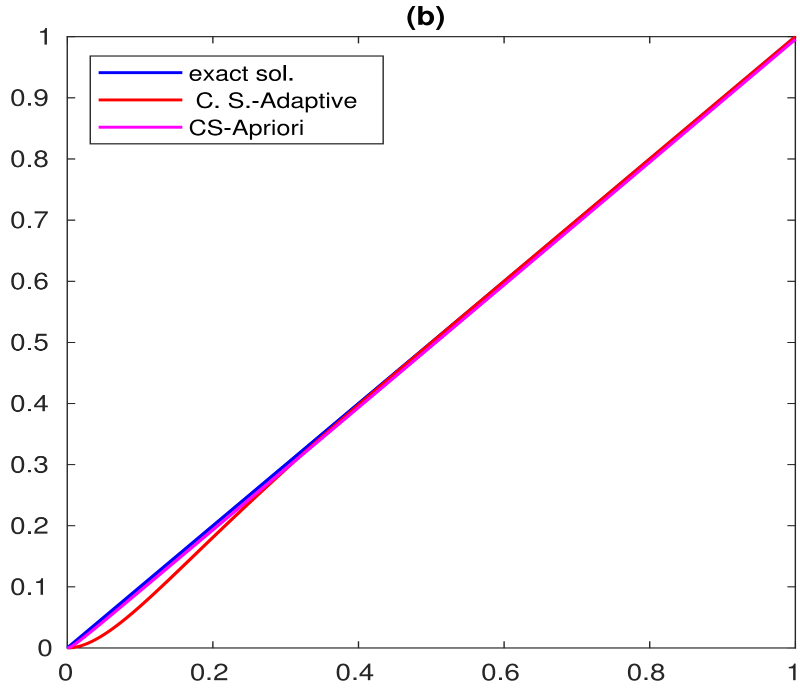

Figure 3.10 (**b**) Solution with $\delta = 0.001$ and $n = 1000$.

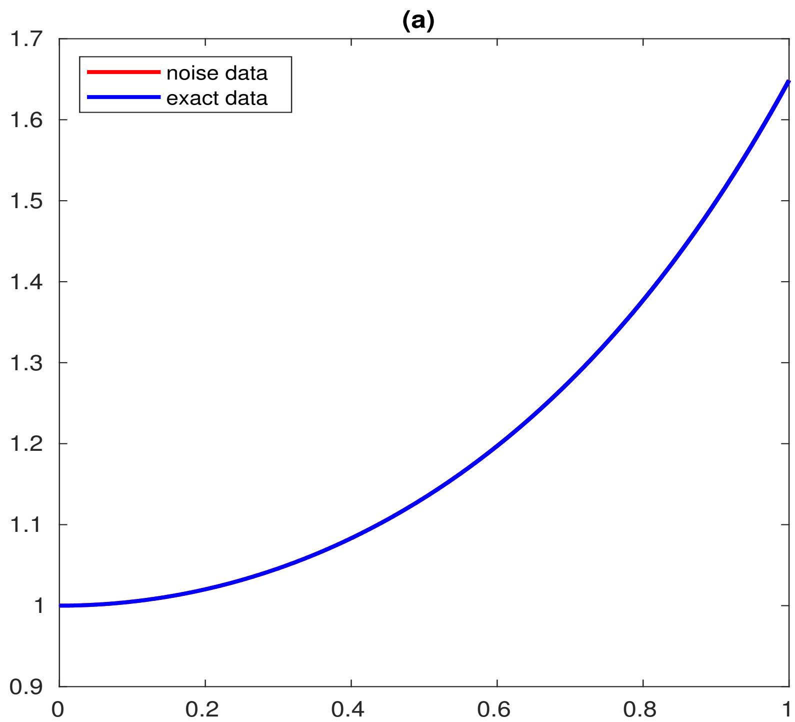

Figure 3.11 (**a**) Data with $\delta = 0.0001$ and $n = 1000$.

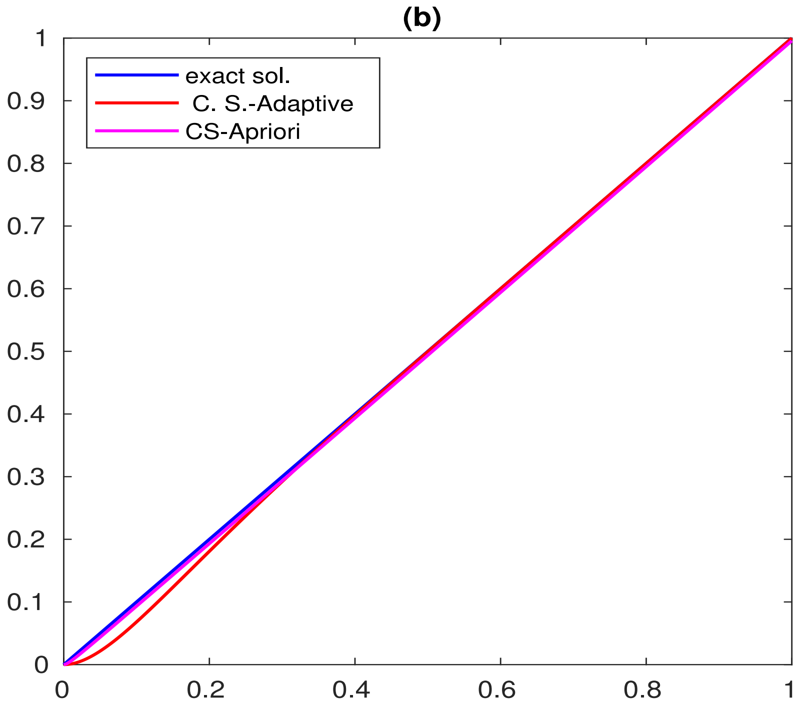

Figure 3.12 (**b**) Solution with $\delta = 0.0001$ and $n = 1000$.

Table 3.1 Relative errors using discrepancy principle.

Method		$n=500$			$n=1000$		
		$\delta=0.01$	$\delta=0.001$	$\delta=0.0001$	$\delta=0.01$	$\delta=0.001$	$\delta=0.0001$
(3.4.3)	α	4.283954×10^{-3}	4.283969×10^{-3}	4.283972×10^{-3}	3.602506×10^{-3}	3.602505×10^{-3}	3.602536×10^{-3}
	E_α	1.225477×10^{-2}	1.225481×10^{-2}	1.225482×10^{-2}	1.036919×10^{-2}	1.036919×10^{-2}	1.036927×10^{-2}
	CT	3.764950×10^{-1}	3.286400×10^{-1}	3.355110×10^{-1}	1.879650×10^{-1}	1.870468×10^{-1}	1.802014×10^{-1}
Adaptive Method	α	1.040604×10^{-4}	1.040604×10^{-6}	1.040604×10^{-8}	1.040604×10^{-4}	1.040604×10^{-6}	1.040604×10^{-8}
	E_α	2.182110×10^{-2}	2.173007×10^{-2}	2.172918×10^{-2}	2.183745×10^{-2}	2.174636×10^{-2}	2.174546×10^{-2}
in George and Nair (2017)	CT	1.246600×10^{-2}	1.159500×10^{-2}	4.501330×10^{-1}	1.352600×10^{-2}	1.191300×10^{-2}	8.252000×10^{-3}

CHAPTER 4

EXTENDING THE APPLICABILITY OF CORDERO TYPE ITERATIVE METHOD

4.1 INTRODUCTION

In this Chapter, the main goal is to obtain convergence order of the iterative method studied in (Cordero et al. (2012)) without using assumptions on the higher-order derivatives. Throughout this chapter $\mathscr{U}, \mathscr{V}$ denote Banach spaces and $\Omega \subset \mathscr{U}$ is a convex set. We are interested in approximating the solution s^* of the equation

$$\mathscr{L}(s) = 0, \tag{4.1.1}$$

where $\mathscr{L} : \Omega \subset \mathscr{U} \to \mathscr{V}$ is a nonlinear operator that is Frèchet differentiable. A considerable number of nonlinear problems of the form (4.1.1) that arise in physics, chemistry, biology, finance, and mathematics are modeled on principles of symmetry. In general, the classical Newton method of second-order defines $\forall k = 0, 1, 2, \ldots,$ by

$$s_{k+1} = s_k - \mathscr{L}_{s_k}^{-1}\mathscr{L}(s_k), \tag{4.1.2}$$

where $\mathscr{L}_{s_k} = \mathscr{L}'(s_k)$, is considered to be the most efficient iterative method to solve (4.1.1). Cordero et al. (2012) modified the classical Newton method by employing Adomian polynomial decomposition and obtained a fourth-order iterative scheme. The iterative scheme in (Cordero et al. (2012)) is defined $\forall k = 0, 1, 2, \ldots,$ by

$$\begin{aligned} t_k &= s_k - \mathscr{L}_{s_k}^{-1}\mathscr{L}(s_k) \\ s_{k+1} &= t_k - (2\mathscr{L}_{s_k}^{-1} - \mathscr{L}_{s_k}^{-1}\mathscr{L}_{t_k}\mathscr{L}_{s_k}^{-1})\mathscr{L}(t_k), \end{aligned} \tag{4.1.3}$$

where $\mathscr{L}_{t_k} = \mathscr{L}'(t_k)$. This new fourth-order Cordero method has better stability than the classical Newton method with higher-order convergence.

A new technique was introduced by Cordero et al. (2012) to improve the convergence order of an iterative method from q to $q+2$ by combining it with the classical Newton method. By using this technique, we modified the fourth-order iterative method (4.1.3) to a sixth-order iterative scheme that is defined $\forall k = 0, 1, 2, \ldots$, by

$$\begin{aligned} t_k &= s_k - \mathscr{L}_{s_k}^{-1}\mathscr{L}(s_k) \\ u_k &= t_k - \mathscr{L}_{s_k}^{-1}(2I - \mathscr{L}_{t_k}\mathscr{L}_{s_k}^{-1})\mathscr{L}(t_k) \\ s_{k+1} &= u_k - \mathscr{L}_{t_k}^{-1}\mathscr{L}(u_k). \end{aligned} \tag{4.1.4}$$

However, the disadvantage of the convergence analysis conducted by Cordero et al. (2012) is that they use Taylor expansion which involves the Fréchet derivative of the function up to order six.

In this Chapter, we could obtain the sixth-order convergence for the method (4.1.4) without using Taylor expansion. We have used the assumptions based on the Fréchet derivative of order one. The novelty of our approach is that it does not require higher-order Fréchet derivatives of the operator and Taylor expansion in the convergence analysis. Thus, we enhance the method's utility. We also modify the last step of the method (4.1.4) and obtain a new eighth-order iterative scheme that is defined $\forall k = 0, 1, 2, \ldots$, by

$$\begin{aligned} t_k &= s_k - \mathscr{L}_{s_k}^{-1}\mathscr{L}(s_k) \\ u_k &= t_k - \mathscr{L}_{s_k}^{-1}(2I - \mathscr{L}_{t_k}\mathscr{L}_{s_k}^{-1})\mathscr{L}(t_k) \\ s_{k+1} &= u_k - \mathscr{L}_{u_k}^{-1}\mathscr{L}(u_k), \end{aligned} \tag{4.1.5}$$

where $\mathscr{L}_{u_k} = \mathscr{L}'(u_k)$.

Parhi and Sharma (2021), proved the convergence of the method (4.1.4) without using Taylor expansion. However, they could not obtain the sixth-order convergence theoretically for method (4.1.4).

In this Chapter, we also estimate the radius of convergence of the methods (4.1.4) and (4.1.5) under assumptions on first-order Fréchet derivative and compute the efficiency indices. We numerically demonstrate that the radius of convergence in our study is superior to the estimates of Parhi and Sharma. We also considered the analogous iterative methods of these two iterative schemes to solve an ill-posed problem in a Hilbert space.

The convergence analysis of methods (4.1.4) and (4.1.5) is provided in Section 4.2.

The radius of convergence and Approximate Computational Order of Convergence (ACOC) is computed numerically in Section 4.3. A numerical example of an ill-posed problem is given in Section 4.4 and the paper concludes in Section 4.6.

4.2 CONVERGENCE ANALYSIS OF (4.1.4) AND (4.1.5)

The following definition and assumptions are used to prove our results.

Assumption 4.2.1. $\exists\ \zeta_1 > 0$ *such that* $\forall s,t \in D(\mathscr{L})$,

$$\|\mathscr{L}'(s)^{-1}(\mathscr{L}'(t) - \mathscr{L}'(s))\| \leq \zeta_1 \|t - s\|.$$

Assumption 4.2.2. $\exists\ \zeta_2 > 0, \rho > 0$ *such that* $\forall s,t \in B(s^*, \rho)$,

$$\|\mathscr{L}'(s)^{-1}\mathscr{L}'(t)\| \leq \zeta_2.$$

The local convergence is based on functions $\phi_i, \psi_i, i = 1,2$, which are defined as follows. Let $\phi_1 : [0,\infty) \to [0,\infty)$ be defined by

$$\phi_1(\tau) = \frac{\zeta_1^3}{32}[16\zeta_1 + 4\zeta_2^2 + 8\zeta_1^2\tau + \zeta_1^3\tau^2]\tau^3$$

and

$$\psi_1(\tau) = \phi_1(\tau) - 1.$$

We observe that $\psi_1(0) = -1$ and $\psi_1(\tau) \to \infty$ as $\tau \to \infty$. So, by intermediate value theorem $\psi_1(\tau) = 0$ has a minimal zero $\rho_1 > 0$. Similarly, define $\phi_2 : [0,\infty) \to [0,\infty)$ by

$$\phi_2(\tau) = \frac{\zeta_1^2}{2}\left(1 + \frac{\phi_1(\tau)}{\zeta_1\tau}\right)\phi_1(\tau)\tau^2$$

and

$$\psi_2(\tau) = \phi_2(\tau) - 1.$$

Furthermore, let $\rho_2 > 0$ be the minimal zero of $\psi_2(\tau) = 0$. Let

$$\rho = min\left\{\frac{2}{\zeta_1}, \rho_1, \rho_2\right\}. \tag{4.2.1}$$

Then, $0 < \phi_1(\tau), \phi_2(\tau) < 1, \forall \tau \in (0,\rho)$. Let $e_n^s = \|s_n - s^*\|, e_n^t = \|t_n - s^*\|$, and $e_n^u = \|u_n - s^*\|, \forall n = 0,1,2,\ldots$

Theorem 4.2.3. ***(Existence and Uniqueness)*** *Let* ρ *be as in (4.2.1). Then* $\{s_k\}$ *defined* *by (4.1.4) with* $s_0 \in B(s^*,\rho) - \{s^*\}$, *converges to* s^* *with order of convergence six, i.e.,*

$$e^s_{k+1} \leq C(e^s_k)^6,$$

where $C = \frac{\zeta_1^5}{64}\left(1 + \frac{\phi_1(\rho)}{\zeta_1\rho}\right)\left(16\zeta_1 + 4\zeta_2^2 + 8\zeta_1^2\rho + \zeta_1^3\rho^2\right)$. *Suppose that (4.1.1) has a simple solution in the set* $S = \Omega \cap \overline{B(s^*,\rho)}$. *Then* s^* *is the unique solution of equation* $\mathscr{L}(s) = 0$ *in the set* S, *provided that* $\zeta_1\rho < 2$.

