A STUDY ON CERTAIN POSITIVITY CLASSES OF OPERATORS IN HILBERT SPACES

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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Dedicated to the loving memory of my father

Athilan Abdurahman (Andru A.)

(05 January 1955 - 03 March 2010)

and to my beloved mother

Fathima C. K. (Chettiyankandy Biyyathu)

DECLARATION

I hereby declare that the research thesis entitled **A STUDY ON CERTAIN POSITIVITY CLASSES OF OPERATORS IN HILBERT SPACES** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this research thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to *certify* that the research thesis entitled **A STUDY ON CERTAIN POSITIVITY CLASSES OF OPERATORS IN HILBERT SPACES** submitted by **Mr. RASHID A** (Register Number : 187124MA010) as the record of the research work carried out by him is *accepted as the research thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

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ABSTRACT

In mathematical optimization theory, the linear complementarity problem, which is stated as, given a vector q in a finite-dimensional real vector space and an $n \times n$ real matrix A, then, finding a vector x that satisfies the system of inequalities $x \geq 0$, $q + Ax \geq 0$, $x^T(q + Ax) = 0$, plays a vital role in many areas such as bimatrix game theory, market equilibrium, computational complexity, and many more. The nature of the solution of the linear complementarity problem can be discussed with the help of the matrix A involved in the problem. The sign-reversing is a property of matrices along with a given vector, which is stated as an $n \times n$ matrix A reverses the sign of a vector x in an n-dimensional real vector space if it satisfies $x_i(Ax)_i \leq 0$ for all index i. The concept of sign-reversing is a useful tool to identify and characterize certain matrix classes involved in linear complementarity problems. The sign-reversing set of a matrix A is defined as $\{x : x_i(Ax)_i \leq 0, \forall i\}$.

In this thesis, we characterize the sign-reversing set of an arbitrary square matrix A in terms of the null spaces of the matrices $DA - A - D$, where D is a diagonal matrix such that $0 \leq D \leq I$. The matrices which have convex sign-reversing sets include large classes of matrices; we discuss a subclass of matrices in which the convexity of the sign-reversing set is characterized.

A real square matrix A is called a P-matrix if all its principal minors are positive. In 2016, Kannan and Sivakumar extended the notion of P-matrix to infinite-dimensional Banach spaces relative to a given Schauder basis by using the sign non-reversal property of matrices. Motivated by their work, we discuss P-operators on separable real Hilbert spaces with the help of the inner product structure of the Hilbert spaces. We also investigate P-operators relative to various orthonormal bases.

In this thesis, we define the concept of a sign-reversing set for operators on separable Hilbert spaces with the help of the inner product structure of the Hilbert spaces relative to a given orthonormal basis. We also generalize some special classes of matrices to operators in infinite-dimensional Hilbert spaces with the help of the sign-reversing property of operators. The thesis also studies the spectral theory of certain positive operators under consideration.

Keywords: *Linear complementarity problems, sign reversing set, sign reversal property, P-matrix, sufficient matrix, adequate matrix, P-operator.*

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LIST OF SYMBOLS AND ABBREVIATIONS

$A[\alpha,\beta]$	:	Sub-matrix of A whose rows and columns are indexed by $\alpha,\beta \subseteq \langle n\rangle$
$A[\alpha]$	:	Principal sub-matrix of A with rows and columns are indexed by α
$A[\alpha,\bullet]$	:	Sub-matrix of A consisting of the rows indexed by α
$A[\bullet,\alpha]$	:	Sub-matrix of A consisting of columns indexed by α
$\overline{\alpha}$	:	For the index set $\alpha \subseteq \langle n\rangle$, $\overline{\alpha} = \langle n\rangle \backslash \alpha$
$\mathscr{B}$	:	Orthonormal basis in Hilbert space
B	:	Schauder basis in Banach space
$\mathscr{B}(\mathscr{H})$	:	The space of all bounded linear operators on $\mathscr{H}$
$\mathbb{C}$	:	Field of complex numbers
C_A	:	The set $\{x \in rev(A) : x_j = 0$ and $(Ax)_j = 0,$ for some index j $\in \langle n\rangle\}$
$\mathscr{C}$	:	$\mathscr{C} = \{A \in \mathbb{R}^{n\times n} : C_A = \{0\}$ and $x * Ay = y * Ax$ for all $x,y \in rev(A)\}$
$c(A,B)$	:	$c(A,B) = \{C : C = AT + B(I-T), T = diag(t_1,\ldots,t_n), t_i \in [0,I]\}$
C_j	:	The set $(N_j^+ \cap M_j^-) \cup (N_j^- \cap M_j^+)$
$det(A)$	:	The determinant of the matrix A
$D = (d_{ii})$	:	Diagonal Matrix/Diagonal operator with diagonal entries are d_{ii}
$D = diag(x,y)$	:	D is a diagonal matrix with x and y are respective diagonal entries
$\{e_i\}_{i=1}^{\infty}$	:	Schauder basis/ Orthonormal basis
$GL_n(\mathbb{R})$	:	The group of $n \times n$ invertible matrices of real numbers
$\mathscr{H}$	:	Separable Hilbert spaces
$h(A,B)$	:	$h(A,B) = \{C : C = tA + (1-t)B, t \in [0,1]\}$
$i(A,B)$	:	$i(A,B) = \{C : C = T \circ A + (J-T) \circ B, T = (t_{ij}), t_{ij} \in [0,1]\}$
$\mathbb{K}$	:	Field $\mathbb{C}$ or $\mathbb{R}$
ker	:	Kernel of a linear operator
$K(\mathscr{H})$	:	Collection of compact operators on $\mathscr{H}$
LCP	:	The linear complementarity problem

$\mathscr{L}^p(\mathbb{R})$	:	$\{f : \mathbb{R} \in \mathbb{C}, f$ measurable $, \int_{\mathbb{R}} \|f(x)\|^p\,dx < \infty\}$
$L(\mathbb{R}^n, \mathbb{R}^m)$	:	The space of all linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$
$L(\mathbb{R}^n)$	:	The space of all linear operators on $\mathbb{R}^n$
ℓ^p	:	$\{\{a_i\}_{i=1}^{\infty} : a_i \in \mathbb{K}, \forall i \in \mathbb{N}, \sum_{i=1}^{\infty} \|a_i\|^p < \infty\}$
ℓ^∞	:	$\{\{a_i\}_{i=1}^{\infty} : a_i \in \mathbb{K}, \forall i \in \mathbb{N}, \sup_{i\in\mathbb{N}} \|a_i\| < \infty\}$
$\mathbb{M}^c$	:	Complement of $\mathbb{M}$
M_j^+	:	The set $\{x \in \mathbb{R}^n : f_j(Ax) \geq 0\}$
M_j^-	:	The set $\{x \in \mathbb{R}^n : f_j(Ax) \leq 0\}$
$\mathbb{N}$	:	Set of natural numbers
N_j^+	:	The set $\{x \in \mathbb{R}^n : f_j(x) \geq 0\}$
N_j^-	:	The set $\{x \in \mathbb{R}^n : f_j(x) \leq 0\}$
$\langle n \rangle$	:	The set $\{1, 2, \ldots, n\}$
$ppt(A, \alpha)$	:	The principal pivot transform of $A \in M_n(\mathbb{C})$ relative to α.
$\mathscr{P}$	:	The set of all complex P-matrices.
P_0	:	The class of all $n \times n$ real matrices with nonnegative principal minors.
$\mathscr{P}\mathscr{M}$	:	Collection of matrices with all of whose powers are P-matrices
$\mathscr{P}_0\mathscr{M}$	:	Collection of matrices with all of whose powers are P_0-matrices
$\mathbb{R}$	:	Field of real numbers
$\mathbb{R}^{n\times n}$	:	The set of all $n \times n$ matrices
$\mathbb{R}^n$	:	Denote $\mathbb{R}^{n\times 1}$
$r(A, B)$	:	The set $\{C : C = TA + (I - T)B, T = diag(t_1, \ldots, t_n), t_i \in [0, I]\}$
$rev(A)$	:	The set $\{x \in \mathbb{R}^n : x_i(Ax)_i \leq 0,$ for all $i \in \langle n \rangle\}$
$rev_{\mathscr{B}}(T)$	:	The set $\{x \in \mathscr{H} : \langle x, e_i \rangle \langle Tx, e_i \rangle \leq 0, \forall i\}$
S	:	The collection of sufficient matrices
T_I	:	The idempotent operator on a Hilbert space
T_L	:	The left shift operator on $\ell_2(\mathbb{N})$
T_R	:	The right shift operator on $\ell_2(\mathbb{N})$
$trace(A)$	:	The trace of the matrix A, that is the sum of diagonal elements of A

$U(A)$	:	$U(A) = (I+A)^{-1}(I-A)$
U	:	The unitary operator on a Hilbert space
x_i	:	The i^{th} component of the vector x
$\langle \cdot, \cdot \rangle$	:	Inner product which is linear in first variable and conjugate linear in second variable
$\lvert . \rvert$	:	Determinant
$\lVert \cdot \rVert$	:	Norm
$\sigma(A)$	:	Spectum of the matrix/operator A
$\sigma_a(T)$	:	Approximate eigen spectum of the operator T
$\sigma_e(T)$	:	Eigen spectum of the operator T
ϕ	:	The empty set
ρ	:	Spectral radius of an operator

CHAPTER 1

INTRODUCTION

1.1 GENERAL INTRODUCTION

A matrix is a rectangular array of numbers or symbols which are generally arranged in rows and columns. The entries of the matrix are known as elements, and the order of a matrix is denoted by $m \times n$, where m is the number of rows and n is the number of columns. Matrices serve as highly versatile structures with applications across a wide spectrum of scientific and mathematical fields. Engineering mathematics, which relies heavily on matrix theory, plays a pivotal role in numerous aspects of our daily lives, including computer graphics, optics, cryptography, economics, chemistry, geology, robotics, animation, wireless communication, signal processing, finance, and so on.

Operator theory, a fundamental branch of mathematics with far-reaching applications in various fields, including physics and engineering, emerged from the rich soil of matrix theory. Matrix theory provided the initial framework and inspiration for the development of operator theory. As mathematicians explored the properties of matrices, they began to realize that these arrays of numbers could be seen as representations of linear transformations between vector spaces. This revelation paved the way for the abstraction of operators, which are generalized transformations that act on more general vector spaces than just finite-dimensional ones. This transition from finite to infinite dimensions allowed for the analysis of more complex and subtle mathematical structures, leading to a deeper understanding of various mathematical phenomena and their real-world applications. Thus, operator theory owes its origins to the insights gained from matrix theory and has grown into a vital field in mathematics and beyond.

1.2 LINEAR COMPLEMENTARITY PROBLEM

The topic we are addressing in this section has undergone numerous name changes. It has been called the "composite problem," the "fundamental problem," and the "complementary pivot problem" (Cottle (1968b); Cottle and Dantzig (1968a)). In 1965, the present name, "linear complementarity problem" (abbreviated as LCP), was proposed by Cottle. In mathematical optimization theory, The linear complementarity problem stands out as one of the most fundamental problems due to its dual traits of linearity and complementarity. These two features endow the problem with the capability to model a wide range of problems found in disciplines such as computer science, economics, physics, and engineering, as evidenced by numerous sources (Björklund et al. (2005); Cottle et al. (1992a); Fridman and Černina (1967); Howson (1972); Lemke (1965); Pardalos and Nagurney (1990); Beckmann (1973)).

One of the earliest documented mentions of an explicitly stated LCP can be found in a publication by Du Val (1940). In that paper, a problem of the form (q,A) is given to find the least element (in the vector sense) of the linear inequality system $q+Az>0$, $z>0$, where A is an $n\times n$ matrix and q is a given vector. Typically, problems of this nature are unsolvable. However, when the matrix A possesses specific special characteristics, a unique solution does exist. However, it began to emerge as its own topic of research in the mid-1960s. This kind of problem was studied in detail by Cottle (1966, 1964); Lemke (1965); Cottle (1968b); Cottle and Dantzig (1968b), and later it was used in the papers by Cottle et al. (1970); Cottle and Dantzig (1970).

The flourishing of the contemporary theory of mathematical programming was spurred by the rise of linear programming and the advancement of computer performance. Mathematical programming explores techniques to achieve the optimal value concerning defined objectives within a mathematical model. Linear programming seeks to identify the optimal solution when the objective is expressed as a linear function, and the model is represented as a set of linear inequalities. More precisely, the goal of linear programming is to seek a real vector x which maximizes $c^T x$ subject to $Ax \leq b$, where A is a real matrix and b and c are real vectors.

Significant research efforts have been dedicated to exploring linear programming since the 1940s and its theoretical and practical importance is now widely recognized. During the 1940s, two pioneers, Kantorovich and Koopmans, independently propelled the field of linear programming forward to tackle economic hurdles, such as optimizing production with available resources and attaining fair income distribution (Kantorovich (1960); Koopmans (1939)). They were awarded the Nobel Prize in Economic Sciences in 1975 in recognition of this contribution. This indicates the substantial impact of linear programming on the advancement of economics. As the realm of linear programming expanded, the theory of quadratic programming underwent rapid development. Quadratic programming involves seeking a vector x that fulfills a linear constraint while maximizing or minimizing a quadratic function of the form $\frac{1}{2}x^T Dx + c^T x$, where D is a square real matrix, and c is a real vector. The exploration of quadratic programming spans as far back in history as linear programming.