Proof. (Existence Part) By induction, we shall prove the following inequalities:

$$\begin{aligned} t_n \in B(s^*,\rho),\, e^t_n &\leq \frac{\zeta_1}{2}(e^s_n)^2, \\ u_n \in B(s^*,\rho),\, e^u_n &\leq \phi_1(e^s_n)e^s_n, \\ s_{n+1} \in B(s^*,\rho),\, e^s_{n+1} &\leq C(e^s_n)^6. \end{aligned}$$

For $s_0 \in B(s^*,r)$, by (4.1.4) we have,

$$\begin{aligned} t_0 - s^* &= s_0 - s^* - \mathscr{L}_{s_0}^{-1}(\mathscr{L}(s_0) - \mathscr{L}(s^*)) \\ &= \left(-\mathscr{L}_{s_0}^{-1}\int_0^1 \mathscr{L}'(s^* + \tau(s_0 - s^*)) - \mathscr{L}_{s_0} d\tau\right)(s_0 - s^*). \end{aligned}$$

So by Assumption 4.2.1, we obtain,

$$e^t_0 \leq \frac{\zeta_1}{2}(e^s_0)^2. \tag{4.2.2}$$

By (4.2.1), $\frac{\zeta_1}{2}(e^s_0)^2 \leq \frac{\zeta_1}{2}\rho^2 \leq \rho$, so we have $t_0 \in B(s^*,\rho)$. Again, from the second step of (4.1.4),

$$\begin{aligned} u_0 - s^* &= t_0 - s^* - \left(\mathscr{L}_{s_0}^{-1}(2I - \mathscr{L}_{t_0}\mathscr{L}_{s_0}^{-1})\right)(\mathscr{L}(t_0) - \mathscr{L}(s^*)) \\ &= \mathscr{L}_{s_0}^{-1}(\mathscr{L}_{s_0}(t_0 - s^*) - (2I - \mathscr{L}_{t_0}\mathscr{L}_{s_0}^{-1}) \\ &\quad \times \int_0^1 \mathscr{L}'(s^* + \tau(t_0 - s^*))(t_0 - s^*)d\tau) \\ &= -\mathscr{L}_{s_0}^{-1}\int_0^1 (\mathscr{L}'(s^* + \tau(t_0 - s^*)) - \mathscr{L}_{s_0})(t_0 - s^*)d\tau \\ &\quad -\mathscr{L}_{s_0}^{-1}(I - \mathscr{L}_{t_0}\mathscr{L}_{s_0}^{-1})\int_0^1 \mathscr{L}'(s^* + \tau(t_0 - s^*))(t_0 - s^*)d\tau \\ &= -\mathscr{L}_{s_0}^{-1}\int_0^1 (\mathscr{L}'(s^* + \tau(t_0 - s^*)) - \mathscr{L}_{s_0})(t_0 - s^*)d\tau \\ &\quad -\mathscr{L}_{s_0}^{-1}(\mathscr{L}_{s_0} - \mathscr{L}_{t_0})\mathscr{L}_{s_0}^{-1}\left(\int_0^1 \mathscr{L}'(s^* + \tau(t_0 - s^*))(t_0 - s^*)d\tau\right). \end{aligned}$$

By adding and subtracting the term $\Gamma = \int_0^1 \mathscr{L}'(s^* + \tau(t_0 - s^*))d\tau$ we get,

$$
\begin{aligned}
u_0 - s^* &= -\mathscr{L}_{s_0}^{-1} \int_0^1 (\mathscr{L}'(s^* + \tau(t_0 - s^*)) - \mathscr{L}_{s_0})(t_0 - s^*)d\tau) \\
&\quad -\mathscr{L}_{s_0}^{-1}(\mathscr{L}_{s_0} + \Gamma - \Gamma - \mathscr{L}_{t_0}) \\
&\quad \times \left(\mathscr{L}_{s_0}^{-1} \left(\int_0^1 \mathscr{L}'(s^* + \tau(t_0 - s^*))(t_0 - s^*)d\tau \right) \right) \\
&= -\mathscr{L}_{s_0}^{-1} \left(\Gamma - \int_0^1 \mathscr{L}_{s_0} d\tau \right) (t_0 - s^*) \\
&\quad -\mathscr{L}_{s_0}^{-1} (\mathscr{L}_{s_0} - \Gamma) \mathscr{L}_{s_0}^{-1} \Gamma (t_0 - s^*) \\
&\quad -\mathscr{L}_{s_0}^{-1} \left(\Gamma - \int_0^1 \mathscr{L}_{t_0} d\tau \right) \mathscr{L}_{s_0}^{-1} \Gamma (t_0 - s^*) \\
&= -\mathscr{L}_{s_0}^{-1} \left(\Gamma - \int_0^1 \mathscr{L}_{s_0} d\tau \right) \left(I - \mathscr{L}_{s_0}^{-1} \Gamma \right) (t_0 - s^*) \\
&\quad -\mathscr{L}_{s_0}^{-1} \left(\Gamma - \int_0^1 \mathscr{L}_{t_0} d\tau \right) \mathscr{L}_{s_0}^{-1} \Gamma (t_0 - s^*) \\
&= -\mathscr{L}_{s_0}^{-1} \left(\Gamma - \int_0^1 \mathscr{L}_{s_0} d\tau \right) \mathscr{L}_{s_0}^{-1} (\mathscr{L}_{s_0} - \Gamma)(t_0 - s^*) \\
&\quad - \left(\mathscr{L}_{s_0}^{-1} \mathscr{L}_{t_0} \mathscr{L}_{t_0}^{-1} \left(\Gamma - \int_0^1 \mathscr{L}_{t_0} d\tau \right) \right) \mathscr{L}_{s_0}^{-1} \Gamma (t_0 - s^*).
\end{aligned}
$$

Therefore, by (4.2.2), Assumptions 4.2.1 and 4.2.2, we obtain

$$
\begin{aligned}
e_0^u &\leq \zeta_1^2 \left(e_0^s + \frac{e_0^t}{2} \right)^2 e_0^t + \frac{\zeta_1 \zeta_2^2}{2} (e_0^t)^2 \\
&= \zeta_1^2 \left((e_0^s)^2 + e_0^s e_0^t + \frac{(e_0^t)^2}{4} \right) e_0^t + \frac{\zeta_1 \zeta_2^2}{2} (e_0^t)^2 \\
&\leq \zeta_1^2 \left((e_0^s)^2 + \frac{\zeta_1}{2} (e_0^s)^3 + \frac{\zeta_1^2}{16} (e_0^s)^4 \right) \frac{\zeta_1^2}{2} (e_0^s)^2 + \frac{\zeta_1 \zeta_2^2}{2} \left(\frac{\zeta_1}{2} (e_0^s)^2 \right)^2 \\
&= \frac{\zeta_1^4}{32} \left(16 + 8\zeta_1 e_0^s + \zeta_1^2 (e_0^s)^2 \right) (e_0^s)^4 + \frac{\zeta_1^3 \zeta_2^2}{8} (e_0^s)^4 \\
&= \frac{\zeta_1^3}{32} \left(16\zeta_1 + 4\zeta_2^2 + 8\zeta_1^2 e_0^s + \zeta_1^3 (e_0^s)^2 \right) (e_0^s)^4 \qquad (4.2.3) \\
&= \phi_1(e_0^s) e_0^s < e_0^s.
\end{aligned}
$$

Thus, $u_0 \in B(s^*, \rho)$. By the third step of (4.1.4) we have,

$$
\begin{aligned}
s_1 - s^* &= u_0 - s^* - \mathscr{L}_{t_0}^{-1} (\mathscr{L}(u_0) - \mathscr{L}(s^*)) \\
&= -\mathscr{L}_{t_0}^{-1} \int_0^1 (\mathscr{L}'(s^* + \tau(u_0 - s^*)) - \mathscr{L}_{t_0})(u_0 - s^*) d\tau.
\end{aligned}
$$

Again, by using Assumption 4.2.1, (4.2.2) and (4.2.3) we get,

$$\begin{aligned}
e_1^s &\leq \zeta_1\left(e_0^t+\frac{e_0^u}{2}\right)e_0^u \\
&\leq \zeta_1\left(\frac{\zeta_1}{2}(e_0^s)^2+\frac{\zeta_1^3}{64}\left(16\zeta_1+4\zeta_2^2+8\zeta_1^2e_0^s+\zeta_1^3(e_0^s)^2\right)(e_0^s)^4\right) \\
&\quad \frac{\zeta_1^3}{32}\left(16\zeta_1+4\zeta_2^2+8\zeta_1^2e_0^s+\zeta_1^3(e_0^s)^2\right)(e_0^s)^4 \\
&= \frac{\zeta_1^5}{64}\left(1+\frac{\zeta_1^2}{32}\left(16\zeta_1+4\zeta_2^2+8\zeta_1^2e_0^s+\zeta_1^3(e_0^s)^2\right)(e_0^s)^2\right) \\
&\quad \left(16\zeta_1+4\zeta_2^2+8\zeta_1^2e_0^s+\zeta_1^3(e_0^s)^2\right)(e_0^s)^6 \\
&= \frac{\zeta_1^5}{64}\left(1+\frac{\phi_1(e_0^s)}{\zeta_1e_0^s}\right)\frac{32}{\zeta_1^3}\phi_1(e_0^s)(e_0^s)^3 \\
&= \phi_2(e_0^s)e_0^s. \qquad (4.2.4)
\end{aligned}$$

Note that,

$$\begin{aligned}
\phi_2(e_0^s) &= \frac{\zeta_1^2}{2}\left(1+\frac{\phi_1(e_0^s)}{\zeta_1e_0^s}\right)(\phi_1(e_0^s))(e_0^s)^2 \\
&= \frac{\zeta_1^5}{64}\left(1+\frac{\phi_1(e_0^s)}{\zeta_1e_0^s}\right)\left(16\zeta_1+4\zeta_2^2+8\zeta_1^2e_0^s+\zeta_1^3(e_0^s)^2\right)(e_0^s)^5.
\end{aligned}$$

So by (4.2.4), we get,

$$\begin{aligned}
e_1^s &= \frac{\zeta_1^5}{64}\left(1+\frac{\phi_1(e_0^s)}{\zeta_1e_0^s}\right)\left(16\zeta_1+4\zeta_2^2+8\zeta_1^2e_0^s+\zeta_1^3(e_0^s)^2\right)(e_0^s)^6 \\
&\leq C(e_0^s)^6.
\end{aligned}$$

Further, since $\phi_2(e_0^s)<1$, we have $s_1\in B(s^*,\rho)$. The induction is complete, by replacing s_0,t_0,u_0,s_1 by s_n,t_n,u_n,s_n+1,, respectively, in the preceding arguments.
(Uniqueness Part) Let $\bar{s}$ be another solution of (4.1.1) in the set S.
Let $T=\int_0^1\mathscr{L}'(s^*+\tau(\bar{s}-s^*))d\tau$. By using Assumption 4.2.1, we have

$$\begin{aligned}
\|\mathscr{L}'(s^*)^{-1}(T-\mathscr{L}'(s^*))\| &\leq \zeta_1\int_0^1\|s^*+\tau(\bar{s}-s^*)-s^*\|d\tau \\
&= \zeta_1\int_0^1\tau\|\bar{s}-s^*\|d\tau \\
&\leq \frac{\zeta_1}{2}\rho<1.
\end{aligned}$$

Therefore, by using Banach lemma Argyros (2008), one can conclude that T is invertible. Hence $\bar{s} = s^*$ follows from $0 = \mathscr{L}(\bar{s}) - \mathscr{L}(s^*) = T(\bar{s} - s^*)$. $\square$

Next, we prove the convergence of method (4.1.5). Let $\tilde{\phi_2} : [0,\infty) \to [0,\infty)$ be defined by

$$\tilde{\phi_2}(\tau) = \frac{\zeta_1}{2}\phi_1(\tau)\tau^4.$$

Again, by intermediate value theorem $\tilde{\psi_2}(\tau) = \tilde{\phi_2}(\tau) - 1 = 0$ has a minimal zero $\tilde{\rho_2} > 0$. Let us define

$$\tilde{\rho} = min\left\{\frac{2}{\zeta_1}, \rho_1, \tilde{\rho_2}\right\}. \tag{4.2.5}$$

Theorem 4.2.4. *Let $\tilde{\rho}$ be as in (4.2.5). Then $\{s_k\}$ defined by (4.1.5) with $s_0 \in B(s^*, \tilde{\rho}) - \{s^*\}$, converges to s^* with the order of convergence eight. i.e.,*

$$e^s_{k+1} \leq \tilde{C}(e^s_k)^8,$$

where $\tilde{C} = \frac{\zeta_1\phi_1(\tilde{\rho})}{2\tilde{\rho}^3}$. Furthermore, s^* is the unique solution of (4.1.1) in the set $S = \Omega \cap \overline{B(s^*, \tilde{\rho})}$ provided that $\zeta_1\tilde{\rho} < 2$.

Proof: By the third sub-step of (4.1.5), we have

$$s_1 - s^* = u_0 - s^* - \mathscr{L}_{u_0}^{-1}(\mathscr{L}(u_0) - \mathscr{L}(s^*))$$

so, by (4.2.3), we get

$$\begin{aligned} e^s_1 &\leq \frac{\zeta_1}{2}(e^u_0)^2 \\ &\leq \frac{\zeta_1}{2}\left(\frac{\zeta_1^3}{32}\left(16\zeta_1 + 4\zeta_2^2 + 8\zeta_1^2 e^s_0 + \zeta_1^3(e^s_0)^2\right)(e^s_0)^4\right)^2 \\ &= \frac{\zeta_1}{2}\left(\frac{\phi_1(e^s_0)}{(e^s_0)^3}\right)(e^s_0)^8 \\ &= \tilde{\phi_2}(e^s_0)e^s_0 < e^s_0. \end{aligned} \tag{4.2.6}$$

From (4.2.6), we get

$$\begin{aligned} e^s_1 &\leq \frac{\zeta_1\phi_1(\tilde{\rho})}{2\tilde{\rho}^3}(e^s_0)^8 \\ &= \tilde{C}(e^s_0)^8. \end{aligned}$$

The rest of the proof proceeds in the same manner as in Theorem 4.2.3.

Remark 4.2.5. *Note that by (4.2.3), we obtain the convergence order four for the Cordero method (4.1.3).*

4.3 ESTIMATION OF RADIUS OF CONVERGENCE AND COMPUTATIONAL ORDER

We estimate the radius of convergence ρ and $\tilde{\rho}$ to validate the theoretical results.

Example 4.3.1. *Let* $\mathscr{U} = \mathscr{V} = \mathbb{R}, s_0 = 1, \Omega = [s_0 - (1-k), s_0 + (1-k)], k \in (2-\sqrt{2}, 1)$ *and* $F : \Omega \to \mathscr{K}$ *be defined by*

$$\mathscr{L}(s) = s^3 - k.$$

We have, $\|\mathscr{L}_{s_0}^{-1}\| = \frac{1}{3}$.

$$\begin{aligned}
\|\mathscr{L}_{s_0}^{-1}(\mathscr{L}'(s) - \mathscr{L}_{s_0})\| &= \frac{1}{3}\|(3s^2-3)\| \\
&\leq \|s+1\|\|s-1\| \\
&= (3-k)(1-k).
\end{aligned}$$

By using Banach Lemma,

$$\begin{aligned}
\|\mathscr{L}'(s)^{-1}\| &\leq \frac{\|\mathscr{L}_{s_0}^{-1}\|}{1 - \|\mathscr{L}_{s_0}^{-1}\mathscr{L}'(s) - I\|} \\
&= \frac{1}{3(1-(3-k)(1-k))}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathscr{L}'(s)^{-1}(\mathscr{L}'(t) - \mathscr{L}'(s))\| &\leq \|\mathscr{L}'(s)^{-1}\|\|3t^2 - 3s^2\| \\
&\leq \frac{3(t+s)(t-s)}{3(1-(3-k)(1-k))} \\
&= \frac{2(2-k)}{(1-(3-k)(1-k))}\|t-s\|.
\end{aligned}$$

Therefore, $\zeta_1 = \frac{2(2-k)}{(1-(3-k)(1-k))}$.

$$\begin{aligned}
\|\mathscr{L}'(s)^{-1}\mathscr{L}''(t)\| &\leq \|\mathscr{L}'(s)^{-1}\|\|\mathscr{L}''(t)\| \\
&\leq \frac{6t}{3(1-(3-k)(1-k))} \\
&= \frac{2(2-k)}{(1-(3-k)(1-k))} = \zeta_2.
\end{aligned}$$

Set $k = 0.85$, *we then get,* $\zeta_1 = \zeta_2 \approx 3.3948, \rho_1 \approx 0.1899, \rho_2 \approx 0.2092, \frac{2}{\zeta_1} = 0.35$, $\rho = min\{\frac{2}{\zeta_1}, \rho_1, \rho_2\} \approx 0.1899$. *Furthermore, we have* $\tilde{\rho}_2 \approx 0.4409$ *and* $\tilde{\rho} = \min\{\frac{2}{\zeta_1}, \rho_1, \tilde{\rho}_2\} =$

0.1899. *Using the convergence analysis in Parhi and Sharma (2021), we obtain the radius* $R = 0.1123$.

Example 4.3.2. *Let* $\mathscr{U} = \mathscr{V} = \mathbb{R}^3, \Omega = B[0,1], s_0 = (0,0,0)^T$. *Define function* $\mathscr{L}$ *on* Ω *for* $s = (s,t,u)^T$ *by*

$$\mathscr{L}(x) = \left(e^s - 1, \frac{e-1}{2}t^2 + t, u\right)^T.$$

Then,

$$\mathscr{L}'(x) = \begin{pmatrix} e^s & 0 & 0 \\ 0 & (e-1)t+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, $\zeta_1 = e - 1$ *and* $\zeta_2 = e$. *Furthermore, we get,* $\frac{2}{\zeta_1} \approx 1.1639, \rho_1 \approx 0.4510, \rho_2 \approx 0.4779$ *and the radius of convergence* $\rho = min\{\frac{2}{\zeta_1}, \rho_1, \rho_2\} \approx 0.4510$. *Furthermore, we have* $\tilde{\rho_2} \approx 0.7154$ *and* $\tilde{\rho} = \min\{\frac{2}{\zeta_1}, \rho_1, \tilde{\rho_2}\} \approx 0.4510$. *Parhi and Sharma Parhi and Sharma (2021) considered this example 4.3.2 and obtained the radius* $R = 0.133649$.