The criteria for an optimal solution to the nonlinear programming problem were established by Kuhn and Tucker (1951); Karush (1939), known as the Karush-Kuhn-Tucker (KKT) conditions. For the existence of a global minimum in quadratic programming, Frank and Wolfe (1956) presented a sufficient condition known as the Frank-Wolfe theorem. Later, in 1959, Wolfe proposed an algorithm similar to the simplex method for addressing nonconvex quadratic programming problems (Wolfe (1959, 1960)). Around 1960, Dennis (1959) and Dorn (1961) investigated duality in the context of convex quadratic programming. Dantzig and Cottle (1967) established both the necessary and sufficient conditions for convex quadratic programming problems to have solutions, consolidating the discoveries made by Dennis and Dorn. In the late 1980s, Kapoor and Vaidya (1986) as well as Ye and Tse (1989) extended an interior point method, initially crafted for linear programming problems, to create a polynomial-time algorithm for convex quadratic programming problems. Notably, Cottle and Dantzig (1970) reached the aforementioned result by extending the convex quadratic programming problem to the linear complementarity problem and researching this generalized problem. In a related work, Kojima et al. (1989) developed an interior-point method for the linear complementarity problem.

The linear complementarity problem is typically categorized within the realm of mathematical programming, commonly linked to finite-dimensional optimization and equilibrium problems in physical or economic contexts (Cottle et al. (1992b)). It entails resolving a system of linear inequalities where the solution vectors adhere to the complementarity condition. This condition dictates that the two solution vectors should be orthogonal to each other. Additionally, if a non-negativity constraint is applied to vector elements, then the products of each equivalently indexed element should be zero.

Let $\mathbb{R}^{n\times n}$ denote the set of all $n \times n$ matrices and $\mathbb{R}^n$ denote $\mathbb{R}^{n\times 1}$. The notation $x \geq 0$ denotes each coordinate of the vector x is non-negative, and x^T denotes the transpose of the vector $x \in \mathbb{R}^n$. Formally the linear complementarity problem is stated as follows:

Definition 1.2.1. *(Cottle et al. (1992a)) Given a vector $q \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$ the linear complementarity problem, abbreviated LCP, is to find a vector $z \in \mathbb{R}$ such that*

$$\begin{aligned} x &\geq 0 \\ q+Ax &\geq 0 \\ x^T(q+Ax) &= 0 \end{aligned}$$

or to show that no such vector x exists. We denote the above LCP by the pair (q,A).

In 1928, the foundation of game theory was laid with the groundbreaking contributions of von Neumann (1928), who notably introduced the minimax theorem, a fundamental concept in game theory. The application of game theory has been expanded into economics by von Neumann and Morgenstern (1944), and they authored a seminal book on this topic, attracting the attention of prominent mathematicians, including John Nash. Around 1950, Nash made a notable contribution by demonstrating that every finite n-person game has a minimum of one Nash equilibrium, a state where no player has a motive to change their strategy alone while others maintain their decisions (Nash (1950, 1951)). This work ultimately earned the Nobel Prize in Economic Science for Nash in 1994, along with two economists, Selten and Harsanyi, for their groundbreaking contributions to the development and application of game theory in economics. It is worth noting that Nash's proof in (Nash (1951)) was nonconstructive and relied on Brouwer's fixed-point Theorem. As mentioned in (Papadimitriou (2001)), finding a Nash equilibrium poses as one of the significant complexity challenges of the present era.

The applications of LCP extend to the first-order optimality conditions of quadratic programming. Indeed, in its initial phases, LCP was intimately connected to linear and quadratic programming. Although it is used in those domains, additional applications have emerged, such as bimatrix games, where two players select from a finite set of options (an LCP formulation for finding a Nash equilibrium of a bimatrix game is discussed in Lemke and Howson (1964)), financial modeling, computational complexity, market equilibrium, and algorithms for nonlinear complementarity problems. For instance, the complementary pivot algorithm, initially devised for solving LCPs, has been extended to tackle Brouwer and Kakutani fixed points utilized in computing economic equilibria, as well as solving systems of nonlinear equations and nonlinear programming problems. Additionally, iterative approaches for LCP resolution exhibit potential for handling extremely large linear programming systems where simplex algorithms are ineffective.

1.3 SIGN-REVERSING PROPERTY OF MATRICES

Matrices are classified into an enormous variety of classes according to different properties coming from the order, elements, and other structures of matrices. Sign-reversing is one of the properties of matrices along with a given vector, which is used to identify different classes of matrices having important applications. The concept of sign-reversing proves to be a valuable technique in delineating specific matrix classes within linear complementarity problems. The formal definition of the sign-reversing property is as follows:

Definition 1.3.1. *(Guu (1996)) The matrix $A \in \mathbb{R}^{n\times n}$ reverses the sign of the vector $x \in \mathbb{R}^n$ if $x_i(Ax)_i \leq 0$, for all $i \in \langle n \rangle$, where x_i is the i-th component of the vector $x \in \mathbb{R}^n$ and $\langle n \rangle = \{1,2,3,\ldots,n\}$.*

Definition 1.3.2. *(Khudalov (1998)) Let $\mathbf{V}$ be a vector space, a non-empty subset $\mathbf{V}^+ \subseteq \mathbf{V}$ is called a cone if for any $x \in \mathbf{V}^+$ and $\lambda \in \mathbb{R}^+ = \{\alpha \in \mathbb{R} : \alpha \geq 0\}$ one has $\lambda x \in \mathbf{V}^+$. The cone $\mathbf{V}^+$ is called pointed if $\mathbf{V}^+ \cap -\mathbf{V}^+ = \{0\}$.*

Proposition 1.3.3. *(Borwein and Moors (2010)) Let $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ and let K be a closed cone (not necessarily convex) in $\mathbb{R}^n$. If $K \cap ker(T) = \{0\}$, then there exists a neighbourhood N of T in $L(\mathbb{R}^n, \mathbb{R}^m)$ such that $S(K)$ is closed in $\mathbb{R}^m$ for each $S \in N$.*

Definition 1.3.4. *(Guu (1996)) The sign-reversing set of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $rev(A)$ and is defined as*

$$rev(A) = \{x \in \mathbb{R}^n : x_i(Ax)_i \leq 0, \text{ for all } i \in \langle n \rangle\}.$$

Notice that $rev(A)$ is a cone containing $ker(A)$ and hence is always non-empty.

The circumstances in which $rev(A)$ and $ker(A)$ coincide can be elucidated by the concept of column adequacy (defined below).

Some classes of matrices that have arisen in connection with the linear complementarity problem (LCP) are P-matrices, totally positive/negative matrices, N-matrices, M-matrices, Q-matrices, Z-matrices and many more (Huang et al. (2006); Berman and Plemmons (1994a); Tsatsomeros (2002); Plemmons (1977, 1976); Karamardian (1976); Sivakumar et al. (2021); Mondal et al. (2022); Sivakumar et al. (2022); Choudhury et al. (2021)). These classes of matrices have the sign-reversing property at different levels.

Remark 1.3.5. *The sign-reversing set of a matrix need not be equal to the sign-reversing set of its transpose. The following example shows this fact.*

Example 1.3.6. *Consider the matrix*

$$A = \begin{pmatrix} 2 & -2 \\ -1 & -1 \end{pmatrix}$$

and the vector $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. *Then, as* $Ax = \begin{pmatrix} -2 \\ -7 \end{pmatrix}$, *we get* $x_i(Ax)_i \leq 0$ *for* $i = 1, 2$, *implies* $x \in rev(A)$. *But, here the transpose of* A *is given by*

$$A^T = \begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix}$$

and $A^T x = \begin{pmatrix} 2 \\ -10 \end{pmatrix}$, *which implies that* $x_i(A^T x)_i > 0$ *for* $i = 1$. *This shows that* $x \notin rev(A^T)$.

Definition 1.3.7. *(Cottle et al. (1992a)) An* $n \times n$ *matrix* A *is said to have a sign non-reversal property if it does not reverse the sign of any non-zero vector.*

Recently, Choudhury and Kannan (2021) established a characterization of almost P-matrices with the help of the sign non-reversal property. It provides a test for an entire class of matrices to be N-matrices/almost P-matrices (see Definition 1.4.2).

1.4 SOME SPECIAL CLASS OF MATRICES

In this section, we explore several classes of matrices that have emerged in relation to the linear complementarity problem.

1.4.1 P-MATRICES

P-matrices represent one category of matrices that are utilized in the study of LCP to discuss the uniqueness of the solution to LCP. An $n \times n$ matrix $A \in M_n(\mathbb{C})$ is called a P-matrix (P_0-matrix) if all its principal minors are positive (non-negative). We recall that a principal minor is the determinant of a submatrix obtained from A when the same set of rows and columns are considered. Fiedler and Pták (1962) have shown that A is a P-matrix if and only if A has sign non-reversal property, that is, if $x \neq 0 \in \mathbb{R}^n$, then there exists some index j for which we have $x_j(Ax)_j > 0$. Gale and Nikaidô (1965) characterized the class of P-matrices in terms of the sign-reversing set. Later, Eaves (1971) characterized the class of column adequate matrices (see Definition 1.4.15 in the following sub-section) in a similar manner. It is shown in (Cottle et al. (1992a)) that given a real square matrix A, the linear complementarity problem $LCP(A,q)$ has a unique solution for each vector $q \in \mathbb{R}^n$ if and only if A is a P-matrix. P-matrices include distinguished categories such as Hermitian positive definite matrices, M-matrices, totally positive matrices, real diagonally dominant matrices with positive diagonal entries, and numerous others.

The initial comprehensive examination of P-matrices can be traced back to the research conducted by Fiedler and Pták (1966). Subsequently, the exploration of P-matrices and their various subclasses has become a fertile area of study, as evidenced by the significant interest they have garnered within the matrix theory community and the motivation they provide for applications across mathematical and social sciences (see Berman and Plemmons (1994b); Fiedler (1986); Horn and Johnson (1991)). The problem of determining whether an $n \times n$ matrix belongs to the P class or not is known as the P-problem. It is receiving attention due to its inherent computational complexity, it is known to be NP-hard (Coxson (1994, 1999)), and due to the connection of P-matrices to the linear complementarity problem and to the self-validating methods for its solution (see Chen et al. (2001); Jansson and Rohn (1999); Rohn and Rex (1996); Rump (2001)). A recursive $O(2^n)$ algorithm for the P-problem that is simple to implement and lends itself to computation in parallel is provided in Tsatsomeros and Li (2000). In Rump (2003), a strategy is provided for detecting P-matrices, which is not a priori exponential, although it can be exponential in the worst case. Gale and Nikaidô (1965) characterized the class of P-matrices in terms of the sign-reversal set.

We use the following notations in the sequel:

1. The set of all complex P-matrices is denoted by $\mathscr{P}$.

2. We denote $\{1,2,3,\ldots,n\}$ by $\langle n \rangle$.

3. We denote the set of all $n \times n$ matrices with complex entries by $M_n(\mathbb{C})$.

4. $A[\alpha,\beta]$ is the submatrix of A whose rows and columns are indexed by $\alpha,\beta \subseteq \langle n \rangle$ respectively. Elements in α,β are assumed to be in ascending order. We abbreviate $A[\alpha,\alpha]$ by $A[\alpha]$ and refer it as a principal submatrix of A. $A[\overline{\alpha}] = A[\langle n \rangle \backslash \alpha, \langle n \rangle \backslash \alpha]$, $A[\overline{\alpha},\alpha] = A[\langle n \rangle \backslash \alpha, \alpha]$ and $A[\alpha,\overline{\alpha}] = A[\alpha, \langle n \rangle \backslash \alpha]$.

5. For $A \in M_n(\mathbb{C})$, $\alpha \subseteq \langle n \rangle$ and $A[\alpha]$ is invertible, we denote $A/A[\alpha]$, the Schur complement of $A[\alpha]$ in A. That is, $A/A[\alpha] = A[\overline{\alpha}] - A[\overline{\alpha},\alpha]A[\alpha]^{-1}A[\alpha,\overline{\alpha}]$.

6. We call a diagonal matrix S whose diagonal entries are ± 1, a signature matrix.

Note that $S^{-1} = S$. Ingleton (1966) called such matrices by sign-changing matrices.

7. $GL_n(\mathbb{R})$ denotes the group of $n \times n$ invertible matrices of real numbers.

8. Let I_M, be the diagonal matrix obtained from the $n \times n$ identity matrix I, by replacing 1 by -1 in the m^{th} row, for each $m \in M \subseteq \langle n \rangle$. Thus $(I_M)_{ii} = 1$ if $i \in M$, $(I_M)_{ii} = -1$ if $i \notin M$ and $(I_M)_{ij} = 0$ if $i \neq j$. For example, $I_\phi = I$, where ϕ is the empty set, and $I_{\langle n \rangle} = -I$.