Remark 4.3.3. *We observe that,* $\tilde{\rho} = \rho$ *in the above examples. Furthermore, note that we can obtain a better radius of convergence than that of in (Parhi and Sharma (2021)).*

4.4 COMPUTATIONAL ORDER AND COMPUTATIONAL EFFICIENCIES OF ITERATIVE METHODS

One can ensure the order of convergence of the iterative methods computationally. We can use the following definitions to validate the theoretical results computationally.

Definition 4.4.1. ***Computational Order of Convergence (COC)*** *of an iterative sequence* $\{s_k\}$*is defined in (Weerakoon and Fernando (2000)), as*

$$\Sigma_2 = \frac{\ln\left(\frac{\|s_{k+1}-s^*\|}{\|s_k-s^*\|}\right)}{\ln\left(\frac{\|s_k-s^*\|}{\|s_{k-1}-s^*\|}\right)},$$

where s_{k+1}, s_k *and* s_{k-1} *are three consecutive terms near to the root* s^*.

Definition 4.4.2. ***Approximate Computational Order of Convergence (ACOC)*** *of an iterative sequence* $\{s_k\}$ *is defined in (Petković et al. (2014)), as*

$$\Sigma_1 = \frac{\ln\left(\frac{\|s_{k+1}-s_k\|}{\|s_k-s_{k-1}\|}\right)}{\ln\left(\frac{\|s_k-s_{k-1}\|}{\|s_{k-1}-s_{k-2}\|}\right)},$$

where s_{k+1}, s_k, s_{k-1} and s_{k-2} are four consecutive terms near to the root s^*.

To ensure the methods (4.1.4) and (4.1.5) attain the order of convergence computationally, we calculated the Approximate Computational Order of Convergence (ACOC) of these iterative methods. We consider the following functions and stopping criterion,

$$\begin{aligned}
\mathscr{L}(a_1,a_2,a_3) &= \left(e^{a_1}-1, \frac{e-1}{2}a_2^2+a_2, a_3\right), & (4.4.1)\\
\mathscr{L}(a_1,a_2) &= (a_1^2-4a_2+a_2^2, 2a_1-a_2^2-2), & (4.4.2)\\
\mathscr{L}(a_1,a_2) &= (a_1^2+a_2^2-1, a_1^2-a_2^2+0.5), & (4.4.3)\\
\mathscr{L}(a_1,a_2) &= (a_1^3-a_2, a_2^3-a_2), & (4.4.4)\\
\mathscr{L}(a_1,a_2) &= (3a_1^2a_2-a_2^3, a_1^3-3a_1a_2^2-1). & (4.4.5)
\end{aligned}$$

Stopping criteria: $\|s_{k+1}-s_k\|+\|\mathscr{L}(s_{k+1})\| \leq 10^{-10}$.

Table 4.1 ACOC for methods (4.1.3), (4.1.4) and (4.1.5)

Eq. No.	s^*	s_0	$\Sigma_1(N)$ CM (4.1.3)	$\Sigma_1(N)$ CM1 (4.1.4)	$\Sigma_1(N)$ CM2 (4.1.5)
(4.4.1)	$(0,0,0)$	$(0.5,0.5,0.5)$	4.3(4)	6.2(4)	7.6(4)
		$(1.1,1.1,1.1)$	3.7(5)	5.9(5)	6.4(4)
(4.4.2)	$(0.3542,1.1364)$	$(0.6,0.7)$	3.5(5)	5.8(4)	8(4)
(4.4.3)	$(\frac{1}{2},\frac{\sqrt{3}}{2})$	$(0.35,0.5)$	3.9(5)	6.3(4)	8.4(4)
		$(0.9,1)$	3.4(4)	5.4(4)	ND(3)
(4.4.4)	$(1,1)$	$(1.1.,0.75)$	3.7(5)	5.8(4)	9.5(4)
(4.4.5)	$(-\frac{1}{2},-\frac{\sqrt{3}}{2})$	$(-0.4,-\frac{1}{2})$	4.2(7)	5.2(8)	7.9(5)

Note that the oscillatory nature of the approximations and slow convergence in the initial stage present the main disadvantages in the computation of ACOC in higher-order iterative methods. In Table 4.1, we observe that the choice of a suitable initial approximation plays a vital role to achieve the maximum order of convergence (see (4.4.1) and (4.4.3). Furthermore, it requires at least four iterations to compute ACOC (see (4.4.3)). Specifically in Table 4.1, we provide ACOC for nonlinear equations using Cordero's fourth-order method (CM) (4.1.3), first extension (CM1) (4.1.4) and second extension (CM2) (4.1.5). Here, N, s^*, and s_0 denote the number of iterations, root, and initial value, respectively.

One of the challenging tasks is to minimize the computational cost of higher order iterative methods. By limiting the number evaluations of functions, higher order Fréchet derivatives and matrix inversions in each iteration, one can reduce the computational cost. One can use informational efficiency or computational efficiency to obtain an estimate on the computational cost of an iterative method.

Definition 4.4.3. ***The informational efficiency*** *I is defined in (Traub (1964)), as*

$$I = \frac{R}{q},$$

where R is the order of convergence of the method and q is the number of functions (and derivative) computed.

Definition 4.4.4. *Recall that (Ostrowski (1973)),* ***efficiency index or computational efficiency*** *is defined as*

$$c_f = R^{\frac{1}{q}}.$$

The efficiency index and informational efficiency of the fourth-order Cordero method (4.1.3) are $c_f = 4^{1/4} = 1.41$ and $I = 4/4 = 1$,, respectively, which coincide with that of the Newton method. Whereas $c_f = 6^{1/5} = 1.43, I = 6/5 = 1.2$ for the sixth-order method (4.1.4) and $c_f = 8^{1/6} = 1.41, I = 8/6 = 1.33$ for the eighth-order method (4.1.5).

4.5 APPLICATION TO ILL-POSED PROBLEM

We consider the ill-posed problem (3.5.3) replacing the variable u with s. We use Lavrentiev regularization method with $\alpha > 0$ (see George et al. (2023) for details), i.e.,

$$\mathscr{H}(s) + \alpha(s - s_0) = y, \tag{4.5.1}$$

to approximate the exact solution $\hat{s}$ of (4.5.1). Let

$$\mathscr{L}(s) = \mathscr{H}(s) + \alpha(s - s_0) - y = 0.$$

We consider the following analogous iterative methods (4.1.2),(4.1.3), (4.1.4) and (4.1.5) defined $\forall k = 0, 1, ...,$ by

$$s_{k+1} = s_k - (\mathscr{L}_{s_k} + \alpha I)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^\delta),$$

$$\begin{aligned} t_k &= s_k - (\mathscr{L}_{s_k} + \alpha I)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^\delta) \\ s_{k+1} &= t_k - (2(\mathscr{L}_{s_k} + \alpha I)^{-1} - ((\mathscr{L}_{s_k} + \alpha I)^{-1} \\ &\quad (\mathscr{L}_{t_k} + \alpha I)(\mathscr{L}_{s_k} + \alpha I)^{-1}(\mathscr{L}(t_k) + \alpha(t_k - s_0) - y^\delta))), \end{aligned}$$

$$
\begin{aligned}
t_k &= s_k - (\mathscr{L}_{s_k} + \alpha I)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^{\delta}) \\
u_k &= t_k - (\mathscr{L}_{s_k} + \alpha I)^{-1}(2I - (\mathscr{L}_{t_k} + \alpha I) \\
&\quad (\mathscr{L}_{s_k} + \alpha I)^{-1})(\mathscr{L}(t_k) + \alpha(t_k - s_0) - y^{\delta}) \\
s_{k+1} &= u_k - (\mathscr{L}_{t_k} + \alpha I)^{-1}(\mathscr{L}(u_k) + \alpha(u_k - s_0) - y^{\delta}),
\end{aligned}
$$

and

$$
\begin{aligned}
t_k &= s_k - (\mathscr{L}_{s_k} + \alpha I)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^{\delta}) \\
u_k &= t_k - (\mathscr{L}_{s_k} + \alpha I)^{-1}(2I - (\mathscr{L}_{t_k} + \alpha I) \\
&\quad (\mathscr{L}_{s_k} + \alpha I)^{-1})(\mathscr{L}(t_k) + \alpha(t_k - s_0) - y^{\delta}) \\
s_{k+1} &= u_k - (\mathscr{L}_{u_k} + \alpha I)^{-1}(\mathscr{L}(u_k) + \alpha(u_k - s_0) - y^{\delta}),
\end{aligned}
$$

respectively.

Remark 4.5.1. *We choose α apriorily which satisfies the following condition;*

$$
\Psi(\alpha, y^{\delta}) := \left\| \alpha^2 \left(\mathscr{H}'(s_0) + \alpha I\right)^{-2} \left(\mathscr{H}(s_0) - y^{\delta}\right) \right\| = d\delta \tag{4.5.2}
$$

for some $d > 1$ with $d\delta \leq \|\mathscr{H}(s_0) - y^{\delta}\|$ (see Chapter 3 for details).

For computation we have taken $s^*(t) = t, s_0(t) = 0$ and $y(t) = e^{\frac{t^2}{2}}$. Table 4.2 provides the relative error $E_\alpha = \frac{\|CS - s^*\|}{\|s^*\|}$ of each iterative method, where CS is the computed solution. We choose α according to the parameter strategy (4.5.2). The accuracy of reconstruction increases as the relative error decreases.

For $\delta = 0.001, 0.0001$, the exact and noisy data are shown in Figure (4.1) and Figure (4.3) the computed solution is in Figure (4.2) and Figure (4.4).

4.6 CONCLUSION

We studied the convergence analysis of a three-step Cordero type method of order six and modified it to a new eighth-order iterative method. The convergence analysis of these methods was studied without using Taylor's expansion. We use assumptions based only on the first-order Fréchet derivative. We computed the radius of convergence and computational efficiencies of these methods. Furthermore, we considered analogous iterative methods to solve an ill-posed problem in a Hilbert space. The developed process can also be applied to any other method using inverses of linear operators with the same benefits. This represents the topic of our future study.

Relative Errors for methods (4.1.2), (4.1.3),(4.1.4), and (4.1.5)

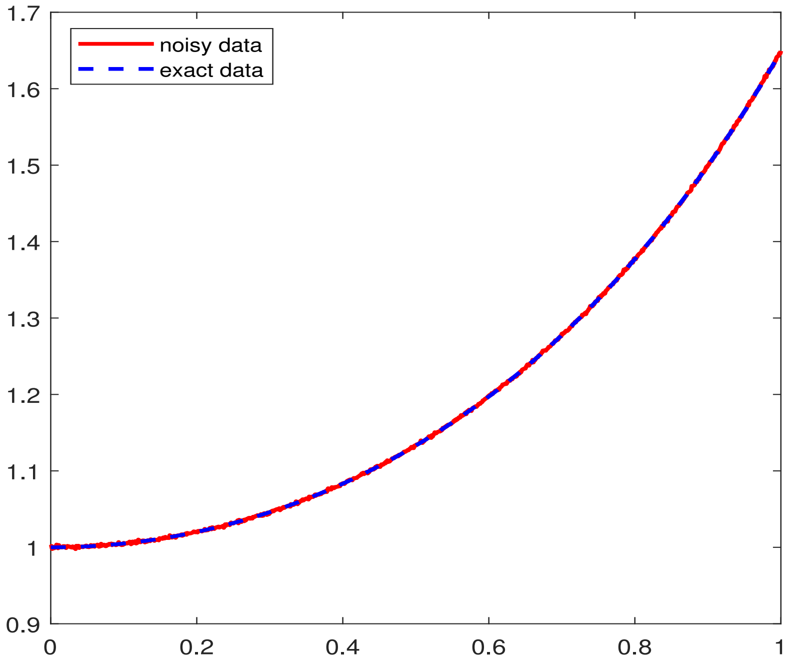

Figure 4.1 Data with $\delta = 0.001$.

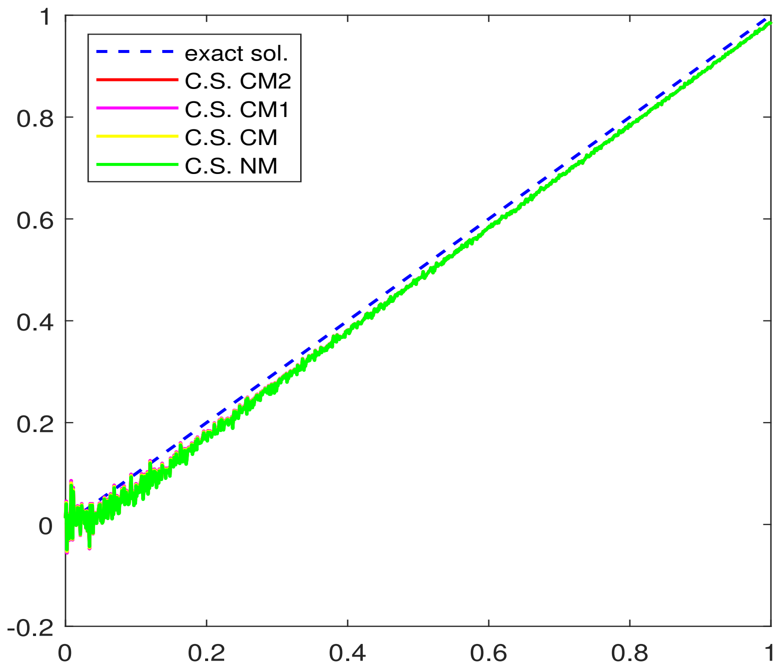

Figure 4.2 Solution with $\delta = 0.001$.

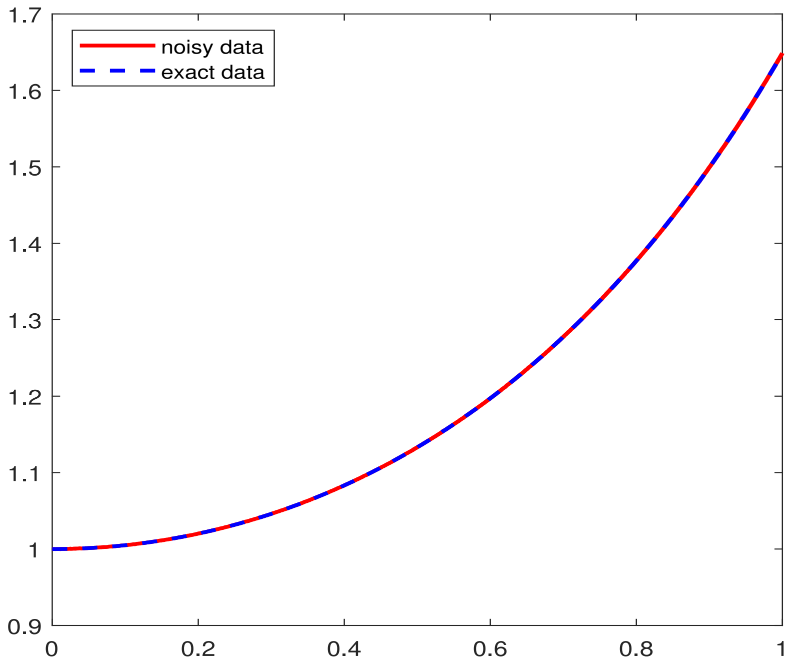

Figure 4.3 Data with $\delta = 0.0001$.