Definition 1.4.1. *(Tsatsomeros (2002)) We call a matrix A semi-positive if there exists $x \geq 0$ such that $Ax > 0$. Notice that by continuity of the map $x \mapsto Ax$, the semipositivity of A is equivalent to the existence of $u > 0$ such that $Au > 0$.*

Definition 1.4.2. *(Choudhury and Kannan (2021))*

1. *If all the principal minors of a matrix $A \in \mathbb{R}^{n \times n}$ are negative, then A is called an N-matrix.*

2. *If all the proper principal minors of a matrix $A \in \mathbb{R}^{n \times n} (n \geq 2)$ are positive, and the determinant of A is negative, then the matrix A is said to be an almost P-matrix.*

Definition 1.4.3. *(Gowda (1986)) Let $(\mathscr{H}, \langle .,. \rangle)$be a real Hilbert space and let T be a bounded linear operator on $\mathscr{H}$. Let K denote a non-empty closed convex cone in $\mathscr{H}$. We say that T is positive definite on K if $\langle Tx, x \rangle > 0$ for all non-zero $x \in K$.*

Definition 1.4.4. *(Tsatsomeros (2002)) Given a non-empty $\alpha \subseteq \langle n \rangle$ and provided that $A[\alpha]$ is invertible, define the principal pivot transform of $A \in M_n(\mathbb{C})$ relative to α as the matrix $ppt(A, \alpha)$ obtained from A by replacing*

$$A[\alpha] \quad by \quad A[\alpha]^{-1} \qquad A[\alpha, \overline{\alpha}] \quad by \quad -A[\alpha]^{-1}A[\alpha, \overline{\alpha}]$$

$$A[\overline{\alpha}, \alpha] \quad by \quad A[\overline{\alpha}, \alpha]A[\alpha]^{-1} \qquad A[\overline{\alpha}] \quad by \quad A/A[\alpha]$$

By convention, $ppt(A, \phi) = A$. To illustrate this definition, when $\alpha = \{1, \dots, k\} (0 < k < n)$,

$$ppt(A, \alpha) = \begin{bmatrix} A[\alpha]^{-1} & A[\alpha]^{-1}A[\alpha, \overline{\alpha}] \\ A[\overline{\alpha}, \alpha]A[\alpha]^{-1} & A/A[\alpha] \end{bmatrix}.$$

Theorem 1.4.5. *(Tsatsomeros (2002)) Let $A \in \mathscr{P}$ be a matrix in $M_n(\mathbb{C})$. Then, the following statements hold.*

1. *The transpose of the matrix A, denoted by A^T always belongs to the collection $\mathscr{P}$.*

2. *For every permutation matrix P the matrix PAP^T belongs to the collection $\mathscr{P}$.*

3. *For every signature matrix S the matrix SAS belongs to the collection $\mathscr{P}$.*

4. *The matrix CAD belongs to the collection $\mathscr{P}$, where C and D are any two diagonal matrices so that CD has positive diagonal entries.*

5. *The matrix $A+D$ belongs to the collection $\mathscr{P}$, for any diagonal matrices D with non-negative entries.*

6. *The principal sub-matrices $A[\beta]$ belongs to the collection $\mathscr{P}$, for every non-empty $\beta \subseteq \langle n \rangle$.*

7. *The Schur complement of $A[\beta]$ in A, that is $A \backslash A[\beta]$ belongs to the collection $\mathscr{P}$, for all $\beta \subseteq \langle n \rangle$.*

8. *For any $\beta \subseteq \langle n \rangle$, the principal pivot transform of A relative to β, that is $ppt(A,\beta)$ belongs to the collection $\mathscr{P}$. In particular, when $\beta = \langle n \rangle$, we obtain that $ppt(A,\beta) = A^{-1} \in \mathscr{P}$.*

9. *The matrices $DI + (I-D)A$ belongs to the collection $\mathscr{P}$, for all diagonal matrices $D \in [0,I]$.*

Definition 1.4.6. *(Johnson and Tsatsomeros (1995)) Let $diag(d_1,\ldots,d_n)$ denote the diagonal matrix whose diagonal entries are $d_1,\ldots,d_n$. Let J denote the all ones matrix, and let $\circ$ denote the Hadamard (entrywise) product of matrices. For any $A,B \in M_n(\mathbb{C})$ we define the following sets:*

1. $h(A,B) = \{M : M = dA + (1-d)B, d \in [0,1]\}$.

2. $r(A,B) = \{M : M = DA + (I-D)B, D = diag(d_1,\ldots,d_n), d_i \in [0,I]\}$.

3. $c(A,B) = \{M : M = AD + B(I-D), D = diag(d_1,\ldots,d_n), d_i \in [0,I]\}$.

4. $i(A,B) = \{M : M = D \circ A + (J-D) \circ B, T = (d_{ij}), d_{ij} \in [0,1]\}$.

Theorem 1.4.7. *(Johnson and Tsatsomeros (1995)) Let $A,B \in M_n(\mathbb{R})$. Then the following are equivalent:*

1. $r(A,B) \subseteq GL_n(\mathbb{R})$.
2. $BA^{-1} \in \mathscr{P}$.

Theorem 1.4.8. *(Johnson and Tsatsomeros (1995)) Let $A,B \in M_n(\mathbb{R})$. Then the following are equivalent:*

1. $c(A,B) \subseteq GL_n(\mathbb{R})$.
2. $B^{-1}A \in \mathscr{P}$.

Theorem 1.4.9. *(Tsatsomeros (2002)) If $A \in M_n(\mathbb{R})$, then A is a P-matrix if and only if for each non-zero $x \in \mathbb{R}^n$, there exists $j \in \langle n \rangle$ such that $x_j(Ax)_j > 0$. That is, A is a P-matrix if and only if $rev(A) = \{0\}$.*

Theorem 1.4.10. *(Tsatsomeros (2002)) Let $A \in M_n(\mathbb{R})$ be a P-matrix. Then A is a semipositive matrix.*

Theorem 1.4.11. *(Tsatsomeros (2002)) Let $A = BC^{-1} \in M_n(\mathbb{R})$. Then $A \in \mathscr{P}$ if and only if the matrix $DB + (I-D)C$ is invertible for every $D \in [0,I]$.*

Theorem 1.4.12. *(Al-Nowaihi (1988)) A is a P-matrix if and only if, for each $M \in \langle n \rangle$, $(I_M A I_M)x$ has a solution $x > 0$.*

Theorem 1.4.13. *(Al-Nowaihi (1988)) Let $A \in M_n(\mathbb{R})$. Then A is a P-matrix if and only if SAS is semipositive, for every signature matrix $S \in M_n(\mathbb{R})$.*

The class of M-matrices was introduced by Ostrowski (7 38). A significant correlation exists between P-matrices and M-matrices. Next, we give the formal definition of M-matrices as follows.

Definition 1.4.14. *(Kannan and Sivakumar (2016)) When all the off-diagonal entries of an $n \times n$ matrix A are non-positive, then the matrix A is called a Z-matrix. We can represent a Z-matrix A as $A = tI - P$, where $t \geq 0$ and $P \geq 0$ (such a P is called a non-negative matrix). The notation $P \geq 0$ denotes that all its entries are non-negative. An M-matrix is a Z-matrix A which can be represented as $A = tI - P$, where $t \geq \rho(P)$ and $P \geq 0$. Here, $\rho(P)$ denotes the spectral radius of P.*

The study of P-matrices has been extended to the idea of P-operator to infinite-dimensional Banach spaces by Kannan and Sivakumar (2016). More properties of P-matrices can be found in Tsatsomeros (2017, 2002); Johnson and Tsatsomeros (1995).

1.4.2 ADEQUATE MATRICES

In this section, we discuss a class of matrices called adequate matrices, which are helpful in discussing the uniqueness of the solution to certain kinds of linear complementarity problems. The concept of this class of matrices was introduced by Ingleton (1966). Later Eaves (1971) given a characterization for the class of column adequate matrices with the help of a sign-reversing set in a similar way as that of P-matrices.

Definition 1.4.15. *(Cottle et al. (1989)) For a given matrix A and index set α, $A[\alpha,\bullet]$ denotes the sub-matrix of A consisting of the rows indexed by α, similarly $A[\bullet,\alpha]$ denotes the sub-matrix of A consisting of columns indexed by α. Its principal submatrices can be expressed as $A[\alpha]$ where $\alpha \subseteq \langle n \rangle$. Let $A \in \mathbb{R}^{n\times n}$ be a matrix with non-negative principal minors. Then A is called*

1. *row adequate if for all $\alpha \subseteq \langle n \rangle$*

$$det\, A[\alpha] = 0 \Leftrightarrow \text{the rows of } A[\alpha,\bullet] \text{ are linearly dependent.}$$

2. *column adequate if for all $\alpha \subseteq \langle n \rangle$*

$$det\, A[\alpha] = 0 \Leftrightarrow \text{the columns of } A[\bullet,\alpha] \text{ are linearly dependent.}$$

A matrix is adequate if it is both row and column adequate.

Theorem 1.4.16. *(Cottle et al. (1989)) Let $A \in \mathbb{R}^{n\times n}$. Then*

1. *A is row adequate if and only if*

$$x_i(A^T x)_i \leq 0 \text{ for all } i \in \langle n \rangle \text{ implies } A^T x = 0.$$

i.e., A is row adequate if and only if $rev(A^T) = ker(A^T)$.

2. *A is column adequate if and only if*

$$x_i(Ax)_i \leq 0 \text{ for all } i \in \langle n \rangle \text{ implies } Ax = 0.$$

i.e., A is column adequate if and only if $rev(A) = ker(A)$.

Proposition 1.4.17. *(Ingleton (1966)) If A is adequate, then so is $A[\alpha,\alpha]$ for every $\alpha \subseteq \langle n \rangle$.*

Proposition 1.4.18. *(Ingleton (1966)) If A is adequate and D is a sign-changing matrix of the same order, then DAD is adequate.*

Theorem 1.4.19. *(Cottle (1968a)) A non-singular matrix is adequate if and only if it has a positive principal minor.*

Theorem 1.4.20. *(Cottle (1968a)) Let A be an adequate matrix of order $n \times n$ and let $y = Ax$. If $x_i y_i \leq 0$ for $i = 1, \ldots, n$, then $y = 0$.*

Theorem 1.4.21. *(Cottle (1968a)) Let A be an adequate matrix and $c = Av$ for some vector v. Then, there exists a unique y such that for some x*

$$y = Ax + c, \quad x \geq 0, \quad y \geq 0, \quad x^T y = 0.$$

When A is non-singular, x is also unique.

Definition 1.4.22. *(Xu (1999)) The matrix A is said to be column competent if $x_i(Ax)_i = 0$, for all $i = 1, 2, \ldots, n$ implies $Ax = 0$.*

Theorem 1.4.23. *(Xu (1999)) The following statements are equivalent:*

1. *A is column competent and A is a P_0-matrix.*
2. *A is column adequate.*

More results and theories about adequate matrices can be found in (Dutta et al. (2022); Xu (1993a); Cottle and Guu (1992)).

1.4.3 SUFFICIENT MATRICES

In this section, we discuss a class of matrices called sufficient matrices, playing an intrinsic role in the theory of the LCP and have an algorithmic significance for the LCP (Cottle et al. (1989), Cottle (1990), Cottle and Chang (1992), Lemke (1965), Cottle and Dantzig (1968a), Pang (1991) and Kojima et al. (1991a)).

Definition 1.4.24. *(Guu (1996)) A matrix $A \in \mathbb{R}^{n \times n}$ is called column sufficient (CSU) if $x_i(Ax)_i \leq 0$ for all $i \in \langle n \rangle$, implies $x_i(Ax)_i = 0$ for all $i \in \langle n \rangle$. The matrix A is said to be row sufficient (RSU) if A^T is column sufficient. A matrix that is both row and column sufficient is simply called sufficient (SU). We denote the class of sufficient matrices by* **S**.

As highlighted by Cottle and Guu (1992), one limitation of these matrices is the absence of established identification methods. In the work of Cottle and Guu (1992) presents two finite tests for determining sufficiency. Additionally, Cottle and Guu (1992); Cottle et al. (1989) demonstrate that all P-matrices and positive semi-definite matrices (regardless of symmetry) are sufficient and that all column (row) adequate matrices are column (row) sufficient.

Proposition 1.4.25. *(Cottle (1990)) Let $A \in \mathbb{R}^{n\times n}$.*

1. *Every principal rearrangement $P^T AP$ of a column (row) sufficient matrix A is column (row) sufficient, where P is any permutation matrix (in the same size as A).*

2. *If A is column (row) sufficient, then so is DAD for any conformable diagonal matrix D.*

3. *Each principal submatrix of a column (or row) sufficient matrix retains its column (or row) sufficiency.*

4. *A matrix of the form*

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

with $b \neq 0$ cannot be column sufficient. Its transpose cannot be row sufficient.

5. *Each principal pivot transform of a column (or row) sufficient matrix preserves its column (or row) sufficiency.*

6. *The formula*

$$(\rho_\alpha(A))^T = E_{\overline{\alpha}}(\rho_\alpha(A^T))E_{\overline{\alpha}}$$

relates the transpose of a principal pivot transform of a matrix A to the corresponding principal pivot transform of A^T. In this formula, $\rho_\alpha(A)$ denotes the principal pivot transform of A relative to the non-singular principal submatrix $A[\alpha]$ and $E_{\overline{\alpha}}$ denotes the diagonal matrix given by

$$E = (e_{ii}) = \begin{cases} 1 & \text{if } i \in \alpha \\ -1 & \text{if } i \in \overline{\alpha}. \end{cases}$$

Theorem 1.4.26. *(Väliaho (1996)) Let $A \in \mathbb{R}^{n\times n}$ be of rank one and have non-negative diagonal. Then*

(i) A is column sufficient if and only if $a_{ii} = 0$ implies $A[\bullet, i] = 0$ for all $i \in \langle n \rangle$.

(ii) A is row sufficient if and only if $a_{ii} = 0$ implies $A[i, \bullet] = 0$ for all $i \in \langle n \rangle$.

(iii) A is sufficient if and only if $a_{ii} = 0$ implies $A[i, \bullet] = 0$ and $A[i, \bullet] = 0$ for all $i \in \langle n \rangle$.

In particular, A is sufficient if it has positive diagonal.

Theorem 1.4.27. *(Väliaho (1996)) Let $x, y \in \mathbb{R}^n \backslash \{0\}$ and $A = yx^T \in \mathbb{R}^{n \times n}$. Then*

(i) A is column sufficient if and only if $x_i = 0$ or $x_i y_i > 0$ for all $i \in \langle n \rangle$.

(ii) A is row sufficient if and only if $y_i = 0$ or $x_i y_i > 0$ for all $i \in \langle n \rangle$.

(iii) A is sufficient if and only if $x_i = y_i = 0$ or $x_i y_i > 0$ for all $i \in \langle n \rangle$.

Theorem 1.4.28. *(Kojima et al. (1991b)) If $A \in \mathbf{S}$, then $PMQ \in \mathbf{S}$ for any diagonal matrices P and Q, where $P_{ii}Q_{ii} > 0$ for all indices i.*

Theorem 1.4.29. *(Väliaho (1997)) The following statements hold good:*

1. *If $A_1, A_2 \in \mathbf{S}$ (possibly with different sizes), then $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in \mathbf{S}$.*

2. *If $A \in \mathbf{S}$, then $\begin{pmatrix} A & I \\ -I & D \end{pmatrix} \in \mathbf{S}$ for any non-negative diagonal matrix D.*

3. *If $M \in \mathbf{S}$, then $M + D \in \mathbf{S}$ for any non-negative diagonal matrix D.*

More results can be found in E.-Nagy et al. (2023); Cottle et al. (1989); Adler et al. (2006); Väliaho (1996); Xu (1993a).

1.5 INCLUSION RELATIONS BETWEEN MATRIX CLASSES

In this section, we discuss some inclusion relations between different matrix classes which are under consideration. We denote P_0 as the collection of matrices having all of its principal minors that are non-negative.

Theorem 1.5.1. *(Xu (1993a)) Let $A \in \mathbb{R}^{n \times n}$ be given by*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $A \in P_0$, then

1. *A is not column sufficient if and only if $c = 0, d = 0, b \neq 0$ or $a = 0, b = 0, c \neq 0$.*

2. *A is not row sufficient if and only if $a = 0, c = 0, b \neq 0$ or $b = 0, d = 0, c \neq 0$.*

Example 1.5.2. *1. The matrices*

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are not column sufficient by Theorem 1.5.1 (1).

2. The matrices

$$B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

are not row sufficient by Theorem 1.5.1 (2).

Remark 1.5.3. *Membership in the collection of column sufficient matrices does not necessarily entail membership in the collection of row sufficient matrices. This distinction is demonstrated in the example below.*

Example 1.5.4. *Consider the matrix*

$$B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

from the Example 1.5.2, we have B_2 is not row sufficient, but we can see that it is column sufficient as follows:

Here for $X = \begin{pmatrix} x \\ y \end{pmatrix}$, the inequalities $X_i(B_2X)_i \leq 0$ for $i = 1,2$ satisfies if and only if $y = 0$. That is for any vectors of the form $X = \begin{pmatrix} x \\ 0 \end{pmatrix}$ the inequalities $X_i(B_2X)_i \leq 0$ and moreover in that case $B_2X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus $X_i(B_2X)_i \leq 0$ for $i = 1,2$ implies $X_i(B_2X)_i = 0$ for $i = 1,2$. This implies B_2 is a column sufficient matrix.

Remark 1.5.5. *A matrix falling within the category of row sufficient matrices does not necessarily fall within the category of column sufficient matrices. This distinction is exemplified in the following instance.*

Example 1.5.6. *Consider the matrix*

$$A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

from the Example 1.5.2, we have A_2 is not column sufficient, but we can see that it is row sufficient as follows:

Here $A_2^T = B_2$ which is column sufficient matrix from previous Example 1.5.4. Thus by the definition of row sufficiency, we see that A_2 is row sufficient.

Remark 1.5.7. *Membership in the class of row adequate matrices does not guarantee inclusion in the class of column sufficient matrices, as illustrated by the following example.*

Example 1.5.8. *Consider the matrix*

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

From the Example 1.5.2, we have A_1 is not column sufficient, but we can see that it is row adequate as follows:

Here, $det(A_1[\{1\}]) = 1$, $det(A_1[\{2\}]) = 0$ and $det(A_1) = 0$. That is, all the principal minors of A_1 are non-negative, and also $det(A_1[\alpha]) = 0$ if and only if rows of $A_1[\alpha, \bullet]$ are dependent. Here for $\alpha = \{2\}$ and $\alpha = \langle n \rangle$ we have $det(A_1[\alpha]) = 0$, in both of these cases the rows of $A_1[\alpha, \bullet]$ are linearly dependent. This shows that A_1 is row adequate.

Remark 1.5.9. *A matrix belonging to the class of column adequate matrices does not imply that it belongs to the class of row sufficient matrices. It is illustrated in the following example.*

Example 1.5.10. *Consider the matrix*

$$B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

From the Example 1.5.2, we have B_1 is not row sufficient, but we can see that it is column adequate as follows:

Here, $det(B_1[\{1\}]) = 1$, $det(B_1[\{2\}]) = 0$ and $det(B_1) = 0$. That is, all the principal minors of B_1 are non-negative, and also $det(B_1[\alpha]) = 0$ if and only if columns of $B_1[\bullet, \alpha]$ are dependent. Here for $\alpha = \{2\}$ and $\alpha = \langle n \rangle$ we have $det(B_1[\alpha]) = 0$, in both of these cases the columns of $B_1[\bullet, \alpha]$ are linearly dependent. This shows that B_1 is column adequate.

Remark 1.5.11. *There are matrices that are neither row adequate nor row sufficient. The next example shows this fact.*

Example 1.5.12. *Consider the matrix*

$$B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

From the Example 1.5.2, we have B_1 is not row sufficient. Also, we can see that it is not row adequate as follows:

Here, $det(B_1[\{1\}]) = 1$, $det(B_1[\{2\}]) = 0$ and $det(B_1) = 0$. That is all the principal minors of B_1 are non-negative, but here the condition $det(B_1[\alpha]) = 0$ if and only if rows of $B_1[\alpha, \bullet]$ are dependent, not satisfying for $\alpha = \{2\}$. Here for $\alpha = \{2\}$ we have $det(B_1[\alpha]) = 0$, but rows of $B_1[\alpha, \bullet]$, that is single row vector $\{[1,0]\}$, is linearly independent. This shows that B_1 is not row adequate.

Remark 1.5.13. *There are matrices that are neither column adequate nor column sufficient. The next example shows this fact.*

Example 1.5.14. *Consider the matrix*

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

From the Example 1.5.2, we have A_1 is not column sufficient. Also, we can see that it is not column adequate as follows:

Here, $det(A_1[\{1\}]) = 1$, $det(A_1[\{2\}]) = 0$ and $det(A_1) = 0$. That is all the principal minors of A_1 are non-negative, but here the condition $det(A_1[\alpha]) = 0$ if and only if columns of $A_1[\bullet, \alpha]$ are dependent, not satisfying for $\alpha = \{2\}$. Here for $\alpha = \{2\}$ we have $det(A_1[\alpha]) = 0$, but columns of $A_1[\bullet, \alpha]$, that is single column vector $\{[1,0]\}$, is linearly independent. This shows that A_1 is not column adequate.

Remark 1.5.15. *A P-matrix is always a sufficient matrix. This we can see as, if A is a P-matrix, then the sign non-reversal property of P-matrix (Theorem 1.4.9) says that $x_i(Ax)_i \leq 0$ for all i implies that $x = 0$, but this will directly imply that $x_i(Ax)_i = 0$ for all i, this implies that A is a column sufficient matrix. Next, by Theorem 1.4.5, we have A^T is also a P-matrix, and thus A^T is column sufficient by the above argument, that is, A is row sufficient. Thus, A is a sufficient matrix. But we can have matrices that may be row sufficient or column sufficient or sufficient matrices which may not be a P-matrix. Next, we give such an example.*

Example 1.5.16. *One trivial example is the zero matrix, which is row sufficient, column sufficient, and sufficient matrix, but clearly, it is not a P-matrix. Next, consider the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

here the inequalities $x_i(Ax)_i \leq 0$ for all i implies that $x_1 = 0, x_2 = 0$ and x_3 any number, that is the vectors of the form $(0,0,1)$ will satisfy $x_i(Ax)_i \leq 0$ for all i, but for such vectors we have $Ax = (0,0,0)$. This says that $x_i(Ax)_i \leq 0$ implies $x_i(Ax)_i = 0$ for all i, this implies that A is column sufficient. Similarly, we can see that A is also row sufficient; thus, A is sufficient also. But, as $|A| = 0$ implies that A is not a P-matrix.

Remark 1.5.17. *A P-matrix is always a row adequate matrix. Indeed, if A is a P-matrix, then all the principal minors of A are positive, which implies that all principal sub-matrices would be non-singular. Hence, all the rows of all the principal sub-matrices would be linearly independent. By the definition of row adequate matrices, we get A as row adequate. Similarly, we can show that A is column adequate. Moreover, we can have matrices which may be row adequate or column adequate matrices which may not be P-matrices.*

Example 1.5.18. *Consider the matrix*

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

then as $|A| = 0$, A is not a P-matrix. But, here we can see that all the principal minors of A are non-negative and $|A| = 0$; in this case, rows of A are linearly dependent; thus, by the definition of row adequate matrices, we can see that A is a row adequate matrix. Similarly, we can see that A is column adequate also.

Remark 1.5.19. *Indeed, the class of adequate matrices properly includes the class of P-matrices, but an adequate matrix in general need not be a P-matrix. The class of adequate matrices has a non-empty intersection with the class of positive semi-definite matrices but does not include the class of positive semi-definite matrices.*

Example 1.5.20. *The matrix*

$$A = \begin{pmatrix} 5 & -10 \\ -2 & 4 \end{pmatrix}$$

is adequate but fails to have positive principal minors and is not positive semi-definite either.

Definition 1.5.21. *(cf. Horn and Johnson (2008))*

1. *An $n \times n$ Hermitian matrix A is said to be positive definite if $x^*Ax > 0$ for all non-zero $x \in \mathbb{C}^n$, where x^* denotes the conjugate transpose of x.*

2. *If the strict inequality required in $x^*Ax > 0$ is weakened to $x^*Ax \geq 0$, then A is said to be positive semi-definite.*

3. *Similarly, the terms negative definite and negative semi-definite may be defined for A by reversing the inequalities in the definitions of positive definite and positive semi-definite.*

4. *If a Hermitian matrix falls into none of the aforementioned classes, it is said to be indefinite.*

Proposition 1.5.22. *A positive definite matrix is a P-matrix.*

Proof. Suppose A is a positive definite matrix. Then, by definition for $x \neq 0$ we have $x^T(Ax) > 0$, implies $\sum_{i=1}^{n} x_i(Ax)_i > 0$, this implies that, there exists some index j, for which $x_j(Ax)_j > 0$, this implies that A is a P-matrix. □

Remark 1.5.23. *A positive semi-definite matrix need not be a P-matrix. The following example shows this fact.*

Example 1.5.24. *The $n \times n$ matrix in which all entries are 1, is a rank one positive semi-definite matrix. Clearly, these matrices are not P-matrices. The matrix*

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

is also a positive semi-definite matrix, but as $|A| = 0$, A is not a P-matrix.

Remark 1.5.25. *A P-matrix does not necessarily have to be a positive semi-definite matrix, as demonstrated by the following example.*

Example 1.5.26. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

As every principal minor of the matrix A is positive, we can see that A is a P-matrix, but $x^T(Ax) < 0$ for $x = (-0.1, 0.2)$, this indicates that A is not positive semi-definite matrix.

Proposition 1.5.27. *Every positive semi-definite matrix is row sufficient as well as column sufficient.*

Proof. Let us assume that A represents a positive semi-definite matrix. Then A is symmetric and for any $x \neq 0$, we have $x^T(Ax) \geq 0$ this implies that $\sum_{i=1}^{n} x_i(Ax)_i \geq 0$. But if $x_i(Ax)_i \leq 0$ for all i implies that $x_i(Ax)_i = 0$ for all i. This shows that A is column sufficient. As A is symmetric, by the definition of row sufficient matrices we get A is row sufficient also. □

We end this section with a diagram that shows the inclusion relation between some of these classes of matrices.

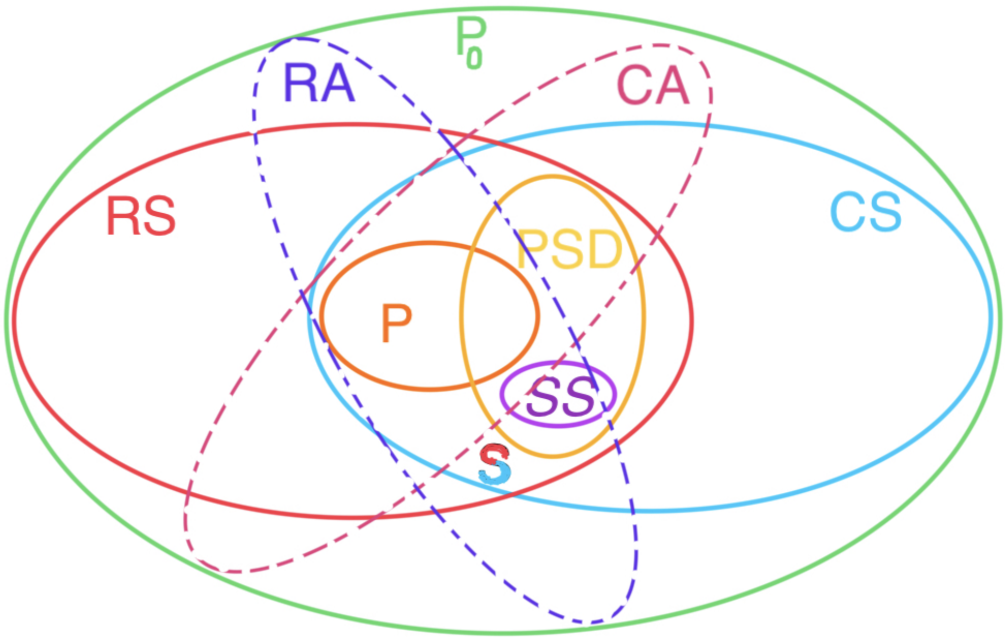


Figure 1.1: The inclusion relations between matrix classes: P=P-matrix class, P_0=P_0-matrix, CS=column sufficient, RS= row sufficient, S= sufficient, SS= skew-symmetric, PSD=positive semidefinite, CA= column adequate, RA= row adequate.

1.6 P-OPERATORS ON BANACH SPACES

The study of P-matrices has been extended to the notion of P-operator to infinite-dimensional Banach spaces having a Schauder basis by Kannan and Sivakumar (2016). In this section, we mainly discuss their work, which motivated us to work on the topic in Hilbert space settings to identify more interesting aspects of the operator.