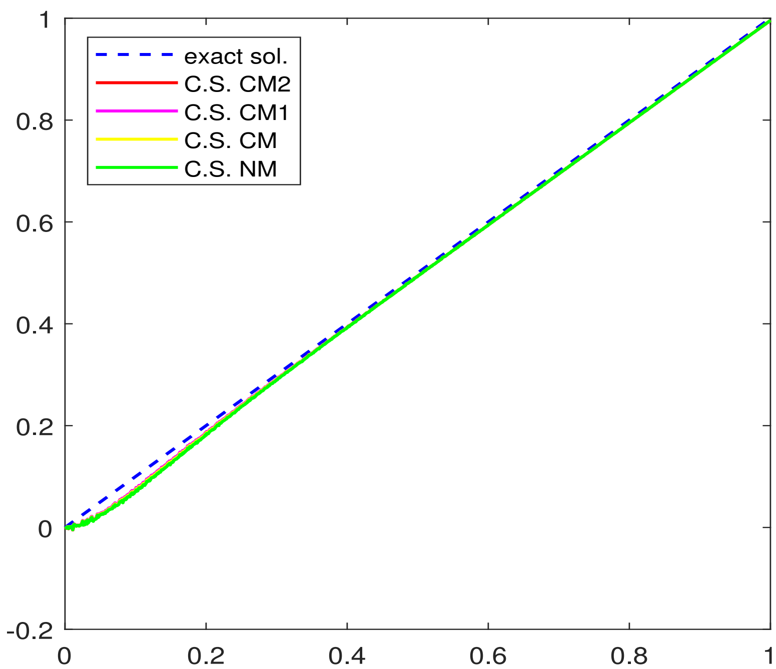

Figure 4.4 Solution with $\delta = 0.0001$.

Table 4.2 Relative errors for Example 3.5.1

Method	α and E_α	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.0001$	$\delta = 0.00001$
	α	3.719646×10^{-2}	1.147951×10^{-2}	3.601858×10^{-3}	1.136730×10^{-3}
(4.1.5)	E_α	1.323726×10^{-1}	3.918406×10^{-2}	1.912899×10^{-2}	1.532976×10^{-2}
stopping index		11	11	11	11
(4.1.4)	E_α	1.323724×10^{-1}	3.918200×10^{-2}	1.912626×10^{-2}	1.532735×10^{-2}
stopping index		15	15	15	15
(4.1.3)	E_α	1.314847×10^{-1}	3.983072×10^{-2}	2.047544×10^{-2}	1.680669×10^{-2}
stopping index		11	11	11	11
(4.1.2)	E_α	1.305499×10^{-1}	4.070776×10^{-2}	2.210599×10^{-2}	1.856438×10^{-2}
stopping index		27	27	27	27

CHAPTER 5

ON NEWTON'S MIDPOINT-TYPE ITERATIVE SCHEME'S CONVERGENCE

5.1 INTRODUCTION

In this paper, we consider a nonlinear equation of the form

$$\mathscr{L}(s) = 0, \tag{5.1.1}$$

where $\mathscr{L} : \Omega \subset \mathscr{U} \to \mathscr{V}$ is an operator between Banach spaces $\mathscr{U}$ and $\mathscr{V}$. Here we assume that $\mathscr{L}$ is a Fréchet differentiable nonlinear operator and Ω is a nonempty open convex set.

The best known single point iterative method used to approximate a solution s^* of (5.1.1) is Newton's method, which converges quadratically. The classical Newton's iterative method (CN) is defined $\forall k = 0, 1, 2, ...$ by

$$s_{k+1} = s_k - \mathscr{L}'(s_k)^{-1}\mathscr{L}(s_k), \tag{5.1.2}$$

where $\mathscr{L}'(s_k)$ is first order Fréchet derivative of the operator $\mathscr{L}$ evaluated at s_k.

To attain the third order convergence, some efficient modifications of classical Newton's method have been introduced. Well known such iterative methods are Arithmetic Newton Method (AN) (also known as Trapezoidal Newton's method)(Weerakoon and Fernando (2000); Özban (2004); Frontini and Sormani (2003); Regmi et al. (2020)), Harmonic mean Newton's method (HN)(Özban (2004))and midpoint Newton's method (MNM),(Özban (2004); Frontini and Sormani (2003, 2004)). Computation of higher order derivatives is not required in these methods unlike other third order classes of Chebyshev-Halley-like iterative procedures (Gutiérrez and Hernández (1997); Zheng and Robbie (1995)).

The convergence analysis of higher order iterative methods used to solve nonlinear equations in Banach space required assumptions on the higher order Fréchet derivatives. Frontini and Sormani developed a third order iterative method, known as the midpoint Newton method (MNM) (Frontini and Sormani (2003, 2004)), from classical Newton's method by employing midpoint integration. Moreover, the convergence analysis of MNM only requires the Fréchet derivative of order up to two, but that of the Arithmetic Newton method in (Weerakoon and Fernando (2000)) relies on the assumptions involving third order Fréchet derivatives. MNM method requires the computations of one function, two first order derivatives, and inversions of two matrices. In (Frontini and Sormani (2003)), Frontini and Sormani proved that Midpoint Newton's method and Arithmetic Newton's method are the most efficient iterative methods among the class of Newton quadrature formula iterative methods. The midpoint Newton's iterative method is defined $\forall k = 0, 1, 2, ...$ by

$$\begin{aligned} t_k &= s_k - \mathscr{L}'(s_k)^{-1}\mathscr{L}(s_k) \\ s_{k+1} &= s_k - \mathscr{L}'\left(\frac{s_k + t_k}{2}\right)^{-1}\mathscr{L}(s_k). \end{aligned} \tag{5.1.3}$$

To enhance the applicability of iterative methods in practical cases, we extended midpoint Newton's method to the following two three step methods of order five and six, respectively. The fifth order iterative method defined for $k = 0, 1, 2, \ldots$ by

$$\begin{aligned} t_k &= s_k - \mathscr{L}'(s_k)^{-1}\mathscr{L}(s_k) \\ u_k &= s_k - \mathscr{L}'\left(\frac{s_k + t_k}{2}\right)^{-1}\mathscr{L}(s_k) \\ s_{k+1} &= u_k - \mathscr{L}'(t_k)^{-1}\mathscr{L}(u_k), \end{aligned} \tag{5.1.4}$$

and the sixth order iterative method defined for $k = 0, 1, 2, \ldots$ by

$$\begin{aligned} t_k &= s_k - \mathscr{L}'(s_k)^{-1}\mathscr{L}(s_k) \\ u_k &= s_k - \mathscr{L}'\left(\frac{s_k + t_k}{2}\right)^{-1}\mathscr{L}(s_k) \\ s_{k+1} &= u_k - \mathscr{L}'(u_k)^{-1}\mathscr{L}(u_k). \end{aligned} \tag{5.1.5}$$

Mostly, the convergence analysis of higher order iterative methods in Banach space uses Taylor's expansion and assumptions on the Fréchet derivatives of higher order. If the higher order derivatives are unbounded, then the assumptions on the higher order derivatives reduce the applicability of these methods. The primary advantage of our convergence analysis is that, we do not use Taylor's expansion and hence, we do not need assumptions on higher order (more than two order) Fréchet derivatives of the operator involved.

Suppose the sequence $\{s_n\}$ converges to s^*. Then, the Taylor expansion (and higher order derivative) is used to prove;

$$\|s_{k+1} - s^*\| \leq \tilde{C}\|s_k - s^*\|^p, \tag{5.1.6}$$

where p is called an order of convergence of the sequence and $\tilde{C}$ is called an asymptotic error constant or rate of convergence. The novelty in our approach is that we obtain the relation (5.1.6) without using Taylor expansion.

The remainder of the Chapter is laid out as follows: We provide the convergence analysis of methods (5.1.4) and (5.1.5) in section 5.2. A Trade off between the computational efficiency and radius of convergence discussed in section 5.3. The numerical examples are given in section 5.4 and Section 5.5 contains the basin of attractions. Finally, Section 5.6 concludes the Chapter.

5.2 LOCAL CONVERGENCE ANALYSIS OF (5.1.4) AND (5.1.5)

Assume that Fréchet derivative of $\mathscr{L}$ satisfies the following assumptions.

(A1) s^* is a simple solution of (5.1.1) and $\mathscr{L}'(s^*)^{-1} \in L(\mathscr{V}, \mathscr{U})$.

(A2) There exists $C_1 > 0$ such that for every $s, t \in \Omega$,

$$\|\mathscr{L}'(s^*)^{-1}(\mathscr{L}'(t) - \mathscr{L}'(u))\| \leq C_1\|t - u\|.$$

(A3) There exists a constant $C_2 > 0$ such that for every $t \in \Omega$,

$$\|\mathscr{L}'(s^*)^{-1}\mathscr{L}''(t)\| \leq C_2.$$

(A4) There exists $C_3 > 0$ such that for every $t \in \Omega$,

$$\|\mathscr{L}'(s^*)^{-1}(\mathscr{L}''(t) - \mathscr{L}''(s^*))\| \leq C_3\|t - s^*\|.$$

Let $h_0, g_0, \phi, \psi : [0, \frac{1}{C_1}) \to \mathbb{R}$ be continuous non decreasing functions defined by

$$\begin{aligned} h_0(\tau) &= \frac{C_1\tau}{2(1-C_1\tau)}, \\ g_0(\tau) &= h_0(\tau)\tau - 1, \\ \phi(\tau) &= \frac{C_1\tau}{2}(1+h_0(\tau)), \\ \psi_(\tau) &= 1-\phi(\tau). \end{aligned}$$

Then, $\psi(0) = 1$, $\psi(\tau) \to \infty$ as $\tau \to \frac{1}{C_1}^-$, so by the Intermediate value theorem ψ has a minimal zero $r_0 \in \left(0, \frac{1}{C_1}\right)$. Let $h_1, h_2, h_3, g_1, g_2, g_3 : [0, r_0) \to \mathbb{R}$ be continuous non decreasing functions defined by

$$\begin{aligned} h_1(\tau) &= \frac{C_3}{2\psi(\tau)}\left(\frac{1}{3}+\frac{h_0(\tau)}{8}\right)+\frac{C_1C_2}{8(1-C_1\tau)\psi(\tau)}, \\ g_1(\tau) &= h_1(\tau)\tau^2 - 1, \\ h_2(\tau) &= \frac{C_1}{1-C_1h_0(\tau)\tau}\left(\frac{h_0(\tau)}{t}+\frac{h_1(\tau)\tau}{2}\right)h_1(\tau) \\ g_2(\tau) &= h_2(\tau)\tau^4 - 1 \\ h_3(\tau) &= \frac{C_1}{2(1-C_1h_1(\tau)\tau^3)}(h_1(\tau))^2 \\ g_3(\tau) &= h_3(\tau)\tau^5 - 1. \end{aligned}$$

Then, $g_1(0) = -1$, $g_1(\tau) \to \infty$ as $\tau \to {r_0}^-$, so by the Intermediate value theorem g_1 has a minimal zero $r_1 \in (0, r_0)$. Similarly, there exists a minimal zero $r_2, r_3 \in (0, r_0)$ for g_2 and g_3 respectively. Let

$$r = \min\left\{1, \frac{2}{3C_1}, r_0, r_1, r_2\right\}. \tag{5.2.1}$$

Next, we provide the convergence analysis of method (5.1.4).

Theorem 5.2.1. *Let r be as in (5.2.1). Then the sequence $\{s_k\}$ defined by (5.1.4) with $s_0 \in B(s^*, r) - \{s^*\}$, converges to s^* such that*

$$\|s_{k+1} - s^*\| \le h_2(r)\|s_k - s^*\|^5.$$

Proof. The proof is by Induction. We prove the following inequalities;

$$\begin{aligned} t_n \in B(s^*, r),\ \|t_n - s^*\| &\le \frac{C_1}{2(1-C_1r)}\|s_n - s^*\|^2. \\ u_n \in B(s^*, r),\ \|u_n - s^*\| &\le h_1(r)\|s_n - s^*\|^3. \\ s_{n+1} \in B(s^*, r),\ \|s_{n+1} - s^*\| &\le h_2(r)\|s_n - s^*\|^5. \end{aligned} \tag{5.2.2}$$

To obtain the above inequalities, we prove that $\mathscr{L}'\left(\frac{s+t}{2}\right)$ and $\mathscr{L}'(s)$ are invertible $\forall s,t \in B(s^*,r)$. By using (A2) we get,

$$\left\|\mathscr{L}'(s^*)^{-1}\left(\mathscr{L}'(s^*)-\mathscr{L}'\left(\frac{s+t}{2}\right)\right)\right\| \leq \frac{C_1}{2}\left(\|s-s^*\|+\|t-s^*\|\right) \quad (5.2.3)$$
$$\leq \frac{C_1}{2}2r \leq C_1 r < 1.$$

By using Banach Lemma on invertible operators and (5.2.3), one can obtain

$$\left\|\mathscr{L}'\left(\frac{s+t}{2}\right)^{-1}\mathscr{L}'(s^*)\right\| \leq \frac{1}{1-\frac{C_1}{2}\left(\|s-s^*\|+\|t-s^*\|\right)}. \quad (5.2.4)$$

Similar manner one can obtain

$$\left\|\mathscr{L}'(s)^{-1}\mathscr{L}'(s^*)\right\| \leq \frac{1}{1-C_1\|s-s^*\|}. \quad (5.2.5)$$

For $s_0 \in B(s^*,r)$, we use Mean value Theorem and obtain

$$\begin{aligned} t_0 - s^* &= s_0 - s^* - \mathscr{L}'(s_0)^{-1}(\mathscr{L}(s_0)-\mathscr{L}(s^*)) \\ &= \mathscr{L}'(s_0)^{-1}\left(\int_0^1 (\mathscr{L}'(s_0)-\mathscr{L}'(s^*+\theta_1(s_0-s^*)))(s_0-s^*)d\theta_1\right) \\ &= \mathscr{L}'(s_0)^{-1}\mathscr{L}'(s^*)\mathscr{L}'(s^*)^{-1} \\ &\quad \times\left(\int_0^1 (\mathscr{L}'(s_0)-\mathscr{L}'(s^*+\theta_1(s_0-s^*)))(s_0-s^*)d\theta_1\right). \end{aligned}$$

Thus by (5.2.5) and $(A2)$ we get,

$$\begin{aligned} \|t_0-s^*\| &\leq \|\mathscr{L}'(s_0)^{-1}\mathscr{L}'(s^*)\|\left\|\mathscr{L}'(s^*)^{-1}\int_0^1 (\mathscr{L}'(s_0)-\mathscr{L}'(s^*+\theta_1(s_0-s^*)))\right. \\ &\quad \left.\times(s_0-s^*)d\theta_1\right\| \\ &\leq \frac{1}{1-C_1\|s_0-s^*\|}\int_0^1 \left\|\mathscr{L}'(s^*)^{-1}(\mathscr{L}'(s_0)-\mathscr{L}'(s^*+\theta_1(s_0-s^*)))\right\| \\ &\quad \times\|s_0-s^*\|d\theta_1, \qquad \text{by (5.2.5)} \\ &\leq \frac{C_1\|s_0-s^*\|}{1-C_1\|s_0-s^*\|}\int_0^1 \|s_0-(s^*+\theta_1(s_0-s^*))\|d\theta_1, \quad \text{by} \quad \text{(A2)} \\ &\leq \frac{C_1\|s_0-s^*\|}{1-C_1\|s_0-s^*\|}\int_0^1 |1-\theta_1|\|s_0-s^*\|d\theta_1 \\ &= \frac{C_1\|s_0-s^*\|^2}{2(1-C_1\|s_0-s^*\|)} \qquad (5.2.6) \\ &= h_0(\|s_0-s^*\|)\|s_0-s^*\| < r. \end{aligned}$$

Since $r < \frac{2}{3C_1}$, we have $t_0 \in B(s^*, r)$. Again by the second substep of method (5.1.4),

$$\begin{aligned}
u_0 - s^* &= s_0 - s^* - \mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1}(\mathscr{L}(s_0) - \mathscr{L}(s^*)) \\
&= \mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1}\int_0^1\left(\mathscr{L}'\left(\frac{s_0+t_0}{2}\right) - \mathscr{L}'(s^* + \theta_1(s_0 - s^*))\right) \\
&\quad \times (s_0 - s^*)d\theta_1. \\
&= \mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1}\int_0^1\int_0^1 \mathscr{L}''(s^* + \theta_1(s_0 - s^*) \\
&\quad + \left(\theta_2\left(\frac{s_0+t_0}{2} - (s^* + \theta_1(s_0 - s^*))\right)\right) \\
&\times \left(\frac{s_0+t_0}{2} - (s^* + \theta_1(s_0 - s^*)\right)(s_0 - s^*)d\theta_2 d\theta_1. \qquad (5.2.7)
\end{aligned}$$

Let

$$\eta(\theta_1, \theta_2) = s^* + \theta_1(s_0 - s^*) + \theta_2\left(\frac{s_0+t_0}{2} - (s^* + \theta_1(s_0 - s^*))\right). \qquad (5.2.8)$$

Also note that,

$$\frac{s_0+t_0}{2} - (s^* + \theta_1(s_0 - s^*)) = \frac{1}{2}((1-2\theta_1)(s_0 - s^*) + (t_0 - s^*)). \qquad (5.2.9)$$