Definition 1.6.1. *(Kannan and Sivakumar (2016)) Let X be a real Banach space. A sequence $\{z_n\}$ in X is said to be a Schauder basis for X if for each $x \in X$, there exists a unique sequence of scalars $\{\alpha_n(x)\}$ in X such that $x = \sum_{n=1}^{\infty} \alpha_n(x) z_n$. In such a case, we denote $x_n = \alpha_n(x)$, $n \in \mathbb{N}$.*

Definition 1.6.2. *(Kannan and Sivakumar (2016)) Let X be a Banach space with a Schauder basis $\boldsymbol{B}$. For $A, B \in \mathscr{B}(X)$, Let $h(A,B) = \{C \in \mathscr{B}(X) : tA + (1-t)B, t \in [0,1]\}$. Let T be a diagonal operator relative to $\boldsymbol{B}$ with diagonal entries in $[0,1]$. Let $c(A,B) = \{C \in \mathscr{B}(X) : C = AT + B(I-T)\}$ and $r(A,B) = \{C \in \mathscr{B}(X) : C = TA + (I-T)B\}$. Let $\mathscr{G}\mathscr{L}(X)$ denote the set of all bounded linear invertible operators on X.*

Definition 1.6.3. *(Kannan and Sivakumar (2016)) Let X be a Banach space with an unconditional Schauder basis $\mathbf{B}$. Let $T \in B(X)$ be partitioned such that*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

where T_{11} is invertible. The principal pivot transform of T relative to T_{11} is defined as

$$ppt(T, T_{11}) = \begin{pmatrix} T_{11}^{-1} & -T_{11}^{-1}T_{12} \\ T_{21}T_{11}^{-1} & (T/T_{11}) \end{pmatrix}.$$

Definition 1.6.4. *(Kannan and Sivakumar (2016)) Let X be a Banach space with a Schauder basis $\boldsymbol{B} = \{z_i\}_{i=1}^{\infty}$. A bounded linear operator $T : X \longrightarrow X$ is said to be a P-operator relative to the given Schauder basis $\boldsymbol{B}$ if for $x \in X$ (with $x = \sum_{i=1}^{\infty} x_i z_i$ and $Tx = \sum_{i=1}^{\infty} (Tx)_i z_i$), the inequalities $x_i(Tx)_i \leq 0$ for all i imply that $x = 0$.*

Lemma 1.6.5. *(Kannan and Sivakumar (2016)) Let $T \in B(X)$ be an invertible P-operator relative to a given Schauder basis of X. Then T^{-1} is a P-operator relative to the same Schauder basis.*

Theorem 1.6.6. *(Kannan and Sivakumar (2016)) Let $A, B \in \mathscr{B}(X)$ be invertible. Then $h(A,B) \subseteq \mathscr{G}\mathscr{L}(X)$ if and only if BA^{-1} has no negative spectral value.*

Lemma 1.6.7. *(Kannan and Sivakumar (2016)) Let X be a Banach space with an unconditional Schauder basis $\boldsymbol{B}$. Let $\boldsymbol{B} = \boldsymbol{B}_1 \cup \boldsymbol{B}_2$ be a partition of $\boldsymbol{B}$ and*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

be the corresponding partition of T. If T is a P-operator relative to the Schauder basis $\boldsymbol{B}$, then T_{11} and T_{22} are P-operators relative to the Schauder bases $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, respectively.

Theorem 1.6.8. *(Kannan and Sivakumar (2016)) Let X be a Banach space with an unconditional Schauder basis* $\boldsymbol{B}$*. Let* $\boldsymbol{B} = \boldsymbol{B}_1 \cup \boldsymbol{B}_2$ *be a partition of* $\boldsymbol{B}$ *such that*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

is a P-operator, with T_{11} *invertible. Then* $ppt(T, T_{11})$ *is a P-operator relative to* $\boldsymbol{B}$*.*

Theorem 1.6.9. *(Kannan and Sivakumar (2016)) Let X be a Banach space with an unconditional Schauder basis* $\boldsymbol{B}$*. Let* $\boldsymbol{B} = \boldsymbol{B}_1 \cup \boldsymbol{B}_2$ *be a partition of* $\boldsymbol{B}$ *such that*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

is a P-operator, with T_{11} *invertible. Then* (T/T_{11}) *is a P-operator relative to* $\boldsymbol{B}$*.*

1.7 RESEARCH OBJECTIVES

The overall objectives can be summarized as follows;

(i) Characterize the sign-reversing set of an arbitrary matrix and identify a class of matrices with convex sign-reversing sets.

(ii) Generalize the concept of P-matrices to Hilbert space settings and examine their structural properties.

(iii) Extend the concept of sign-reversing sets from matrices to operators in separable Hilbert spaces and utilize this framework to generalize certain matrix classes, such as sufficient matrices, into operators within an infinite-dimensional Hilbert space.

(iv) Investigate the spectral results of specific classes of positive operators under consideration, focusing on characterizing their spectral properties and analyzing the behavior of their various eigenvalues and eigenvectors.

1.8 ORGANIZATION OF THE THESIS

The organization of the thesis is as follows: This thesis consists of six chapters. In Chapter 1, the introduction, preliminaries, literature review, and mathematical backgrounds are given, which are essential for the subsequent chapters.

In Chapter 2, the sign-reversing set of an arbitrary matrix A is characterized in terms of the null space of the matrices $DA - A - D$, where D represents diagonal matrices with diagonal elements in $[0,1]$. Then, a subclass of matrices having convex sign-reversing sets is investigated. Moreover, the sign-reversing set of several classes of matrices, including P-matrix, sufficient matrices, and adequate matrices, is studied.

In Chapter 3, the concept of P-matrix is generalized to separable Hilbert space settings by implementing the sign-reversing property with the help of the inner-product structure of the Hilbert space relative to an orthonormal basis. It is observed that an operator can be a P-operator relative to several orthonormal bases under certain conditions. Some properties and results of such operators are discussed in this chapter.

In Chapter 4, the concept of the sign-reversing property and sign-reversing set for operators on separable Hilbert spaces relative to a given orthonormal basis is defined. The class of column sufficient matrices and row sufficient matrices in separable Hilbert spaces is also generalized, and some observations on these classes of operators are obtained in this chapter.

In Chapter 5, the spectral properties such as eigenvalues, approximate eigenvalues, and spectral values associated with certain classes of positive operators under consideration are discussed.

The conclusion of the thesis and the scope for future work are presented in the final Chapter 6.

CHAPTER 2

ON A SUB-CLASS OF MATRICES HAVING CONVEX SIGN-REVERSING SET

2.1 INTRODUCTION

The sign-reversing set of a matrix forms a cone, but its convexity is not assured, a requisite for many cone theory applications. However, matrices possessing convex sign-reversing sets encompass large classes of matrices. Characterizing an $n \times n$ matrix A to have $rev(A)$ as a convex set is an open problem. We have seen that the sign reversing set $rev(A)$ of a P-matrix A is $\{0\}$, which is trivially convex. But for a general matrix A, the sign reversing set $rev(A)$ need not be convex.

Example 2.1.1. *Let $A \in \mathbb{R}^{2\times 2}$ be given by*

$$A = \begin{pmatrix} -1 & 1 \\ \frac{-5}{2} & \frac{-13}{2} \end{pmatrix}.$$

Then we can see that $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{R}^2$ and $y = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ are elements in $rev(A)$, because $Ax = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, and $x_i(Ax)_i \leq 0$ for $i = 1$ and 2. Also $Ay = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and $y_i(Ay)_i \leq 0$ for $i = 1$ and 2. But $x + y \notin rev(A)$, because $x + y = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $A(x+y) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $(x+y)_i(A(x+y))_i > 0$ for $i = 2$. Therefore, $rev(A)$ is not convex.

In this case, an alternative way to see the convexity of $rev(A)$ is that the inequalities $x_i(Ax)_i \leq 0$ for $i = 1$ and 2 can be plotted together on a Euclidean plane $\mathbb{R}^2$ as shown in the following figure. The grey-shaded region indicates $rev(A)$, which is not convex.

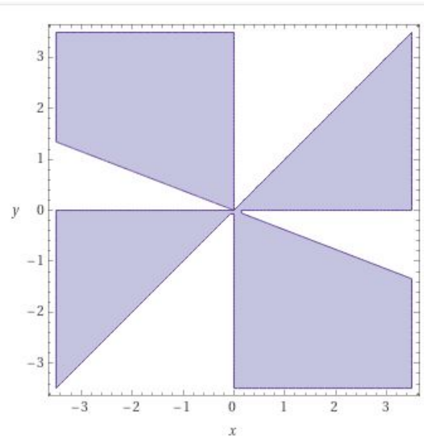


Figure 2.1: $rev(A)$

In this chapter, we give a class of matrices which are having $rev(A)$ convex, and we also observed that this class has a plethora of important classes of matrices such as P-matrices, row and column adequate matrices, and sufficient matrices under certain conditions. The notation $0 \leq D \leq I$ stands for $D = (d_{ii})$ a diagonal matrix such that the diagonal entries d_{ii} satisfy $0 \leq d_{ii} \leq 1$ for all i.

2.2 CHARACTERIZATIONS OF THE SIGN-REVERSING SET

In this section, we characterize the sign-reversing set of a general matrix, for that first we prove a lemma.

Lemma 2.2.1. *Let $A \in \mathbb{R}^{n\times n}$ and let $x \in \mathbb{R}^n$ be non-zero. If $x \in rev(A)$, then there exists a diagonal matrix $D = (d_{ii})$, with $0 \leq d_{ii} \leq 1$ such that $x \in ker(DA - A - D)$.*

Proof. To prove the lemma, we define the appropriate diagonal elements d_{ii} of D which can be done as follows :

1. Suppose that $x_i(Ax)_i = 0$. If both x_i and $(Ax)_i$ are zero, then any $d_{ii} \in [0,1]$ will do. Otherwise, there are two possibilities.

   (a) If $x_i = 0$, take $d_{ii} = 1$.

   (b) If $x_i \neq 0$, and hence $(Ax)_i = 0$, take $d_{ii} = 0$.

2. Suppose that $x_i(Ax)_i < 0$. Then, x_i and $(Ax)_i$ are non-zero and of opposite sign. Take $d_{ii} = \frac{-(Ax)_i}{x_i-(Ax)_i} = \frac{-x_i(Ax)_i}{x_i^2-x_i(Ax)_i}$. In each case, we have $0 \leq d_{ii} \leq 1$ and $d_{ii}(Ax)_i - (Ax)_i - d_{ii}x_i = 0$ for all $i \in \langle n \rangle$. The proof is now finished.

□

Next, we prove a useful characterization for the sign-reversing set.

Theorem 2.2.2. *Let $A \in \mathbb{R}^{n\times n}$. Then*

$$rev(A) = \bigcup_{0\leq D\leq I} ker(DA - A - D).$$

Proof. By Lemma 2.2.1, we have $rev(A) \subseteq \bigcup_{0\leq D\leq I} ker(DA - A - D)$.

Conversely, let D be a diagonal matrix such that $0 \leq D \leq I$. Let $x \in ker(DA - A - D)$. For each $i \in \langle n \rangle$, we have $d_{ii}(Ax)_i - (Ax)_i - d_{ii}x_i = 0$ and consequently $d_{ii}x_i(Ax)_i - x_i(Ax)_i - d_{ii}x_i^2 = 0$. From this, we deduce the following:

(i) If $d_{ii} = 0$, then $x_i(Ax)_i = 0$.

(ii) If $d_{ii} = 1$, then $-x_i^2 = 0$, thus $x_i(Ax)_i = 0$.

(iii) If $0 < d_{ii} < 1$, then $d_{ii}x_i(Ax)_i - x_i(Ax)_i - d_{ii}x_i^2 = 0$ which implies that $x_i(Ax)_i = \frac{d_{ii}x_i^2}{(d_{ii}-1)} < 0$.

Hence, $x \in rev(A)$. □

Corollary 2.2.3. *Let $A \in \mathbb{R}^{n\times n}$.*

1. *If A is a non-P-matrix, then*

$$rev(A) = \bigcup_{\substack{0\leq D\leq I \\ |DA-A-D|=0}} ker(DA - A - D).$$

2. *If A is a P-matrix, then*

$$rev(A) = ker(DA - A - D)$$

for any diagonal matrix D such that $0 \leq D \leq I$.

Proof. 1. Suppose A is a non P-matrix. Then $rev(A) \neq \{0\}$, by Theorem 2.2.2, we have

$$\{0\} \neq rev(A) = \bigcup_{0\leq D\leq I} ker(DA - A - D).$$

The matrices with $|D'A - A - D'| \neq 0$ will have $ker(D'A - A - D') = \{0\}$. As $ker(DA - A - D)$ is a subspace, $ker(D'A - A - D') \subseteq ker(DA - A - D)$. Thus, it is enough to consider only those kernels with $|DA - A - D| = 0$, that is,

$$rev(A) = \bigcup_{\substack{0 \leq D \leq I \\ |DA - A - D| = 0}} ker(DA - A - D).$$

2. Suppose A is a P-matrix. Then $rev(A) = \{0\}$, by Theorem 2.2.2, we have

$$\{0\} = rev(A) = \bigcup_{0 \leq D \leq I} ker(DA - A - D).$$

This implies that in the union, each of the kernels will be equal to 0. Thus, for any diagonal matrix D such that $0 \leq D \leq I$, we get $rev(A) = ker(DA - A - D)$.

□

2.3 A CLASS OF MATRICES WITH CONVEX SIGN-REVERSING SET

We now discuss the class of matrices which are having convex sign-reversing set. We start with the definition of Hadamard product of two matrices, especially two vectors, multiplied together element-wise.