Then from (5.2.7),(5.2.8) and (5.2.9) we can write,

$$\begin{aligned}
u_0 - s^* &= \frac{1}{2}\mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1}\int_0^1\int_0^1 \mathscr{L}''(\eta(\theta_1,\theta_2))((1-2\theta_1)(s_0 - s^*) \\
&\quad +(t_0 - s^*))(s_0 - s^*))d\theta_2 d\theta_1. \\
&= \Gamma_1 + \Gamma_2, \qquad (5.2.10)
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_1 &= \frac{1}{2}\mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1}\int_0^1\int_0^1 \mathscr{L}''(\eta(\theta_1,\theta_2))(1-2\theta_1)(s_0 - s^*)^2 d\theta_2 d\theta_1, \\
\Gamma_2 &= \frac{1}{2}\mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1}\int_0^1\int_0^1 \mathscr{L}''(\eta(\theta_1,\theta_2))(t_0 - s^*)(s_0 - s^*)d\theta_2 d\theta_1.
\end{aligned}$$

Consider,

$$\|\Gamma_1\| = \left\|\frac{1}{2}\mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1}\int_0^1\int_0^1 \mathscr{L}''(\eta(\theta_1,\theta_2))(1-2\theta_1)(s_0 - s^*)^2 d\theta_2 d\theta_1\right\|$$

$$
\begin{aligned}
&= \left\| \frac{1}{2}\mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1} \int_0^1\int_0^1 (\mathscr{L}''(\eta(\theta_1,\theta_2)) - \mathscr{L}''(s^*) \right.\\
&\quad \left. + \mathscr{L}''(s^*))(1-2\theta_1)(s_0-s^*)^2 d\theta_2 d\theta_1 \right\| \\
&= \left\| \frac{1}{2}\mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1} \int_0^1\int_0^1 (\mathscr{L}''(\eta(\theta_1,\theta_2)) - \mathscr{L}''(s^*)) \right.\\
&\quad \times (1-2\theta_1)(s_0-s^*)^2 d\theta_2 d\theta_1 \\
&\quad \left. + \frac{1}{2}\mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1} \int_0^1\int_0^1 \mathscr{L}''(s^*)(1-2\theta_1)(s_0-s^*)^2 d\theta_2 d\theta_1 \right\| \\
&\leq \frac{1}{2}\int_0^1\int_0^1 \left\| \mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1} \mathscr{L}'(s^*) \right\| \|\mathscr{L}'(s^*)^{-1}(\mathscr{L}''(\eta(\theta_1,\theta_2)) - \mathscr{L}''(s^*)\| \\
&\quad \left(|1-2\theta_1|\|s_0-s^*\|^2\right) d\theta_1 d\theta_2 \\
&\quad + \left\| \frac{1}{2}\mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1} \mathscr{L}''(s^*) \int_0^1\int_0^1 (1-2\theta_1)(s_0-s^*) d\theta_1 d\theta_2 \right\| \\
&\leq \frac{1}{2}\int_0^1\int_0^1 \frac{C_3}{1-\frac{C_1}{2}\left(\|s_0-s^*\|+\|t_0-s^*\|\right)} \|\eta(\theta_1,\theta_2)-s^*\| \\
&\quad \times |1-2\theta_1|\|s_0-s^*\|^2 d\theta_2 d\theta_1, \qquad \text{by (A4) and (5.2.4)} \\
&= \frac{C_3}{2\left(1-\frac{C_1}{2}\left(\|s_0-s^*\|+\|t_0-s^*\|\right)\right)} \int_0^1\int_0^1 \left\| \theta_1(s_0-s^*) \right.\\
&\quad \left. + \theta_2\left(\frac{s_0+t_0}{2} - (s^*+\theta_1(s_0-s^*))\right) \right\| \\
&\quad \times \left(|1-2\theta_1|\|s_0-s^*\|^2\right) d\theta_2 d\theta_1, \qquad \text{by (5.2.8)} \\
&= \frac{C_3}{2\left(1-\frac{C_1}{2}\left(\|s_0-s^*\|+\|t_0-s^*\|\right)\right)} \int_0^1\int_0^1 \left\| \theta_1(s_0-s^*) \right.\\
&\quad \left. + \frac{\theta_2}{2}(1-2\theta_1)(s_0-s^*) + \frac{\theta_2}{2}(t_0-s^*) \right\| \\
&\quad \times \left(|1-2\theta_1|\|s_0-s^*\|^2\right) d\theta_2 d\theta_1, \qquad \text{by (5.2.9)} \\
&\leq \frac{C_3}{2\left(1-\frac{C_1}{2}\left(\|s_0-s^*\|+\|t_0-s^*\|\right)\right)} \int_0^1\int_0^1 \left(|\theta_1|\|s_0-s^*\| \right.\\
&\quad \left. + \frac{|\theta_2|}{2}|1-2\theta_1|\|s_0-s^*\| + \frac{|\theta_2|}{2}\|t_0-s^*\| \right) \\
&\quad \times \left(|1-2\theta_1|\|s_0-s^*\|^2\right) d\theta_2 d\theta_1 \\
&\leq \frac{C_3}{2\left(1-\frac{C_1}{2}\left(\|s_0-s^*\|+\frac{C_1\|s_0-s^*\|^2}{2(1-C_1\|s_0-s^*\|)}\right)\right)} \\
&\quad \times \int_0^1\int_0^1 |\theta_1||1-2\theta_1|\|s_0-s^*\|^3 + |\theta_2||1-2\theta_1|^2\|s_0-s^*\|^3
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{|\theta_2||1-2\theta_1|}{2}\frac{C_1\|s_0-s^*\|^4}{2(1-C_1\|s_0-s^*\|)}d\theta_2 d\theta_1, \qquad \text{by (5.2.6)}\\
&= \frac{C_3}{2\left(1-\frac{C_1}{2}\left(\|s_0-s^*\|+\frac{C_1\|s_0-s^*\|^2}{2(1-C_1\|s_0-s^*\|)}\right)\right)}\\
&\quad \times\left(\frac{1}{3}+\frac{C_1\|s_0-s^*\|}{16(1-C_1\|s_0-s^*\|)}\right)\|s_0-s^*\|^3 \qquad (5.2.11)
\end{aligned}
$$

Also by (A3) and (5.2.6) we get,

$$
\begin{aligned}
\|\Gamma_2\| &= \frac{1}{2}\left\|\int_0^1\int_0^1 \mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1}\mathscr{L}''(\eta(\theta_1,\theta_2))(t_0-s^*)(s_0-s^*)d\theta_2 d\theta_1\right\|\\
&= \frac{1}{2}\left\|\int_0^1\int_0^1 \mathscr{L}'\left(\frac{s_0+t_0}{2}\right)^{-1}\mathscr{L}'(s^*)\mathscr{L}'(s^*)^{-1}\mathscr{L}''(\eta(\theta_1,\theta_2))(t_0-s^*)\right.\\
&\quad \times(s_0-s^*)d\theta_2 d\theta_1\|\\
&\leq \frac{C_2}{2\left(1-\frac{C_1}{2}(\|s_0-s^*\|+\|t_0-s^*\|)\right)}\|t_0-s^*\|\|s_0-s^*\|\\
&= \frac{C_1C_2}{8\left(1-\frac{C_1}{2}\left(\|s_0-s^*\|+\frac{C_1\|s_0-s^*\|^2}{2(1-C_1\|s_0-s^*\|)}\right)\right)}\frac{\|s_0-s^*\|^3}{(1-C_1\|s_0-s^*\|)}. \qquad (5.2.12)
\end{aligned}
$$

Therefore by (5.2.10), (5.2.11) and (5.2.12) we get,

$$
\begin{aligned}
\|u_0-s^*\| &\leq \left(\frac{C_3}{2\left(1-\frac{C_1}{2}\left(\|s_0-s^*\|+\frac{C_1\|s_0-s^*\|^2}{2(1-C_1\|s_0-s^*\|)}\right)\right)}\right.\\
&\quad \times\left(\frac{1}{3}+\frac{C_1\|s_0-s^*\|}{16(1-C_1\|s_0-s^*\|)}\right)\\
&\quad \left.+\frac{C_1C_2}{8\left(1-\frac{C_1}{2}\left(\|s_0-s^*\|+\frac{C_1\|s_0-s^*\|^2}{2(1-C_1\|s_0-s^*\|)}\right)\right)(1-C_1\|s_0-s^*\|)}\right)\\
&\quad \times\|s_0-s^*\|^3, \qquad (5.2.13)\\
&= g_1(\|s_0-s^*\|)\|s_0-s^*\| < r.
\end{aligned}
$$

so $u_0 \in B(s^*, r)$. By the third step of method (5.1.4), we have

$$
\begin{aligned}
s_1-s^* &= u_0-s^*-\mathscr{L}'(t_0)^{-1}(\mathscr{L}(u_0)-\mathscr{L}(s^*))\\
&= \mathscr{L}'(t_0)^{-1}\left(\int_0^1\left(\mathscr{L}'(t_0)-\mathscr{L}'(s^*+\theta_1(u_0-s^*))\right)(u_0-s^*)d\theta_1\right).\\
&= \mathscr{L}'(t_0)^{-1}\mathscr{L}'(s^*)\mathscr{L}'(s^*)^{-1}\Big(\int_0^1\left(\mathscr{L}'(t_0)-\mathscr{L}'(s^*+\theta_1(u_0-s^*))\right)
\end{aligned}
$$

$$(u_0 - s^*)d\theta_1\Big).$$

By using Assumption , we get

$$\begin{aligned}
\|s_1 - s^*\| &\leq \|\mathscr{L}'(t_0)^{-1}\mathscr{L}'(s^*)\| \left\| \int_0^1 \mathscr{L}'(s^*)^{-1} \left(\mathscr{L}'(t_0) - \mathscr{L}'(s^* + \theta_1(u_0 - s^*))\right) \right. \\
&\quad (u_0 - s^*)d\theta_1\| \\
&\leq \frac{C_1}{1 - C_1(\|t_0 - s^*\|)} \int_0^1 \|t_0 - (s^* + \theta_1(u_0 - s^*))\| \|u_0 - s^*\| d\theta_1, \\
&\qquad \text{by (A2) and (5.2.5),} \\
&\leq \frac{C_1}{1 - C_1(\|t_0 - s^*\|)} \left(\|t_0 - s^*\| + \frac{\|u_0 - s^*\|}{2} \right) \|u_0 - s^*\| \\
&= \frac{C_1}{1 - C_1\left(\frac{C_1\|s_0 - s^*\|^2}{2(1 - C_1\|s_0 - s^*\|)}\right)} \left(\frac{C_1}{2(1 - C_1\|s_0 - s^*\|)} \|s_0 - s^*\|^2 \right. \\
&\quad \left. + \frac{h_1(\|s_0 - s^*\|)}{2} \|s_0 - s^*\|^3 \right) h_1(\|s_0 - s^*\|) \|s_0 - s^*\|^3, \\
&\qquad \text{by (5.2.6) and (5.2.13)} \\
&= \frac{C_1}{1 - C_1\left(h_0(\|s_0 - s^*\|)\|s_0 - s^*\|\right)} \\
&\quad \times \left(\frac{h_0(\|s_0 - s^*\|)}{\|s_0 - s^*\|)} + \frac{h_1(\|s_0 - s^*\|)}{2} \|s_0 - s^*\| \right) h_1(\|s_0 - s^*\|) \\
&\quad \times \|s_0 - s^*\|^5 \qquad (5.2.14) \\
&= g_2(\|s_0 - s^*\|)\|s_0 - s^*\| < r.
\end{aligned}$$

Thus, $s_1 \in B(s^*, r)$. The induction is complete, by replacing s_0, t_0, u_0, s_1 by s_n, t_n, u_n, s_{n+1}, respectively in the preceeding arguments. □

Next, we prove the convergence for method (5.1.5).

Let

$$\bar{r} = \min\left\{1, \frac{2}{3C_1}, r_0, r_1, r_3\right\}. \qquad (5.2.15)$$

Theorem 5.2.2. *Let $\bar{r}$ be as in (5.2.15). Then the sequence $\{s_k\}$ defined by (5.1.5) with $s_0 \in B(s^*, \bar{r}) - \{s^*\}$, converges to s^* such that*

$$\|s_{k+1} - s^*\| \leq h_3(r)\|s_k - s^*\|^6.$$

Proof. Note that by the third substep of (5.1.5), we have

$$\begin{aligned}
s_1 - s^* &= u_0 - s^* - \mathscr{L}'(u_0)^{-1}(\mathscr{L}(u_0) - \mathscr{L}(s^*)) \\
&= -\mathscr{L}'(u_0)^{-1} \left(\int_0^1 (\mathscr{L}'(s^* + \theta_1(u_0 - s^*)) - \mathscr{L}'(u_0))(u_0 - s^*)d\theta_1 \right),
\end{aligned}$$

so, by Assumption (A2) and (5.2.13), we get

$$\begin{aligned}\|s_1 - s^*\| &\leq \frac{C_1}{2(1 - C_1\|u_0 - s^*\|)}\|u_0 - s^*\|^2 \\ &= \frac{C_1}{2(1 - C_1 h_1(\|s_0 - s^*\|)\|s_0 - s^*\|^3)}(h_1(\|s_0 - s^*\|))^2\|s_0 - s^*\|^6 \\ &= g_3(\|s_0 - s^*\|)\|s_0 - s^*\| < \bar{r}.\end{aligned}$$

The remaining part of the proof proceeds in the same manner as Theorem 5.2.1. □

Proposition 5.2.3. *Suppose $\mathscr{L}(s) = 0$ has a simple solution in the set $S = \Omega \cap \overline{B(s^*, r)}$ and Assumption (A2) holds. Then s^* is the unique solution of equation $\mathscr{L}(s) = 0$ in the set S, provided $C_1 r < 2$.*

Proof. Let $\tilde{s}$ be a solution of the equation $\mathscr{L}(s) = 0$ in the set S. We are setting $A = \int_0^1 \mathscr{L}'(s^* + \theta_1(\tilde{s} - s^*))d\theta_1$. By using Assumption (A2), we have

$$\begin{aligned}\|\mathscr{L}'(s^*)^{-1}(A - \mathscr{L}'(s^*))\| &\leq C_1 \int_0^1 \|s^* + \theta_1(\tilde{s} - s^*) - s^*\| d\theta_1 \\ &= C_1 \int_0^1 \theta_1 \|\tilde{s} - s^*\| d\theta_1 \\ &\leq \frac{C_1}{2} r < 1.\end{aligned}$$

By using Banach lemma Argyros (2008), one can conclude that A is invertible. Therefore, $\tilde{s} = s^*$ follows from $0 = \mathscr{L}(\tilde{s}) - \mathscr{L}(s^*) = A(\tilde{s} - s^*)$. □

5.3 COMPUTATIONAL EFFICIENCY AND RADIUS OF CONVERGENCE

By Definition 4.4.3 and Definition 4.4.4, we obtained the informational efficiency and computational efficiency of method (5.1.4) as $5/5 = 1$ and $5^{1/5} = 1.3797$, respectively. But, that of (5.1.5) are $6/5 = 1.2$ and $6^{1/5} = 1.4309$, respectively. One can see that order of convergence, computational efficiency and informational efficiency of methods (5.1.5) are better than that of method (5.1.4). In this section, we estimate the radius of convergence of some examples to validate the theoretical results in previous section.

Example 5.3.1. *Let $\mathscr{U} = \mathscr{V} = \mathbb{R}, \Omega = [k, 2 - k], k \in (2 - \sqrt{2}, 1)$ and $\mathscr{L} : \Omega \to \mathscr{V}$ be* *defined by*

$$\mathscr{L}(s) = s^3 - k.$$

Here, $s^* = k^{1/3}$. $C_1 = \frac{2(2-k)}{k^{2/3}}, C_2 = \frac{2(2-k)}{k^{2/3}}$, *and* $C_3 = \frac{2}{k^{2/3}}$. *For* $k = 1$, *from Table 5.1, we obtain of* $r = 0.329122$, *and* $\bar{r} = 0.333333$.

Example 5.3.2. *Let* $\mathscr{U} = \mathscr{V} = \mathbb{R}^3, D = \bar{B}(0,1), s^* = (0,0,1)^T$. *Define function* $\mathscr{L}$ *on* D *for* $s = (a_1, a_2, a_3)^T$ *by*

$$\mathscr{L}(s) = \left(sina_1, \frac{(a_2)^2}{5} + a_2, a_3 \right)^T.$$

Then, the Fréchet-derivatives are given by

$$\mathscr{L}'(s) = \begin{bmatrix} cosa_1 & 0 & 0 \\ 0 & \frac{2a_2}{5} + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

and

$$\mathscr{L}''(s) = \left[\begin{array}{ccc|ccc|ccc} -sina_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Then, $C_1 = C_3 = 1$ *and* $C_2 = \frac{2}{5}$. *Therefore,* $r = \bar{r} = 0.666667$.