Definition 2.3.1. *(Halmos (1942)) Let the operation $*$ denote the component-wise multiplication on $\mathbb{R}^n$ as $x * y = [x_1y_1, x_2y_2, \ldots, x_ny_n]$, where $x = [x_1, x_2, \ldots, x_n]$ and $y = [y_1, y_2, \ldots, y_n]$.*

Proposition 2.3.2. *Let $x, y, z \in \mathbb{R}^n$. Then, the following are true.*

1. $x * y = y * x$.
2. $\alpha(x * y) = \alpha x * y = x * \alpha y$, *where* $\alpha \in \mathbb{R}$.
3. $x * (y + z) = x * y + x * z$.
4. $x * Dy = Dx * y = D(x * y)$, *where D is a diagonal matrix.*
5. $x * (y * z) = (x * y) * z$.

Proof. The proofs are trivial. □

We define a class of matrices in which each matrix has a convex sign-reversing set.

Definition 2.3.3. *Let $A \in \mathbb{R}^{n\times n}$. We define the collection as follows:*

$$\mathscr{C} = \left\{A \in \mathbb{R}^{n\times n} : C_A = \{0\} \text{ and } x * Ay = y * Ax \text{ for all } x, y \in rev(A)\right\}$$

where

$$C_A = \{x \in rev(A) : x_j = 0 \text{ and } (Ax)_j = 0, \text{ for some index } j \in \langle n \rangle\}.$$

Theorem 2.3.4. *Let $A \in \mathbb{R}^{n\times n}$. Then $A \in \mathscr{C}$ if and only if there exists a diagonal matrix D such that $0 \leq D \leq I$ with $rev(A) = ker(DA - A - D)$.*

Proof. Let $A \in \mathscr{C}$. If $rev(A) = \{0\}$, then by Corollary 2.2.3, any diagonal matrix D such that $0 \leq D \leq I$ will do. If $x \neq 0 \in rev(A)$ we define the diagonal matrix D as follows:

$$D = (d_{ii}) = \begin{cases} \frac{-x_i(Ax)_i}{x_i^2 - x_i(Ax)_i} & \text{if } x_i \neq 0 \\ 1 & \text{if } x_i = 0. \end{cases}$$

As $x \in rev(A)$, we have $0 \leq D \leq I$, thus by Theorem 2.2.2, we see that $\ker(DA - A - D) \subseteq \text{rev}(A)$. Also we can see that $d_{ii}(Ax)_i - (Ax)_i - d_{ii}x_i = 0$ for all $i \in \langle n \rangle$, that is, $x \in \ker\,(DA - A - D)$. Now if $y \in rev(A)$, then as $A \in \mathscr{C}$ we have $d_{ii}(Ay)_i - (Ay)_i - d_{ii}y_i = 0$ for all $i \in \langle n \rangle$, that is, $y \in \ker\,(DA - A - D)$, thus we get $rev(A) \subseteq \ker(DA - A - D)$ and hence $rev(A) = \ker\,(DA - A - D)$.

On other hand, suppose that A is an $n \times n$ matrix such that $\text{rev(A)} = \ker(DA - A - D)$ for some diagonal matrix D such that $0 \leq D \leq I$. Thus for any $x, y \in \text{rev(A)}$, we have $DAx - Ax - Dx = 0$ and $DAy - Ax - Dy = 0$, which give us the following relations:

$$Ax = DAx - Dx, \tag{2.3.1}$$

$$Ay = DAy - Dy. \tag{2.3.2}$$

By using Proposition 2.3.2 and the above two equations, for $x, y \in rev(A)$, we have

$$\begin{aligned} x * Ay &= x * (DAy - Dy) \\ &= x * D(Ay - y) \\ &= Dx * (Ay - y) \\ &= (DAx - Ax) * (Ay - y) \\ &= DAx * (Ay - y) - Ax * (Ay - y) \\ &= DAx * Ay - DAx * y - Ax * Ay + Ax * y \\ &= Ax * DAy - Ax * Dy - Ax * Ay + Ax * y \end{aligned}$$

$$
\begin{aligned}
&= Ax*(DAy - Dy) - Ax*Ay + Ax*y \\
&= Ax*Ay - Ax*Ay + Ax*y \\
&= Ax*y \\
&= y*Ax.
\end{aligned}
$$

Thus for any $x, y \in \text{rev(A)}$, we have $x*Ay = y*Ax$. Also, if $x \neq 0 \in C_A \subseteq rev(A)$, then there exists some index $j \in \langle n \rangle$ for which we have $x_j = 0$ and $(Ax)_j = 0$. In this case, the Equation 2.3.1 shows that d_{jj}, the j-th component of the diagonal matrix D is not defined, so such an element $x \neq 0$ does not exist in $C_A \subseteq rev(A)$. Therefore $C_A = \{0\}$, which implies that $A \in \mathscr{C}$. This completes the proof. □

Remark 2.3.5. *1. As the set $ker(DA - A - D)$ is always convex, each member of $\mathscr{C}$ will have a convex sign-reversing set.*

2. Note that all P-matrices belong to the class $\mathscr{C}$ as the sign-reversing set of P-matrix is $\{0\}$ which is equal to $ker(DA - A - DA)$ for any diagonal matrix D such that $0 \leq D \leq I$. Moreover, Theorem 1.4.16 indicates that the class of row adequate matrices and column adequate matrices also belong to the class $\mathscr{C}$.

The next proposition shows that the scalar multiple of $A \in \mathscr{C}$ and certain derived matrices of $A \in \mathscr{C}$ with the help of some diagonal matrices are also elements of the collection $\mathscr{C}$.

Proposition 2.3.6. *Let $A \in \mathbb{R}^{n \times n}$ be an element in $\mathscr{C}$. Then the following are true:*

1. *$\alpha A \in \mathscr{C}$, for any positive real scalar α.*
2. *$D + EA \in \mathscr{C}$, for any positive diagonal matrices D and E.*
3. *$I - D + DA \in \mathscr{C}$, for any diagonal matrix D such that $0 \leq D \leq I$.*
4. *$DA - A - D \in \mathscr{C}$, for any diagonal matrix D such that $0 < D < I$.*

Proof. 1. One can see that $rev(\alpha A) = rev(A)$ for $\alpha > 0$, thus $C_{\alpha A} = \{0\}$ and also for $x, y \in rev(\alpha A)$, we have

$$
\begin{aligned}
x*(\alpha A)y &= \alpha x*Ay \\
&= \alpha(x*Ay) \\
&= \alpha(y*Ax) \quad (since\, A \in \mathscr{C}) \\
&= (\alpha y*Ax) \\
&= y*(\alpha A)x.
\end{aligned}
$$

This shows that $\alpha A \in \mathscr{C}$.

2. First we observe that $rev(D+EA) \subseteq rev(A)$ for any positive diagonal matrices $D = (d_{ii})$ and $E = (e_{ii})$, because if $x \in rev(D+EA)$, then $x_i((D+EA)x)_i \leq 0$ for all $i \in \langle n \rangle$. That is, $x_i(Dx)_i + x_i(EAx)_i \leq 0$ for all $i \in \langle n \rangle$, hence $x_i d_{ii} x_i + x_i e_{ii}(Ax)_i \leq 0$ for all $i \in \langle n \rangle$, which implies $e_{ii} x_i (Ax)_i \leq 0$ for all $i \in \langle n \rangle$ as $d_{ii} x_i^2 \geq 0$ for all $i \in \langle n \rangle$, thus $x \in rev(A)$. Therefore $rev(D+EA) \subseteq rev(A)$ and hence $C_{D+EA} = \{0\}$. Now for any $x, y \in rev(D+EA)$, we have

$$\begin{aligned} x*(D+EA)y &= x*(Dy+EAy) \\ &= x*Dy + x*EAy \\ &= (y*Dx) + Ex*Ay \\ &= y*Dx + E(x*Ay) \\ &= y*Dx + E(y*Ax) \quad \text{(since } A \in \mathscr{C} \text{)} \\ &= y*Dx + y*EAx \\ &= y*(D+EA)x. \end{aligned}$$

Now, suppose that there exists $x \neq 0 \in C_{D+EA}$. Then there exists some index $j \in \langle n \rangle$ with $x_j = 0$ and $((D+EA)x)_j = 0$, this implies $x_j = 0$ and $d_{jj}x_j + e_{jj}(Ax)_j = 0$, which gives $x_j = 0$ and $(Ax)_j = 0$ which is a contradiction to $C_A = \{0\}$. Hence, there does not exist a non-zero vector in C_{D+EA}. Thus $C_{D+EA} = \{0\}$. Therefore $D+EA \in \mathscr{C}$.

3. First we observe that $rev(I-D+DA) \subseteq rev(A)$ for any diagonal matrix $D = (d_{ii})$ such that $0 \leq d_{ii} \leq 1$. Now for any $x, y \in rev(I-D+DA)$, we have

$$\begin{aligned} x*(I-D+DA)y &= x*(y-Dy+DAy) \\ &= x*y - x*Dy + x*DAy \\ &= x*y - x*Dy + D(x*Ay) \\ &= x*y - x*Dy + D(y*Ax) \text{ (since } A \in \mathscr{C} \text{)} \\ &= y*x - y*Dx + D(y*Ax) \\ &= y*(I-D+DA)x. \end{aligned}$$

Suppose that there exists $x \neq 0 \in C_{I-D+DA}$. Then there exists some index $j \in \langle n \rangle$ with $x_j = 0$ and $((I-D+DA)x)_j = 0$, this implies $x_j = 0$ and $x_j - d_{jj}x_j + d_{jj}(Ax)_j = 0$, hence $x_j = 0$ and $(Ax)_j = 0$ which is a contradiction to $C_A = \{0\}$. So there does not exist a non-zero vector in C_{I-D+DA}. Thus $C_{I-D+DA} = \{0\}$. Therefore $I-D+DA \in \mathscr{C}$.

4. First we observe that $rev(DA-A-D) \subseteq rev(A)$ for any diagonal matrix D such that $0 < D < I$. Now for any $x, y \in rev(DA-A-D)$, we have

$$\begin{aligned}
x*(DA-A-D)y &= x*(DAy-Ay-Dy) \\
&= x*DAy - x*Ay - x*Dy \\
&= D(x*Ay) - x*Ay - D(y*x) \\
&= D(y*Ax) - y*Ax - D(x*y) \text{ (since } A \in \mathscr{C}) \\
&= y*DAx - y*Ax - y*Dx \\
&= y*(AD-A-D)x.
\end{aligned}$$

Suppose that there exists $x \neq 0 \in C_{DA-A-D}$. Then there exists some index $j \in \langle n \rangle$ with $x_j = 0$ and $((DA-A-D)x)_j = 0$, this implies $x_j = 0$ and $d_{jj}(Ax)_j - (Ax)_j - d_{jj}x_j = 0$. Hence $x_j = 0$ and $d_{jj}(Ax)_j - (Ax)_j = 0$, since $d_{jj} < 1$, so $x_j = 0$ and $(Ax)_j = 0$, which is a contradiction to $C_A = \{0\}$. Hence, there does not exist a non-zero vector in C_{DA-A-D}. Thus $C_{DA-A-D} = \{0\}$. Therefore $DA-A-D \in \mathscr{C}$.

□

In general, the collection $\mathscr{C}$ is not closed under addition and multiplication, which is illustrated in the following example. But under certain conditions, the addition and multiplication are closed in $\mathscr{C}$, as we discuss in the proposition followed by the example.

Example 2.3.7. *Consider the matrices A and B given by*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is observed that $rev(A) = \mathbb{R}^2$ and $rev(B) =$ y-axis. So the sum $A+B$ of these two matrices A and B given by

$$A+B = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

has the sign-reversing set

$$rev(A+B) = \{(x,y) \in \mathbb{R}^2 : y > x,\, y > 0\} \cup \{(x,y) \in \mathbb{R}^2 : y < x,\, y < 0\} \cup \{(x,y) : xy \leq 0\},$$

and the product $AB \in \mathbb{R}^{2\times 2}$ of these two matrices A and B be given by

$$AB = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$$

has the sign-reversing set

$$rev(AB) = \{(x,y) : xy \geq 0\}.$$

It is clear that $C_A = C_B = C_{A+B} = C_{AB} = \{0\}$. Since $x = (x_1, y_1), y = (x_2, y_2) \in rev(A)$, we have $x * Ay = (x_1, y_1) * (-x_2, -y_2) = (-x_1x_2, -y_1y_2) = y * Ax$. Similarly, $x' * By' = y' * Bx'$ for all $x', y' \in rev(B)$. Hence $A, B \in \mathscr{C}$, but for $x = (1, -2)$ and $y = (1, -3) \in rev(A+B)$, it is not true that $x * (A+B)y = y * (A+B)x$. Thus $A + B \notin \mathscr{C}$. Also $x' = (1,2)$ and $y' = (1,3) \in rev(AB)$, it does not satisfy $x' * ABy' = y' * ABx'$. Thus $AB \notin \mathscr{C}$.

Proposition 2.3.8. *Let $A, B \in \mathscr{C}$. The following are true:*

1. *If $rev(A+B) \subseteq rev(A) \cap rev(B)$ and $C_{A+B} = \{0\}$, then $A + B \in \mathscr{C}$.*
2. *If $rev(AB) \subseteq rev(A) \cap rev(B)$, $AB = BA$, $range(A) \subseteq rev(B)$, $range(B) \subseteq rev(A)$ and $C_{AB} = \{0\}$, then $AB \in \mathscr{C}$.*

Proof. 1. As $A, B \in \mathscr{C}$ and $rev(A+B) \subseteq rev(A) \cap rev(B)$, for $x, y \in rev(A+B)$, we have

$$\begin{aligned} x * (A+B)y &= x * (Ay + By) \\ &= x * Ay + x * By \\ &= y * Ax + y * Bx \\ &= y * (Ax + Bx) \\ &= y * (A+B)x. \end{aligned}$$

Hence by assumption, we have $C_{A+B} = \{0\}$. This proves that the sum of A and B belongs to the collection $\mathscr{C}$.

2. As $rev(AB) \subseteq rev(A) \cap rev(B)$, we have $C_{AB} = \{0\}$. Now let $x, y \in rev(AB)$,

$$\begin{aligned} x * ABy &= x * A(By) \\ &= By * Ax \ \text{ (since } By, x \in rev(A) \text{ and } A \in \mathscr{C}) \\ &= y * B(Ax) \ \text{ (since } y, Ax \in rev(B) \text{ and } B \in \mathscr{C}) \\ &= y * BAx \\ &= y * ABx \ \text{ (since } AB = BA). \end{aligned}$$

Hence by assumption, we have $C_{AB} = \{0\}$. This shows that $AB \in \mathscr{C}$.

□

Next, we give an example that illustrates the above proposition.

Example 2.3.9. *Consider two matrices in* $\mathbb{R}^{2\times 2}$ *given by*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then it is observed that $rev(A) = \mathbb{R}^2 = rev(B)$. *The product* $AB \in \mathbb{R}^{2\times 2}$ *is given by*

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

has $rev(AB) = \{(0,y) \in \mathbb{R}^{2\times 2} : y \in \mathbb{R}\}$ *and the sum* $A+B \in \mathbb{R}^{2\times 2}$ *is given by*

$$A+B = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

has $rev(A+B) = \mathbb{R}^2$. *Now, for* $x = (x_1,y_1), y = (x_2,y_2) \in rev(A)$ *we have* $x * Ay = (x_1,y_1) * (-x_2,0) = (-x_1x_2,0) = y * Ax$ *and* $C_A = \{0\}$. *Similarly,* $x' * By' = y' * Bx'$ *for all* $x',y' \in rev(B)$ *and* $C_B = \{0\}$. *This shows that* $A,B \in \mathscr{C}$ *satisfying* $rev(A+B) \subseteq rev(A) \cap rev(B)$ *and* $rev(AB) \subseteq rev(A) \cap rev(B)$. *Also* $AB = BA$ *and* $range(A) \subseteq rev(B)$ *and* $range(B) \subseteq rev(A)$. *Note that for any* $x,y \in rev(A+B)$, *we have* $x * (A+B)y = y * (A+B)x$ *and* $C_{A+B} = \{0\}$. *Thus* $A+B \in \mathscr{C}$. *Also for* $x,y \in rev(AB)$, *we have* $x * ABy = y * ABx$ *and* $C_{AB} = \{0\}$, *so* $AB \in \mathscr{C}$.

Remark 2.3.10. *The conditions* $rev(A+B) \subseteq rev(A) \cap rev(B)$ *and* $rev(AB) \subseteq rev(A) \cap rev(B)$ *are only sufficient for* $A+B$ *and* AB *to be a member of the collection* $\mathscr{C}$ *respectively. The conditions are not necessary for the sum and product to be an element of* $\mathscr{C}$. *We provide an example to illustrate this fact.*

Example 2.3.11. *Consider the matrices* $A,B \in \mathbb{R}^{2\times 2}$ *given by*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is observed that $rev(A) = \{(0,y) \in \mathbb{R}^{2\times 2} : y \in \mathbb{R}\}$ *and* $rev(B) = \{0\}$. *The sum* $A+B \in \mathbb{R}^{2\times 2}$ *of the two matrices A and B given by*

$$A+B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

has $rev(A+B) = \{(0,y) \in \mathbb{R}^{2\times 2} : y \in \mathbb{R}\}$*. Here* $A, B \in \mathscr{C}$ *and the sum* $A+B$ *is also in* $\mathscr{C}$*, but* $rev(A+B) \nsubseteq rev(A) \cap rev(B)$*. Similarly, the product* $AB \in \mathbb{R}^{2\times 2}$ *of the matrices* A *and* B *given by*

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

has $rev(AB) = \{(0,y) \in \mathbb{R}^{2\times 2} : y \in \mathbb{R}\}$*, and* $AB \in \mathscr{C}$*, here* $rev(AB) \nsubseteq rev(A) \cap rev(B)$*.*

2.4 SIGN-REVERSING SET OF SUFFICIENT MATRICES

Next, we discuss the sign-reversing set of the column sufficient matrices.

Theorem 2.4.1. *Let* $A \in \mathbb{R}^{n\times n}$*. If* A *is a column sufficient matrix, then* $rev(A) = rev(DA - A - D)$*, for any diagonal matrix* D *such that* $0 < D < I$*.*

Proof. By Theorem 2.3.6(4), we have $rev(DA-A-D) \subseteq rev(A)$. To get the other inclusion, let $x \in rev(A)$, then $x_i(Ax)_i \leq 0$ for all $i \in \langle n\rangle$. As A is a column sufficient matrix; we have $x_i(Ax)_i = 0$ for all $I \in \langle n\rangle$. Thus $(d_{ii}-1)x_i(Ax)_i = 0$ for all $i \in \langle n\rangle$. Now $-d_{ii}x_i^2 \leq 0$ for all $i \in \langle n\rangle$, which together with the above relation we get $(d_{ii} - 1)x_i(Ax)_i - d_{ii}x_i^2 \leq 0$ for all $i \in \langle n\rangle$, which implies $d_{ii}x_i(Ax)_i - x_i(Ax)_i - d_{ii}x_i^2 \leq 0$ for all $i \in \langle n\rangle$ implies $x_i((DA-A-D)x)_i \leq 0$ for all $i \in \langle n\rangle$, which shows that $x \in rev(DA - A - D)$. This completes the proof. □

Theorem 2.4.2. *Let* $A \in \mathbb{R}^{n\times n}$ *be a matrix satisfying* $(x * Ay) + (y * Ax) \geq 0$*, for all* $x, y \in rev(A)$*. Then, the following are true.*

1. *If* A *is column sufficient, then* A *satisfies* $x * Ay = -y * Ax$ *for all* $x, y \in rev(A)$*.*
2. *If* $A \in \mathscr{C}$*, then* A *is a column sufficient matrix.*

Proof. 1. Suppose that A is a column sufficient matrix satisfying $(x*Ay)+(y*Ax) \geq 0$ for all $x, y \in rev(A)$. Note that $(x*Ay)+(y*Ax) \geq 0$, for all $x, y \in rev(A)$. Hence $x_i(Ay)_i + y_i(Ax)_i \geq 0$ for all $i \in \langle n\rangle$ and for all $x, y \in rev(A)$. As A is a column sufficient matrix, we have $x_i(Ax)_i = 0$ for all $i \in \langle n\rangle$ and for all $x \in rev(A)$. Let $x, y \in rev(A)$. Then

$$\begin{aligned} (x-y)*A(x-y) &= (x-y)*(Ax-Ay) \\ &= (x-y)*Ax - (x-y)*Ay \\ &= x*Ax - y*Ax - x*Ay + y*Ay \\ &= -(x*Ay + y*Ax) \leq 0. \end{aligned}$$

Hence $(x-y)_i(A(x-y))_i \leq 0$ for all $i \in \langle n \rangle$, and thus $x-y \in rev(A)$. As A is a column sufficient matrix, $x-y \in rev(A)$ implies $(x-y)_i(A(x-y))_i = 0$, for all $i \in \langle n \rangle$. For $x, y \in rev(A)$, we have

$$\begin{aligned} x*Ay &= x*Ay - y*Ay \\ &= (x-y)*Ay \\ &= (x-y)*Ay + (x-y)*(A(x-y)) \\ &= (x-y)*(Ay + Ax - Ay) \\ &= (x-y)*(Ax) \\ &= x*Ax - y*Ax \\ &= -y*Ax. \end{aligned}$$

Thus for all $x, y \in rev(A)$, $x*Ay = -y*Ax$.

2. Suppose that $A \in \mathscr{C}$ satisfies $(x*Ay)+(y*Ax) \geq 0$ for all $x, y \in rev(A)$. Since $A \in \mathscr{C}$, we have for all $x, y \in rev(A)$, $x*Ay = y*Ax$. Taking $x = y$, we get $x*Ay = y*Ax = x*Ax$. But $x \in rev(A)$, so $x*Ax \leq 0$. As A satisfies $(x*Ay)+(y*Ax) \geq 0$ for all $x, y \in \mathscr{C}$, we have $2(x*Ax) \geq 0$, hence $x*Ax \geq 0$. Thus, both direction inequalities give $x*Ax = 0$ for all $x \in rev(A)$. Hence, A is a column sufficient matrix.

□

Theorem 2.4.3. *Let $A \in \mathbb{R}^{n \times n}$. Then the following statements are equivalent:*

1. *A is column sufficient.*

2. *For each $x \in ker(DA - A - D)$ with $0 \leq D \leq I$, if $0 < d_{ii} < 1$, then $x_i = 0$.*

Proof. Suppose that $A \in CSU$ and there exists $0 \leq D \leq I$ with $x \in ker(DA-A-D)$, such that $x_i \neq 0$ when $0 < d_{ii} < 1$. Since $d_{ii}x_i(Ax)_i - x_i(Ax)_i - d_{ii}x_i^2 = 0$, $(d_{ii}-1)x_i(Ax)_i = d_{ii}x_i^2$, hence $x_i(Ax)_i = \frac{d_{ii}x_i^2}{(d_{ii}-1)} < 0$, which is impossible. This contradiction shows that x_i must be equal to 0.

Conversely, suppose that A is not column sufficient. Then there exists an $x \in rev(A)$ and an index set α such that $\overline{\alpha} \neq \phi$ and

$$\begin{aligned} x_i(Ax)_i &= 0, i \in \alpha, \quad \text{and} \\ x_i(Ax)_i &< 0, i \in \overline{\alpha}. \end{aligned}$$

Define the diagonal matrix D as

$$D = (d_{ii}) = \begin{cases} 0 & \text{if } i \in \alpha \text{ and } (Ax)_i = 0 \\ 1 & \text{if } i \in \alpha \text{ and } (Ax)_i \neq 0, x_i = 0 \\ \frac{-x_i(Ax)_i}{x_i^2 - x_i(Ax)_i} & \text{if } i \in \overline{\alpha}. \end{cases}$$

Then $0 \leq D \leq I$ and $(DA - A - D)x = 0$ implies $x \in ker(DA - A - D)$. However $0 < d_{ii} < 1$ and $x_i \neq 0$ for each $i \in \overline{\alpha}$, so the result follows by method of contrapositive. □

2.5 P-MATRICES VIA THE SIGN-REVERSING SET

One of the fundamental research problems in the theory of P-matrices is to identify whether a given matrix is a P-matrix or not because, for an $n \times n$ matrix, there are $2^n - 1$ principal minors in total. Hence, it is difficult to check whether all these principal minors are positive or not when n is large. Tsatsomeros (2002) discussed some results that help to identify the P-property of certain derived matrices. In this section, we revisit some results using the sign-reversing set of the matrix.

Theorem 2.5.1. *Let $A \in \mathbb{R}^{n\times n}$. Then A is a P-matrix if and only if $A + D$ is a P-matrix for any positive diagonal matrix D.*

Proof. First, suppose that A is a P-matrix. Then, by definition, we have $rev(A) = \{0\}$. Now if $x \in rev(A+D)$, where $D = (d_{jj})$ is a positive diagonal matrix. Then by definition of $rev(A+D)$, we have $x_j((A+D)x)_j \leq 0$ for all $j \in \langle n \rangle$ implies $x_j(Ax)_j + x_j(Dx)_j \leq 0$ for all $j \in \langle n \rangle$, this is equivalent to $x_j(Ax)_j + x_j d_{jj} x_j \leq 0$ for all $j \in \langle n \rangle$ which implies $x_j(Ax)_j + (x_j)^2 d_{jj} \leq 0$ for all $j \in \langle n \rangle$, but $(x_j)^2 \geq 0$ and $d_{jj} \geq 0$, so $(x_j)^2 d_{jj} \geq 0$, thus $x_j(Ax)_j \leq 0$ for all $j \in \langle n \rangle$. Hence, $x \in rev(A)$. That is, $rev(A+D) \subseteq rev(A)$. But $\{0\} = rev(A) \subseteq rev(A+D)$. Thus $rev(A+D) = rev(A) = \{0\}$, so that $A+D$ is a P-matrix.

On the other hand, suppose that $A+D$ is a P-matrix for any positive diagonal matrix D, so that $rev(A+D) = \{0\}$. If $x \in rev(A)$, that is, if $x_j(Ax)_j \leq 0$ for all $j \in \langle n \rangle$, then we can see that there exists a positive diagonal matrix $D' = (d'_{jj})$, so that $Ax = -D'x$. Since $A = (a_{ij})$ and $x_j(Ax)_j \leq 0$ for all $j \in \langle n \rangle$, we have $x_j \sum_{i=1}^n a_{ij} x_i \leq 0$ for all $j \in \langle n \rangle$. Then, we can choose the positive diagonal matrix D' as

$$-d'_{jj} = \begin{cases} a_{1j}\frac{x_1}{x_j} + a_{2j}\frac{x_2}{x_j} + a_{3j}\frac{x_3}{x_j} + \cdots + a_{nj}\frac{x_n}{x_j} & \text{if } x_j \neq 0 \\ 0 & \text{if} x_j = 0. \end{cases}$$

Clearly, D' is a positive diagonal matrix satisfying $Ax = -D'x$, so that $(A + D')x = 0$. Thus $x \in rev(A + D')$, that is, $x = 0$ implies $rev(A) = \{0\}$, hence A is a P-matrix. $\square$

Lemma 2.5.2. *Let $A \in \mathbb{R}^{n\times n}$. If A is a P-matrix, then the determinant of A is positive.*

Proof. Suppose that A is a P-matrix. Then $rev(A) = \{0\}$. That is, $x_j(Ax)_j \leq 0$ for all $j \in \langle n \rangle$ implies $x = 0$. Then for $x \neq 0$, there exists some index $j \in \langle n \rangle$, for which $x_j(Ax)_j > 0$. Then we can see that all the real eigenvalues of A are positive, as an eigenvalue is the number λ such that, for $x \neq 0$, we have $Ax = \lambda x$ so that $(Ax)_j = (\lambda x)_j$. Hence $x_j(Ax)_j = \lambda(x_j)^2$ as $x_j(Ax)_j > 0$, we have $\lambda(x_j)^2 > 0$ implies $\lambda > 0$, that is, any real eigenvalue is positive. As complex eigenvalues occur in complex conjugate pairs, we can see that the product of eigenvalues is positive. Thus, the determinant of A is positive. $\square$

Theorem 2.5.3. *Let $A \in \mathbb{R}^{n\times n}$. Then A is a P-matrix if and only if A^T is a P-matrix.*

Proof. Suppose that A is a P-matrix. Let $x \in rev(A^T)$ so that $x_j(A^Tx)_j \leq 0$ for all $j \in \langle n \rangle$. Then, as in the proof of Theorem 2.5.1, there exists a positive diagonal matrix D such that $A^Tx = -Dx$, that is, $(A^T + D)x = 0$. Now, if $x \neq 0$, then the determinant $|A + D| = |(A + D)^T| = |A^T + D| = 0$, but as A is a P-matrix, from the Theorem 2.5.1 we have $A + D$ is also P-matrix, but $|A + D| = 0$, a contradiction to the Lemma 2.5.2, hence $x = 0$, so that $rev(A^T) = \{0\}$, that is, A^T is a P-matrix.