Example 5.3.3. *Define* $\mathscr{L}$ *on* $\Omega = [-1, 1]$ *as*

$$\mathscr{L}(s) = sins$$

$s^* = 0$. *We obtain* $C_1 = C_2 = C_3 = 1$. *Consequently,* $r = 0.656735$, *and* $\bar{r} = 0.666667$.

Example 5.3.4. *Let* $\mathscr{U} = \mathscr{V} = \mathbb{R}, \mathscr{L} : [-\frac{1}{2}, \frac{5}{2}] \to \mathbb{R}$ *defined by*

$$\mathscr{L}(s) = \begin{cases} s^3 log(s^2) + s^5 - s^4 & s \neq 0 \\ 0 & s = 0. \end{cases}$$

Then we have $s^* = 1$ *and* $C_1 = C_2 = C_3 = 44.4234$. *Thus we get* $r = 0.014849$ *and* $\bar{r} = 0.015007$.

Table 5.1 Parameters of the Examples 5.3.1, 5.3.2, 5.3.3 and 5.3.4

Examples	$\frac{2}{3C_1}$	r_0	r_1	r_2	r_3
Example 5.3.1	0.333333	0.381966	0.353932	0.329122	0.352868
Example 5.3.2	0.666667	0.7639324	0.739260	0.685814	0.737396
Example 5.3.3	0.666667	0.763932	0.707208	0.656735	0.705050
Example 5.3.4	0.015007	0.017197	0.015948	0.014849	0.015901

Next in Table (5.2), we have tabulated the ACOC of the iterative methods for the examples considered in the previous Chapter to ensure we can obtain the order of convergence of the proposed schemes computationally. We used stopping criterion $\|x_{k+1}-x_k\|+\|\mathscr{L}(s_{k+1})\| \leq 10^{-8}$.

5.4 AN ILLUSTRATION OF BASIN OF ATTRACTION

Basin of attraction is an effective tool for evaluating the stability and reliability of the iterative methods. Basins of attraction is a set of initial points from which the iterative method converges to a solution of an equation. In the literature, many authors compared the order of convergence and basin of attraction of the iterative methods using graphic visualization (Amat et al. (2004); Chun et al. (2012); Cordero et al. (2014); Scott et al. (2011); Varona (2002)).

Informally, the Fatou set of a function consists of values which behave similarly under repeated iteration of the function, and the Julia set consists of values which change drastically under small perturbation. Recall that for a sequence $\{s_k\}$ produced by the above methods starting with s_0 converging to s^* , the set $S=\{s_0 \in \mathbb{R}^n : s_k$ converges to the zero s^* as i tends to $\infty\}$ is called the **Basins of Attraction (BA)** or **Fatou sets** (Magreñán and Gutiérrez (2015)) and S^c the complement of S is known as a **Julia set**. We considered two system of non linear polynomial equations in two variables and illustrated (Figure 5.1 - Figure 5.6) the basins of attractions associated to the system using iterative methods (5.1.3), (5.1.4), and (5.1.5).

Example 5.4.1.

$$3s^2t-t^3=0$$
$$s^3-3st^2-1=0$$

with solutions $\{(\frac{-1}{2},\frac{-\sqrt{3}}{2}),(\frac{-1}{2},\frac{\sqrt{3}}{2}),(1,0)\}$.

Example 5.4.2.

$$s^3 - t = 0$$
$$t^3 - s = 0$$

with solutions $(-1,-1),(0,0),(1,1)$.

Let $R = \{(s,t) \in \mathbb{R}^2, -2 \leq s,t \leq 2\}$ be a rectangular region which contains all the roots of the examples. As initial points (say s_0) we choose an equidistant grid of 401×401 points in R with constant tolerance 10^{-8} and also we perform maximum of 200 iterations. Different colors are assigned to each attracting basins corresponding to different roots. We used black color to indicate the initial points which does not converge to any of the roots within the desired tolerance and fixed iterations. This black region in sub figures indicate the Julia set.

The figure presented in this work is performed in a 4-core 64bit Windows machine with Intel Core i5 processor using MATLAB programming language.

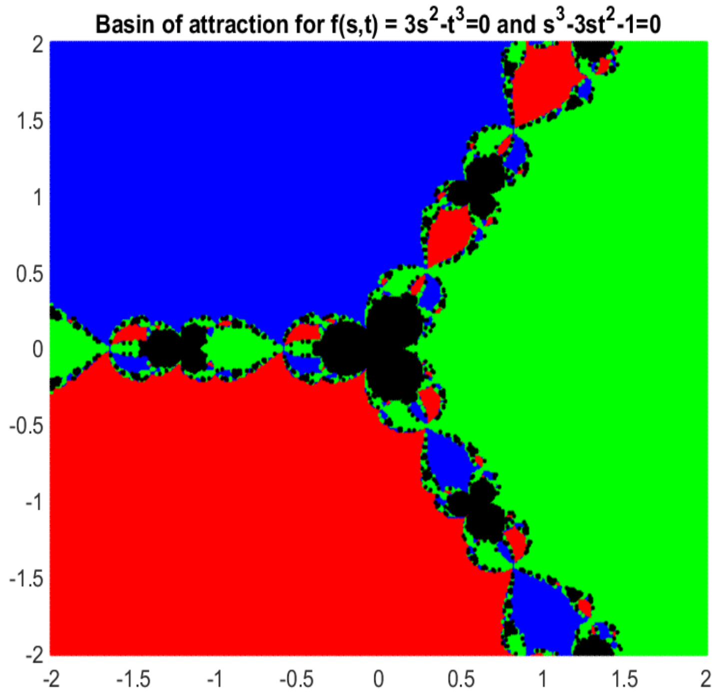

Figure 5.1 Dynamical plane of the method (5.1.3) with basins of attraction for the Example 5.4.1.

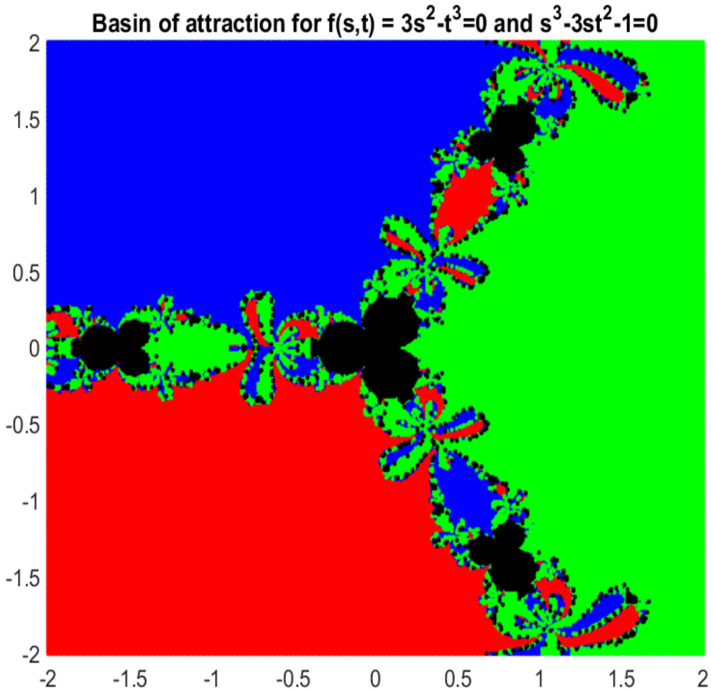

Figure 5.2 Dynamical plane of the method (5.1.4) with basins of attraction for the Example 5.4.1.

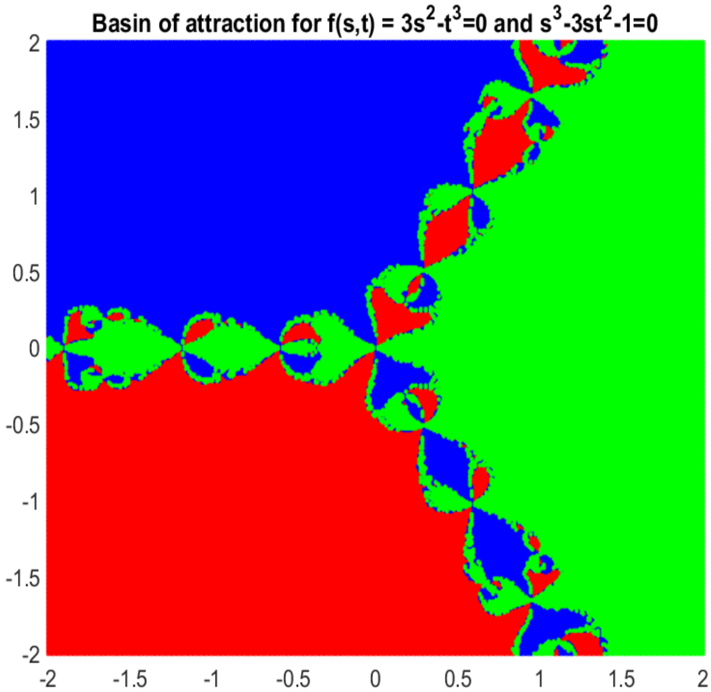

Figure 5.3 Dynamical plane of the method (5.1.5) with basins of attraction for the Example 5.4.1.

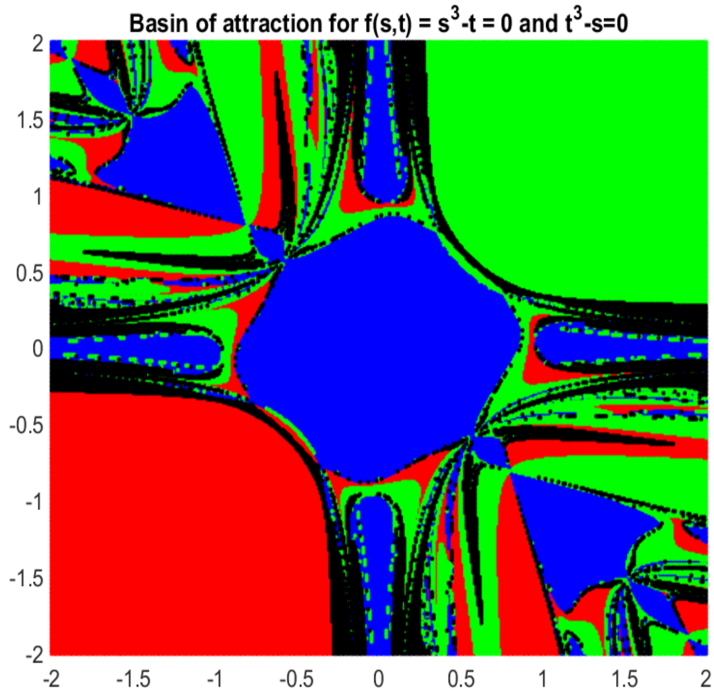

Figure 5.4 Dynamical plane of the method (5.1.3) with basins of attraction for the Example 5.4.2.

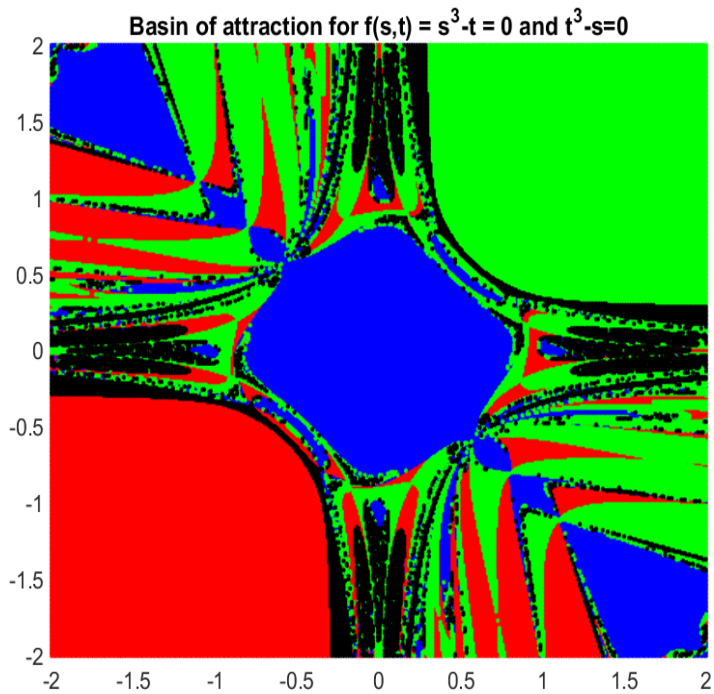

Figure 5.5 Dynamical plane of the method (5.1.4) with basins of attraction for the Example 5.4.2.

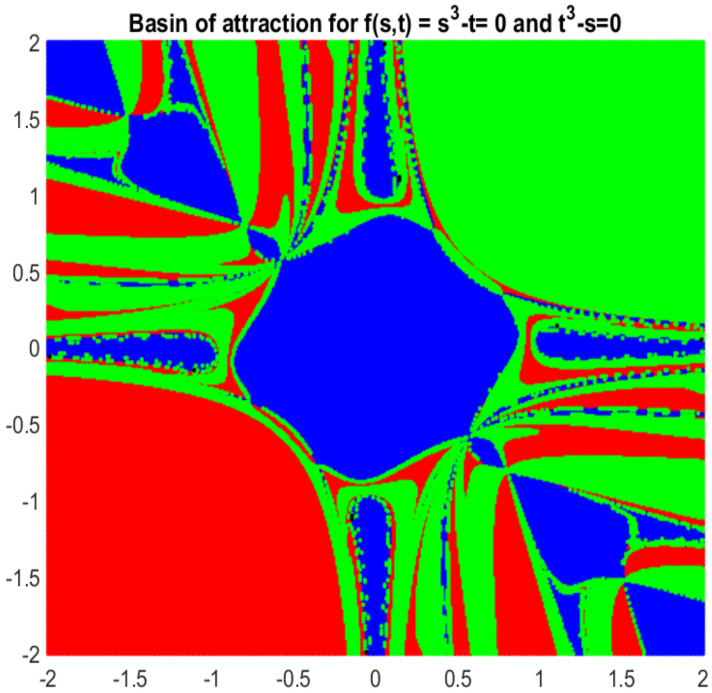

Figure 5.6 Dynamical plane of the method (5.1.5) with basins of attraction for the Example 5.4.2.

5.5 APPLICATION TO ILL-POSED PROBLEM

We consider the ill-posed problem (3.5.3) replacing the variable u with s. We use Lavrentiev regularization method with $\alpha > 0$ as in (4.5.1). The analogous of the iterative methods (5.1.2),(5.1.3), (5.1.4)and(5.1.5) are defined as follows,

$$s_{k+1} = s_k - (\mathscr{L}'(s_k) + \alpha I)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^\delta),$$

$$\begin{aligned} t_k &= s_k - (\mathscr{L}'(s_k) + \alpha I)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^\delta) \\ s_{k+1} &= s_k - \left(\mathscr{L}'\left(\frac{s_k + t_k}{2}\right) + \alpha I\right)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^\delta), \end{aligned}$$

$$\begin{aligned} t_k &= s_k - (\mathscr{L}'(s_k) + \alpha I)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^\delta) \\ u_k &= s_k - \left(\mathscr{L}'\left(\frac{s_k + t_k}{2}\right) + \alpha I\right)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^\delta) \\ s_{k+1} &= u_k - (\mathscr{L}'(t_k) + \alpha I)^{-1}(\mathscr{L}(u_k) + \alpha(u_k - s_0) - y^\delta), \end{aligned}$$

and

$$\begin{aligned} t_k &= s_k - (\mathscr{L}'(s_k) + \alpha I)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^\delta) \\ u_k &= s_k - \left(\mathscr{L}'\left(\frac{s_k + t_k}{2}\right) + \alpha I\right)^{-1}(\mathscr{L}(s_k) + \alpha(s_k - s_0) - y^\delta) \\ s_{k+1} &= u_k - (\mathscr{L}'(u_k) + \alpha I)^{-1}(\mathscr{L}(u_k) + \alpha(u_k - s_0) - y^\delta), \end{aligned}$$

respectively. Table 5.3 provides the relative error $E_\alpha = \frac{\|CS - s^*\|}{\|s^*\|}$ of each iterative method, where CS is the computed solution. For $\delta = 0.001, 0.0001$, the exact and noisy data are shown the following figures.

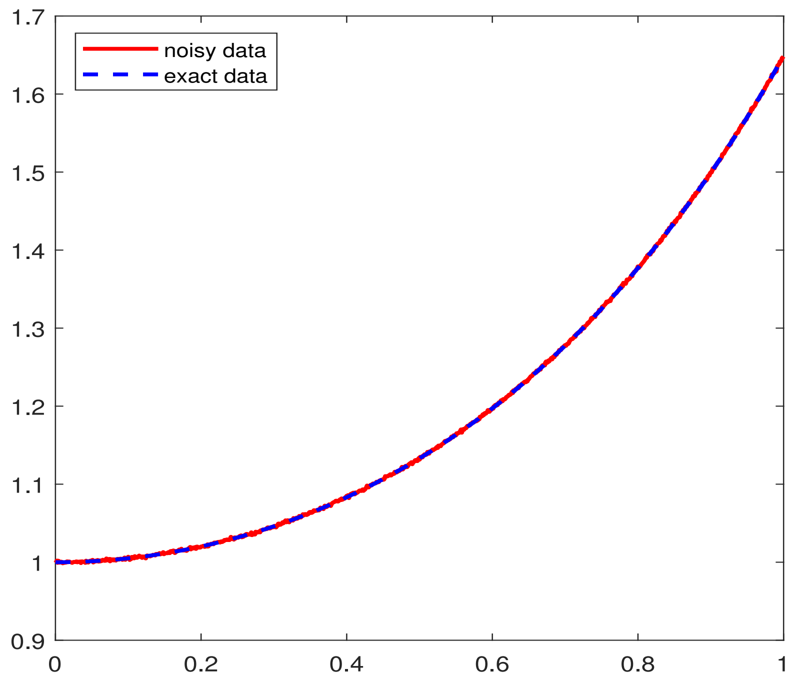

Figure 5.7 Data with $\delta = 0.001$.

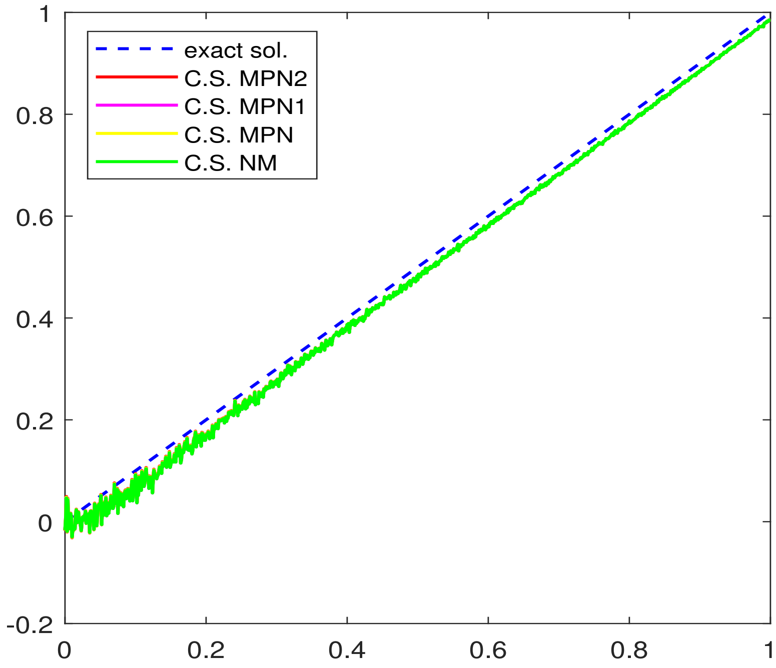

Figure 5.8 Solution with $\delta = 0.001$.

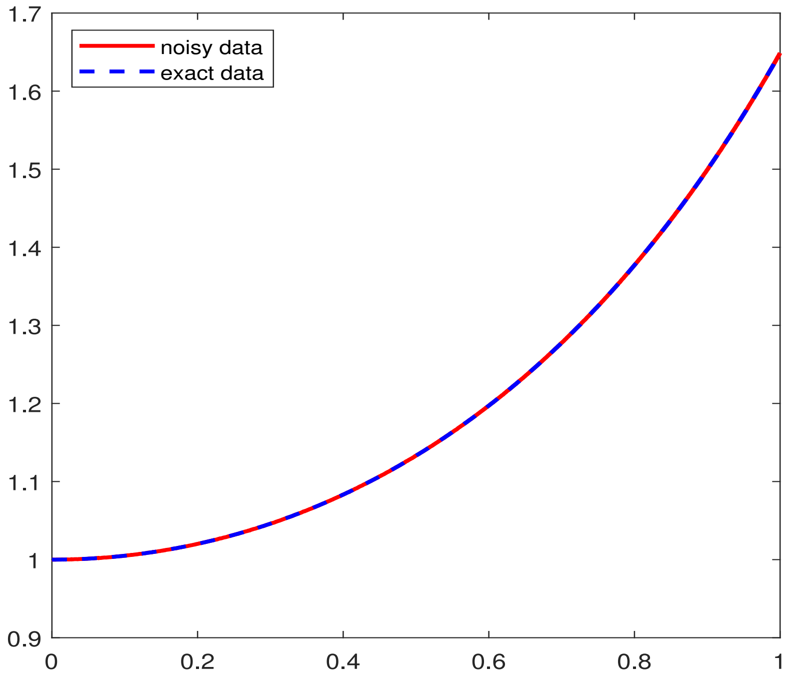

Figure 5.9 Data with $\delta = 0.0001$.

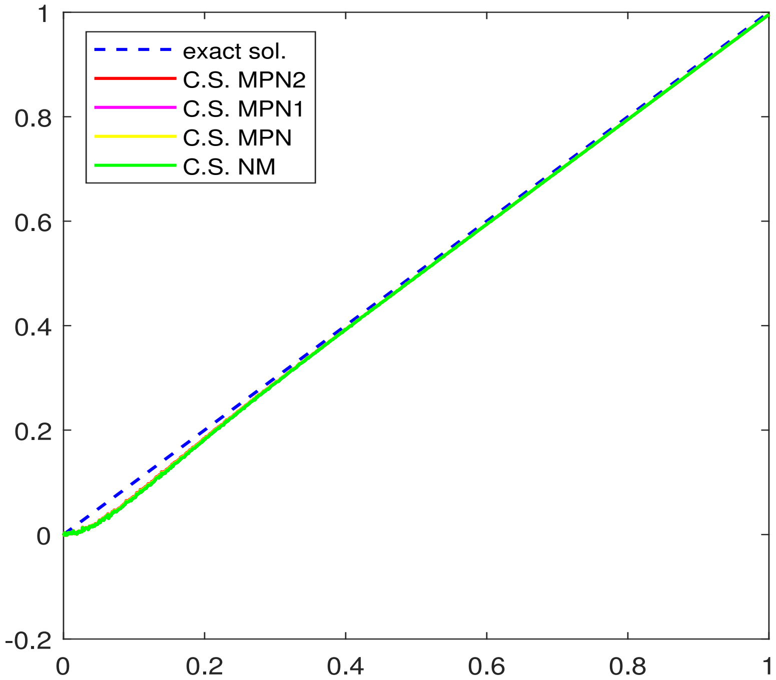

Figure 5.10 Solution with $\delta = 0.0001$.

5.6 CONCLUSION

We introduced two Newton's midpoint-type methods of order five and six. Unlike other higher order iterative methods, the convergence analysis of these methods studied without using Taylor's expansion. We use assumptions based on Fréchet derivative upto order two. We computed the radius of convergence and computational efficiencies of these methods. Furthermore, we considered analogous iterative methods to solve an ill-posed problem in a Hilbert space. The developed process can also be applied to any other method using inverses of linear operators with the same benefits. This represents the topic of our future study.

Table 5.2 ACOC for methods (5.1.2), (5.1.3), (5.1.4) and (5.1.5).

Eq. No.	x^*	x_0	$\Sigma_1(N)$ (5.1.2) (5.1.2)	$\Sigma_1(N)$ MPN (5.1.3)	$\Sigma_1(N)$ MPN1 (5.1.4)	$\Sigma_1(N)$ MPN2 (5.1.5)
(4.4.1)	$(0,0,0)$	$(0.5,0.5,0.5)$ $(0.03,0.03,0.03)$	2(7) 1.96(5)	2.9(5) 2.8(4)	4.2(4) 4.8(4)	5.24(4) ND(3)
(4.4.2)	$(0.3542,1.1364)$	$(0.6,1)$	2(9)	2.7(6)	5(5)	5.9(4)
(4.4.3)	$(\frac{1}{2},\frac{\sqrt{3}}{2})$	$(0.5,0.5)$	2(6)	3(5)	5.3(4)	4.6(4)
(4.4.4)	$(1,1)$	$(-1.5.,-0.5)$ $(1.2,1.2)$	2(6) 2(6)	3(6) 2.8(4)	5.5(5) 4.8(4)	6.1(4) ND(3)
(4.4.5)	$(-\frac{1}{2},-\frac{\sqrt{3}}{2})$	$(-\frac{1}{2},\frac{1}{2})$	2(7)	3(5)	5.6(4)	6.4(4)

Table 5.3 Relative errors for Example 3.5.1

Method	α and E_α	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.0001$	$\delta = 0.00001$
	α	3.721045×10^{-2}	1.147853×10^{-2}	3.601906×10^{-3}	1.136730×10^{-3}
(5.1.5) stopping index	E_α	1.367671×10^{-1} 14	3.852467×10^{-2} 15	2.058719×10^{-2} 15	1.680118×10^{-2} 15
(5.1.4) stopping index	E_α	1.367656×10^{-1} 14	3.948512×10^{-2} 14	2.220931×10^{-2} 14	1.855303×10^{-2} 14
(5.1.3) stopping index	E_α	1.367687×10^{-1} 27	3.898162×10^{-2} 28	2.136944×10^{-2} 28	1.764721×10^{-2} 28
(5.1.2) stopping index	E_α	1.367673×10^{-1} 27	3.949300×10^{-2} 27	2.221818×10^{-2} 27	1.856052×10^{-2} 27

CHAPTER 6

CONCLUSION AND FUTURE SCOPE

This thesis got inspired by the idea of approximating solutions to ill-posed problems in finite dimensional space and solving nonlinear problems using higher order iterative methods. It's always necessary to approximate the solution of a linear ill-posed problem without losing the inherent details of the exact solution. Tikhonov regularization usually over-smooth the solution of the ill-posed problem which could cause the loss of inherent details of the solution in practical cases. Therefore, we considered the Fractional Tikhonov regularization (FTR) to rectify this problem in Chapter 2. Moreover, we studied the finite dimensional realization of the FTR method with the Raus and Gfrerer type discrepancy principle for choosing the regularization parameter α. As mentioned in Remark 2.2.4, it is difficult to choose parameter $\beta \in (-\frac{1}{2}, 0]$ to obtain a better error estimate. But we observed that the relative errors $e_{\alpha,\beta} < e_{\alpha,0}$ hold when α was chosen according to the Raus and Gfrerer type discrepancy principle for $\beta \in (-\frac{1}{2}, 0]$. This demonstrated that the FTR approach provides a better error estimate than the standard Tikhonov regularization method.

In Chapter 3, we considered the finite dimensional study of the iterative regularization method discussed in George and Nair (2017) to approximate the solution of nonlinear ill-posed problems. We introduced a new source condition, namely $u_0 - \hat{u} = A^{\nu}z, \|z\| =\leq \rho, 0 < \nu \leq 1$ (see Chapter 3) and a new parameter-choice strategy. The proposed new parameter-choice strategy was independent of the unknown parameter ν. This new strategy provided the optimal order $O(\delta^{\frac{\nu}{\nu+1}})$, for $0 \leq \nu \leq 1$. We observed in our example that the relative error obtained for our new method (3.4.3) is lesser than that of the adaptive method in (George and Nair (2017)) for various δ values. As the relative error decreases, the accuracy of reconstruction increases.

In Chapter 4, we concentrated on solving the nonlinear problems in Banach space. It was always challenging to eliminate the use of Taylor's expansion in the local convergence analysis of higher order iterative methods. We studied Cordero's methods

(Cordero et al. (2009, 2012)) and modified the sixth order method to the eighth order iterative scheme. Moreover, we used the assumptions based on first order Fréchet derivatives of the operator. We could obtain the order of convergence without using Taylor's expansion.

In Chapter 5, we proposed Newton's midpoint type iterative methods of orders five and six. Similar to the previous Chapter, we eliminated the use of Taylor's expansion in the convergence analysis. We used the assumptions based on Fréchet derivatives of order up to two. Further we applied these iterative schemes to solve nonlinear ill-posed problems, and used proposed parameter choice strategy to compute the regularization parameter. The numerical results show that the relative errors decrease as the order of the method increases.

During the study, we came across the following problems, where further research may be possible.

1. We have studied weighted or fractional regularization methods for linear operator equations. So, can we extend the weighted or fractional regularization method to non-linear ill-posed operator equations?

2. Is there any way to find the optimal value for parameter $\beta \in (-\frac{1}{2}, 0]$ to obtain a better error estimate in the FTR method remains an open problem?

3. We have introduced a new source condition and parameter choice strategy for Lavrentiev regularization. Can we introduce these for a Tikhonov regularization to solve nonlinear ill-posed problems?

4. We have studied local convergence analysis of higher order iterative methods without using Taylor's expansion. So can we extend these ideas into semi local convergence analysis of other higher order iterative methods?

Appendix A

ESTIMATES FOR $\|\Gamma\|$

Note that,

$$\begin{aligned}
\Gamma &= ((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}[(T_h^*T_h)^{1+\beta}((T^*T)^{1+\beta}+\alpha I)\\
&\quad -((T_h^*T_h)^{1+\beta}+\alpha I)(T^*T)^{1+\beta}]((T^*T)^{1+\beta}+\alpha I)^{-1}\hat{x}\\
&= ((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}\alpha[(T_h^*T_h)^{1+\beta}-(T^*T)^{1+\beta}]((T^*T)^{1+\beta}+\alpha I)^{-1}\hat{x}.
\end{aligned}$$

Let $Z=((T^*T)^{1+\beta}+\alpha I)^{-1}\hat{x}$. Then, we have by (2.2.3);

$$\begin{aligned}
\|\Gamma\| &= \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}[((T_h^*T_h)^2)^{\frac{1+\beta}{2}}-((T^*T)^2)^{\frac{1+\beta}{2}}]Z\|\\
&\le \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\int_0^\infty t^{\frac{1+\beta}{2}}\|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}\\
&\quad \times((T_h^*T_h)^2+tI)^{-1}((T^*T)^2-(T_h^*T_h)^2)((T^*T)^2+tI)^{-1}Z\|dt\\
&= \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\Big[\int_0^\infty t^{\frac{1+\beta}{2}}\|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)\\
&\quad \times[(I-P_h)T^*T+P_hT^*T(I-P_h)]((T^*T)^2+tI)^{-1}Z\|dt\\
&\quad +\int_0^\infty t^{\frac{1+\beta}{2}}\|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}\\
&\quad \times((T_h^*T_h)^2+tI)^{-1}[(I-P_h)T^*T\\
&\quad +P_hT^*T(I-P_h)](T^*T)((T^*T)^2+tI)^{-1}Z\|dt\Big]\\
&= \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\Big[\int_0^\infty t^{\frac{1+\beta}{2}}\|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)\\
&\quad P_hT^*T(I-P_h)((T^*T)^2+tI)^{-1}Z\|dt\\
&\quad +\int_0^\infty t^{\frac{1+\beta}{2}}\|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}((T_h^*T_h)^2+tI)^{-1}\\
&\quad P_hT^*T(I-P_h)(T^*T)((T^*T)^2+tI)^{-1}Z\|dt\\
&\le: \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}[I_1+I_2],
\end{aligned}$$

where, $I_1 = \int_0^\infty t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)P_hT^*T(I-P_h)$ $((T^*T)^2+tI)^{-1}Z\|dt$, and $I_2 = \int_0^\infty t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}((T_h^*T_h)^2+tI)^{-1}P_h$ $T^*T(I-P_h)(T^*T)((T^*T)^2+tI)^{-1}Z\|dt$. Note that, we have used relation $((T_h^*T_h)^2$ $+tI)^{-1}(I-P_h)T^*T=0$ and $((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)(I-P_h)T^*T=0$ to obatin the above inequality. Next we shall obtain and estimate for $I_1$and I_2.

$$\begin{aligned}
I_1 \;\leq\; & \int_0^1 t^{\frac{1+\beta}{2}} \Big\|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)P_hT^*T(I-P_h) \\
& \times((T^*T)^2+tI)^{-1}Z\Big\|\,dt \\
& +\int_1^\infty t^{\frac{1+\beta}{2}} \Big\|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)P_hT^*T(I-P_h) \\
& \times((T^*T)^2+tI)^{-1}Z\Big\|\,dt \\
\leq\; & \int_0^1 t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\frac{1}{2}+\beta} \\
& \times((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)^{1-\beta}\|\|T(I-P_h)\|\|((T^*T)^2+tI)^{-1}(T^*T)^{\gamma}(T^*T)^{-\gamma}Z\|dt \\
& +\int_1^\infty t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\frac{1}{2}+\beta} \\
& \times((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)^{1-\beta}\|\|T(I-P_h)\|\|((T^*T)^2+tI)^{-1}Z\|dt \\
\leq\; & \alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}} \int_0^1 t^{\frac{1+\beta}{2}-\frac{1+\beta}{2}+\frac{\gamma}{2}-1}\varepsilon_h dt\|(T^*T)^{-\gamma}Z\| \\
& +\alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}} \int_1^\infty t^{\frac{1+\beta}{2}-\frac{\beta}{2}-2}\|T_h^*T_h\|\varepsilon_h dt\|Z\| \\
\leq\; & \alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}} \left[\int_0^1 t^{\frac{\gamma}{2}-1}\varepsilon_h dt\|(T^*T)^{-\gamma}Z\| + \int_1^\infty t^{-\frac{3}{2}}\varepsilon_h dt\|T_h^*T_h\|\|Z\|\right] \\
\leq\; & \alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}} \left[\frac{2}{\gamma}[\|(T^*T)^{-\gamma}Z\| + 2\|T^*T\|\|Z\|\right]\varepsilon_h,
\end{aligned}$$

where, we used the estimates $\|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\frac{1}{2}+\beta}\| \leq \alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}}$, $\|((T_h^*T_h)^2+tI)^{-\frac{1}{2}}(T_h^*T_h)^{1-\beta}\| \leq t^{-\frac{1+\beta}{2}}$, $\|((T_h^*T_h)^2+tI)^{-\frac{1}{2}}(T_h^*T_h)^{1-\beta}\| \leq t^{-\frac{\beta}{2}}\|T_h^*T_h\|$, $\|((T^*T)^2+tI)^{-1}\| \leq t^{-1}$, $\|T_h^*T_h\| \leq \|T^*T\|$ and $\|((T^*T)^2+tI)^{-1}(T^*T)^{\gamma}\| \leq t^{\frac{\gamma}{2}-1}$. Again, we have

$$\begin{aligned}
I_2 \;\leq\; & \int_0^1 t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}((T_h^*T_h)^2+tI)^{-1}P_hT^*T(I-P_h)(T^*T) \\
& \times((T^*T)^2+tI)^{-1}Z\|dt + \int_1^\infty t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}((T_h^*T_h)^2+tI)^{-1}P_hT^*T \\
& \times(I-P_h)(T^*T)((T^*T)^2+tI)^{-1}Z\|dt \\
\leq\; & \int_0^1 t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\frac{1}{2}+\beta} \\
& \times((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)^{-\beta}T(I-P_h)(T^*T)((T^*T)^2+tI)^{-1}Z\|dt
\end{aligned}$$

$$
\begin{aligned}
& +\int_1^\infty t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\frac{1}{2}+\beta} \\
& \times((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)^{-\beta}T(I-P_h)(T^*T)((T^*T)^2+tI)^{-1}Z\|dt \\
\leq\ & \int_0^1 t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\frac{1}{2}+\beta}\| \\
& \times\|((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)^{-\beta}\|\|T(I-P_h)\|\|((T^*T)^2+tI)^{-1}(T^*T)^{1+\gamma} \\
& \times(T^*T)^{-\gamma}Z\|dt+\int_1^\infty t^{\frac{1+\beta}{2}} \|\alpha((T_h^*T_h)^{1+\beta}+\alpha I)^{-1}(T_h^*T_h)^{\frac{1}{2}+\beta}\| \\
& \times\|((T_h^*T_h)^2+tI)^{-1}(T_h^*T_h)^{-\beta}\|\|T(I-P_h)\|\|((T^*T)^2+tI)^{-1}(T^*T)Z\|dt \\
\leq\ & \alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}}\left[\int_0^1 t^{\frac{1+\beta}{2}-\frac{\beta}{2}-1+\frac{\gamma-1}{2}}\varepsilon_h\|(T^*T)^{-\gamma}Z\|+\int_1^\infty t^{\frac{1+\beta}{2}-\frac{\beta}{2}-2}\varepsilon_h\|(T^*T)\|\|Z\|\right] \\
\leq\ & \alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}}\varepsilon_h\left[\frac{2}{\gamma}\|(T^*T)^{-\gamma}Z\|+2\|T^*T\|\|Z\|\right],
\end{aligned}
$$

where, we used the estimates $\|((T^*T)^2+tI)^{-1}(T^*T)^{1+\gamma}\| \leq t^{\frac{\gamma-1}{2}}$. Further, observe that

$$
\begin{aligned}
\alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}}\|(T^*T)^{-\gamma}Z\| &= \alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}}\|(T^*T)^{-\gamma}((T^*T)^{1+\beta}+\alpha I)^{-1}\hat{x}\| \\
&\leq \alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}}\sup_{\lambda\in\sigma(T^*T)}\left|\frac{\lambda^{\nu-\gamma}}{\lambda^{1+\beta}+\alpha}\right|\|z\| \\
&\leq c_4\alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}}\alpha^{\frac{\nu-\gamma}{1+\beta}-1} \\
&\leq c_4\frac{1}{\alpha^{\frac{\frac{1}{2}+\gamma-\nu}{1+\beta}}}\leq c_4\frac{1}{\alpha^{\frac{1}{2(1+\beta)}}}. \qquad (A.0.1)
\end{aligned}
$$

Similarly, one can prove

$$
\alpha^{\frac{\frac{1}{2}+\beta}{1+\beta}}\|Z\| \leq c_5\frac{1}{\alpha^{\frac{\frac{1}{2}-\nu}{1+\beta}}}\leq c_5\frac{1}{\alpha^{\frac{1}{2(1+\beta)}}}.
$$

Therefore , by we have

$$
\|\Gamma\| \leq 4\left[\frac{1}{\gamma}c_4+\|T^*T\|c_5\right]\frac{\varepsilon_h}{\alpha^{\frac{1}{2(1+\beta)}}}.
$$

Appendix B

ESTIMATES FOR $\|\Gamma_1\|$ and $\|\Gamma_2\|$

Note that

$$\begin{aligned}
\|\Gamma_1\| &\leq \|\alpha\frac{1}{\pi}\int_0^\infty u^{\frac{1}{2}}((T_hT_h^*)^{1+\beta}+(\alpha+u)I)^{-1} \\
&\quad [(TT^*)^{1+\beta}-(T_hT_h^*)^{1+\beta}]Z_1du\| \\
&\leq \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\alpha\frac{1}{\pi}\Big\|\int_0^\infty u^{\frac{1}{2}}\frac{1}{\alpha+u} \\
&\quad \int_0^\infty t^{\frac{1+\beta}{2}}((T_hT_h^*)^2+tI)^{-1}((TT^*)^2-(T_hT_h^*)^2) \\
&\quad \times((TT^*)^2+tI)^{-1}Z_1dudt\Big\|,
\end{aligned} \tag{B.0.1}$$

where $Z_1=((TT^*)^{1+\beta}+(\alpha+u)I)^{-1}((TT^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}y$. Note that as in appendix A, one can write

$$\begin{aligned}
&\Big\|\int_0^\infty t^{\frac{1+\beta}{2}}((T_hT_h^*)^2+tI)^{-1}((TT^*)^2-(T_hT_h^*)^2)((TT^*)^2+tI)^{-1}Z_1dt\Big\| \\
\leq\; &\Big\|\int_0^\infty t^{\frac{1+\beta}{2}}((T_hT_h^*)^2+tI)^{-1}(T_hT_h^*)T(I-P_h)T^*((TT^*)^2+tI)^{-1}Z_1dt\Big\| \\
&+\Big\|\int_0^\infty t^{\frac{1+\beta}{2}}((T_hT_h^*)^2+tI)^{-1}T(I-P_h)T^*(TT^*)((TT^*)^2+tI)^{-1}Z_1dt\Big\| \\
\leq\; &\Big\|\int_0^1 t^{\frac{1+\beta}{2}}((T_hT_h^*)^2+tI)^{-1}(T_hT_h^*)T(I-P_h)T^*((TT^*)^2+tI)^{-1}Z_1dt\Big\| \\
&+\Big\|\int_1^\infty t^{\frac{1+\beta}{2}}((T_hT_h^*)^2+tI)^{-1}(T_hT_h^*)T(I-P_h)T^*((TT^*)^2+tI)^{-1}Z_1dt\Big\| \\
&+\Big\|\int_0^1 t^{\frac{1+\beta}{2}}((T_hT_h^*)^2+tI)^{-1}T(I-P_h)(TT^*)T^*((TT^*)^2+tI)^{-1}Z_1dt\Big\| \\
&+\Big\|\int_1^\infty t^{\frac{1+\beta}{2}}((T_hT_h^*)^2+tI)^{-1}T(I-P_h)T^*(TT^*)((TT^*)^2+tI)^{-1}Z_1dt\Big\| \\
\leq\; &\int_0^1 t^{\frac{1+\beta}{2}-\frac{1}{2}}\|T(I-P_h)\|\|((TT^*)^2+tI)^{-1}(TT^*)^{\frac{1}{2}-\beta}\|
\end{aligned}$$

$$
\begin{aligned}
& \times\|(TT^*)^{\beta}Z_1 dt\| \\
& + \int_1^{\infty} t^{\frac{1+\beta}{2}} \|((T_hT_h^*)^2+tI)^{-1}(T_hT_h^*)\|\|T(I-P_h)\|\|((TT^*)^2+tI)^{-1}(TT^*)^{\frac{1}{2}-\beta}\| \\
& \times\|(TT^*)^{\beta}Z_1 dt\| \\
& + \int_0^{1} t^{\frac{1+\beta}{2}} \|((T_hT_h^*)^2+tI)^{-1}\|\|T(I-P_h)\|((TT^*)^2+tI)^{-1}(TT^*)^{\frac{3}{2}-\beta}\| \\
& \times\|(TT^*)^{\beta}Z_1 dt\| \\
& + \int_1^{\infty} t^{\frac{1+\beta}{2}} \|((T_hT_h^*)^2+tI)^{-1}\|\|T(I-P_h)\|\|((TT^*)^2+tI)^{-1}(TT^*)^{-\beta}(TT^*)^{\frac{3}{2}}\| \\
& \times\|(TT^*)^{\beta}Z_1 dt\| \\
\leq\ & \int_0^{1} t^{\frac{1+\beta}{2}-\frac{1}{2}-\frac{\beta}{2}-\frac{3}{4}} \|T(I-P_h)\|\|(TT^*)^{\beta}Z_1\|dt \\
& + \int_1^{\infty} t^{\frac{1+\beta}{2}-1-\frac{\beta}{2}-1} \|T(I-P_h)\|\|T_hT_h^*\|\|TT^*\|^{\frac{1}{2}}\|(TT^*)^{\beta}Z_1\|dt \\
& + \int_0^{1} t^{\frac{1+\beta}{2}-1-\frac{1}{4}-\frac{\beta}{2}} \|T(I-P_h)\|\|(TT^*)^{\beta}Z_1\|dt \\
& + \int_1^{\infty} t^{\frac{1+\beta}{2}-\frac{\beta}{2}-2} \|T^*T\|^{\frac{3}{2}}\|T(I-P_h)\|\|(TT^*)^{\beta}Z_1\|dt \\
\leq\ & (8\varepsilon_h + 4\varepsilon_h\|TT^*\|^{\frac{3}{2}})\|(TT^*)^{\beta}Z_1\|, \qquad \text{(B.0.2)}
\end{aligned}
$$

where we used the estimate

$$
\|((T_hT_h^*)^2+tI)^{-1}T_h^*\| \leq t^{-\frac{3}{4}},
$$

and

$$
\|((TT^*)^2+tI)^{-1}(TT^*)^{\frac{1}{2}-\beta}\| \leq t^{-\frac{\beta}{2}-\frac{3}{4}}.
$$

Note that $\|(TT^*)^{\beta}Z_1\| \leq \frac{1}{\alpha+u}\|(TT^*)^{\beta}((TT^*)^{1+\beta}+\alpha I)^{-\frac{3}{2}}T\hat{x}\|$.

Let $\tau = \begin{cases} 1, & \text{if } \nu+\beta \geq \frac{1}{2} \\ \frac{1}{2}+\nu+\beta, & \text{if } \nu+\beta < \frac{1}{2}. \end{cases}$ Then, since $y = T\hat{x}$, we have by (2.1.8);

$$
\begin{aligned}
\|(TT^*)^{\beta}((TT^*)^{1+\beta}+\alpha I)^{-1}y\| &= \|T(T^*T)^{\beta}((T^*T)^{1+\beta}+\alpha I)^{-1}\hat{x}\| \\
&\leq \|((T^*T)^{1+\beta}+\alpha I)^{-1}(T^*T)^{\tau}z\|.
\end{aligned}
$$

for some $z \in X$. Hence

$$
\begin{aligned}
\sqrt{\alpha}\|(TT^*)^{\beta}((TT^*)^{1+\beta}+\alpha I)^{-1}y\| &\leq \sqrt{\alpha} \sup_{\lambda\in\sigma(TT^*)} \left|\frac{\lambda^{\tau}}{\lambda^{1+\beta}+\alpha}\right| \|z\| \\
&\leq c_7\alpha^{\frac{\tau}{1+\beta}-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad c_7 \begin{cases} \alpha^{\frac{\frac{1}{2}-\beta}{1+\beta}}, & \nu \geq \frac{1}{2} \\ \alpha^{\frac{\nu}{1+\beta}}, & \nu < \frac{1}{2} \end{cases} \\
&\leq \quad c_8 \ (\text{since } \alpha \leq \alpha_0). \qquad \text{(B.0.3)}
\end{aligned}
$$

Therefore, we have by (B.0.1), (B.0.2) and (B.0.3),

$$
\|\Gamma_1\| \leq \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\frac{1}{\pi}\int_0^\infty \sqrt{\alpha}\frac{u^{\frac{1}{2}}}{(\alpha+u)^2}du(8\varepsilon_h + 4\varepsilon_h\|TT^*\|^{\frac{3}{2}})c_8,
$$

and

$$
\int_0^\infty \sqrt{\alpha}\frac{u^{\frac{1}{2}}}{(\alpha+u)^2}du = \frac{\pi}{2},
$$

we have,

$$
\|\Gamma_1\| \leq \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}(4\varepsilon_h + 2\varepsilon_h\|TT^*\|^{\frac{3}{2}})c_8. \qquad \text{(B.0.4)}
$$

Simillarly, one can prove that

$$
\|\Gamma_2\| \leq \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}(4\varepsilon_h + 2\varepsilon_h\|TT^*\|^{\frac{3}{2}})c_9, \qquad \text{(B.0.5)}
$$

for some constant $c_9 > 0$.

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PUBLICATIONS

1. Remesh, Krishnendu, Ioannis K. Argyros, Muhammed Saeed K, Santhosh George, and Jidesh Padikkal, *Extending the Applicability of Cordero Type Iterative Method*, Symmetry 14, no. 12: 2495, 2022. https://doi.org/10.3390/sym14122495.

2. Krishnendu, R., Saeed, M., George, S., and Jidesh, P. , *On Newton's Midpoint-Type Iterative Scheme's Convergence*, International Journal of Applied and Computational Mathematics, 8(5), 1-11, 2022. https://doi.org/10.1007/s40819-022-01468-1

3. George, Santhosh, Jidesh Padikkal, Krishnendu Remesh, and Ioannis K. Argyros, *A New Parameter Choice Strategy for Lavrentiev Regularization Method for Nonlinear Ill-Posed Equations*,Mathematics 10, no. 18: 3365, 2022. https://doi.org/10.3390/math10183365

4. George, S., Jidesh, P., and Krishnendu, R. , *Finite dimensional realization of the FTR method with Raus and Gfrerer type discrepancy principle*,Rend. Circ. Mat. Palermo, II. Ser., 2023. https://doi.org/10.1007/s12215-022-00858-0.

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Other Publications :

- Saeed K, Muhammed, Krishnendu Remesh, Santhosh George, Jidesh Padikkal, and Ioannis K. Argyros. (2023). *Local Convergence of Traub's Method and Its Extensions*, Fractal and Fractional 7, no. 1: 98. https://doi.org/10.3390/fractalfract7010098 (SCIE).
- Muhammed Saeed, K., Krishnendu, R., George, S., and Padikkal, J. (2022). *On the convergence of Homeier method and its extensions*, The Journal of Analysis, 1-12. https://doi.org/10.1007/s41478-022-00449-3 (SCOPUS).