On the other hand suppose that A^T is a P-matrix. Since $(A^T)^T = A$ and A^T is a P-matrix, we get A is a P-matrix. $\square$

Theorem 2.5.4. *Let $A \in \mathbb{R}^{n\times n}$. If A is a positive definite matrix, then A is a P-matrix.*

Proof. Suppose that $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Then for all $x \neq 0$, we have $x^TAx > 0$. Now let $x \in rev(A)$, then $x_j(Ax)_j \leq 0$ for all $j \in \langle n \rangle$, implies $\sum_{j=1}^{n} x_j(Ax)_j \leq 0$, that is, $x^TAx \leq 0$, but as A is positive definite from the later inequality we get $x = 0$, that is, $rev(A) = \{0\}$, hence A is a P-matrix. $\square$

2.6 GEOMETRY OF THE SIGN-REVERSING SET

In this section, we discuss some geometrical aspects of the sign-reversing set of a matrix. First, we re-define the sign-reversing set $rev(A)$ of an $n \times n$ matrix A through a sequence of cones obtained by introducing a sequence of linear functionals.

Definition 2.6.1. *Let $A \in \mathbb{R}^{n\times n}$ and consider the linear maps $f_j : \mathbb{R}^n \to \mathbb{R}$ defined by $f_j(x) = x_j$, where x_j is the jth component of $x \in \mathbb{R}^n$ for $j \in \langle n \rangle$. Then, we define the following sets:*

1. $N_j^+ = \{x \in \mathbb{R}^n : f_j(x) \geq 0\}$

2. $N_j^- = \{x \in \mathbb{R}^n : f_j(x) \leq 0\}$

3. $M_j^+ = \{x \in \mathbb{R}^n : f_j(Ax) \geq 0\}$

4. $M_j^- = \{x \in \mathbb{R}^n : f_j(Ax) \leq 0\}$

5. $C_j = (N_j^+ \cap M_j^-) \cup (N_j^- \cap M_j^+)$.

Then $rev(A) = \cap_{j=1}^n C_j$.

Remark 2.6.2. *Note that for each j, all the sets* N_j^+, N_j^-, M_j^+ *and* M_j^- *are closed cones, and thus* C_j *is also a closed cone. The following proposition shows that* $rev(A)$ *can be written as the intersection of two closed cones. This representation will help us to have a more geometrical view of the set* $rev(A)$.

Proposition 2.6.3. *Let* $A \in \mathbb{R}^{n\times n}$. *Then* $rev(A) = D^+ \cap D^-$, *where* D^+ *and* D^- *are closed cones.*

Proof. From the definition of $rev(A)$ and set-theoretic properties of union and intersection, we have

$$\begin{aligned}
rev(A) &= \cap_{i=1}^n C_j \\
&= \cap_{i=1}^n [(N_j^+ \cap M_j^-) \cup (N_j^- \cap M_j^+)] \\
&= \cap_{i=1}^n [((N_j^+ \cap M_j^-) \cup N_j^-) \cap ((N_j^+ \cap M_j^-) \cup M_j^+)] \\
&= \cap_{i=1}^n [(N_j^+ \cup N_j^-) \cap (M_j^- \cup N_j^-) \cap (N_j^+ \cup M_j^+) \cap (M_j^- \cup M_j^+)] \\
&= \{\cap_{i=1}^n [(N_j^+ \cup N_j^-)]\} \cap \{\cap_{i=1}^n [(M_j^- \cup N_j^-)]\} \cap \\
&\quad \{\cap_{i=1}^n [(N_j^+ \cup M_j^+)]\} \cap \{\cap_{i=1}^n [(M_j^- \cup M_j^+)]\} \\
&= \mathbb{R}^n \cap \{\cap_{j=1}^n (M_j^- \cup N_j^-)\} \cap \{\cap_{j=1}^n (N_j^+ \cup M_j^+)\} \cap \mathbb{R}^n \qquad (1)
\end{aligned}$$

Since the set $\{\cap_{i=1}^n [(N_j^+ \cup N_j^-)]\}$ includes all $x \in \mathbb{R}^n$ have negative or positive components, hence all $x \in \mathbb{R}^n$. Similarly, the set $\{\cap_{i=1}^n [(M_j^- \cup M_j^+)]\}$ includes all $x \in \mathbb{R}^n$. So from the equation (1), we have

$$\begin{aligned}
rev(A) &= \{\cap_{j=1}^n (M_j^- \cup N_j^-)\} \cap \{\cap_{j=1}^n (N_j^+ \cup M_j^+)\} \\
&= D^+ \cap D^-
\end{aligned}$$

where $D^- = \cap_{j=1}^n (M_j^- \cup N_j^-) = \{x \in \mathbb{R}^n : x_j \leq 0 \; or \; (Ax)_j \leq 0 \; for \; all \; j = 1,2,3,\ldots,n\}$ and $D^+ = \cap_{j=1}^n (M_j^+ \cup N_j^+) = \{x \in \mathbb{R}^n : x_j \geq 0 \; or \; (Ax)_j \geq 0 \; for \; all \; j = 1,2,3,\ldots,n\}$.

Also, we can see D^+ is a cone, as for any $x \in D^+$ and $\alpha > 0$, then clearly $\alpha x \in D^+$. Similarly, D^- is also a cone. The closedness of these sets is straightforward.

□

Remark 2.6.4. *The sets D^+ and D^- defined in Proposition 2.6.3 are not pointed cones, in general. To see this, we give a counter-example. Let us consider the matrix*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can see that $x = [-1,-1]^T \in D^-$. Also we can see that $-x = [1,1]^T \in D^-$, that is $-(-x) \in -D^-$, hence $x \in -D^-$. Therefore $x \in D^- \cap (-D^-)$ and thus $D^- \cap (-D^-) \neq \{0\}$. That is, D^- is not a pointed cone. Similarly, for the same matrix, one can observe that D^+ is also not a pointed cone. Moreover, we can see that A is not a P-matrix, as $0 \neq x \in D^+ \cap D^-$.

Remark 2.6.5. *The sets D^+ and D^- defined in Proposition 2.6.3 are not convex. We give a counter-example to see that D^- is not convex. Let us consider the matrix*

$$A = \begin{pmatrix} -1 & 0 \\ \frac{3}{5} & \frac{-8}{5} \end{pmatrix}$$

and $x = (-1,-1)$, $y = (2, \frac{-1}{2}) \in D^-$ (since, $Ax = (-1,-1)$ and $Ay = (-2,2)$, so that $x_1 \leq 0$ or $Ax_1 \leq 0$, $x_2 \leq 0$ or $Ax_2 \leq 0$ satisfied and similarly for y also). But here $x+y = (3,1/2)$ and $A(x+y) = (-3,1)$, hence as $(x+y)_2 \geq 0$ and $(A(x+y))_2 \geq 0$ implies $x+y \notin D^-$, so that D^- is not convex.

Theorem 2.6.6. *For any invertible matrix A, there exists a neighbourhood N of A in $L(\mathbb{R}^n)$ such that $S(D^+)$ and $S(D^-)$ are closed in $\mathbb{R}^n$, for each $S \in N$.*

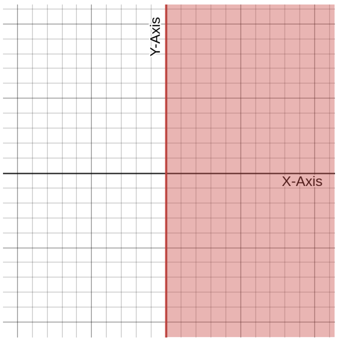
Proof. From the definition of D^+ and D^-, it is clear that $ker(T) \in D^+$ and also $ker(T) \in D^-$. Now if A is invertible, then $ker(T) = \{0\}$ and hence $D^+ \cap ker(T) = \{0\}$ and $D^- \cap ker(T) = \{0\}$. By Proposition 1.3.3, we conclude that $S(D^+)$ and $S(D^-)$ are closed in $\mathbb{R}^n$ for each $S \in N$. □

Example 2.6.7. *We now give an example that illustrates the sign-reversing set by the sets defined in Definition 2.6.1. Let us consider the matrix $A \in \mathbb{R}^{2\times 2}$ given by*

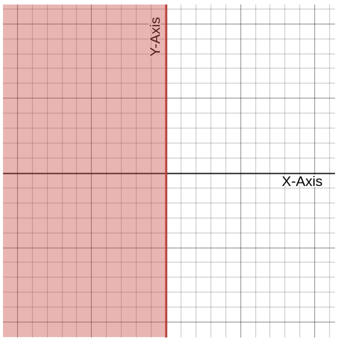
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

and for $x = (x_1, x_2) \in \mathbb{R}^2$. Then we have $Ax = (x_1, x_1) \in \mathbb{R}^2$. It can be seen that the sign-reversing set of A is given by $rev(A) = \{(0, y) \in \mathbb{R}^{2\times 2} : y \in \mathbb{R}\}$. In terms of the new definition of the sign-reversing set defined in this section, we have $rev(A) = D^- \cap D^+$, where $D^- = (N_1^- \cup M_1^-) \cap (N_2^- \cup M_2^-)$ and $D^+ = (N_1^+ \cup M_1^+) \cap (N_2^+ \cup M_2^+)$.

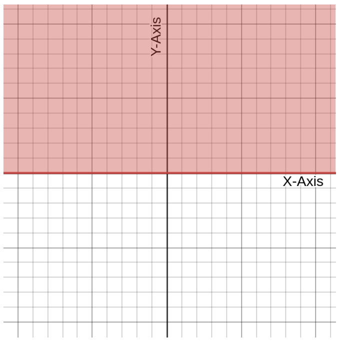
Now we can see the following geometrical interpretation of these individual sets for each of these cases to get $rev(A)$.



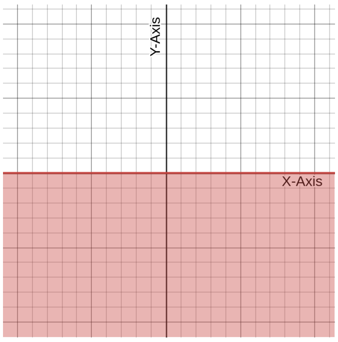
N_1^+



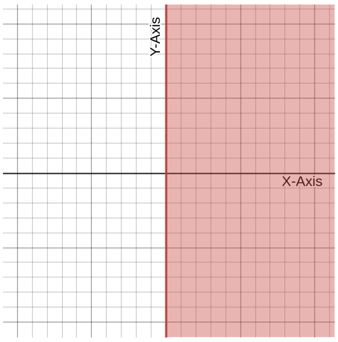
N_1^-



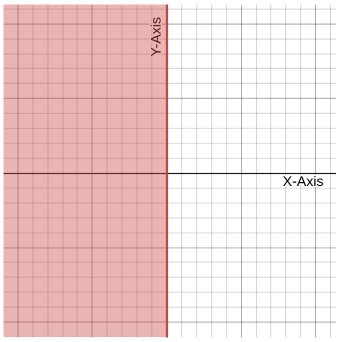
N_2^+



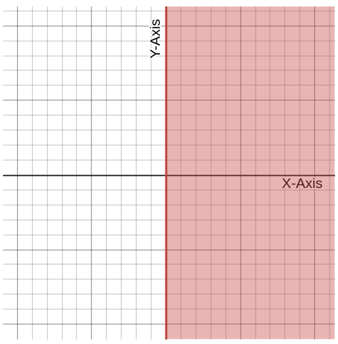
N_2^-



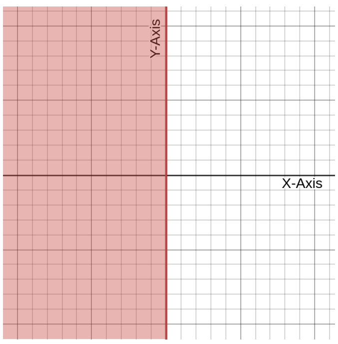
M_2^+



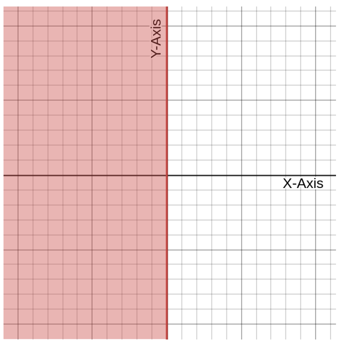
M_2^-



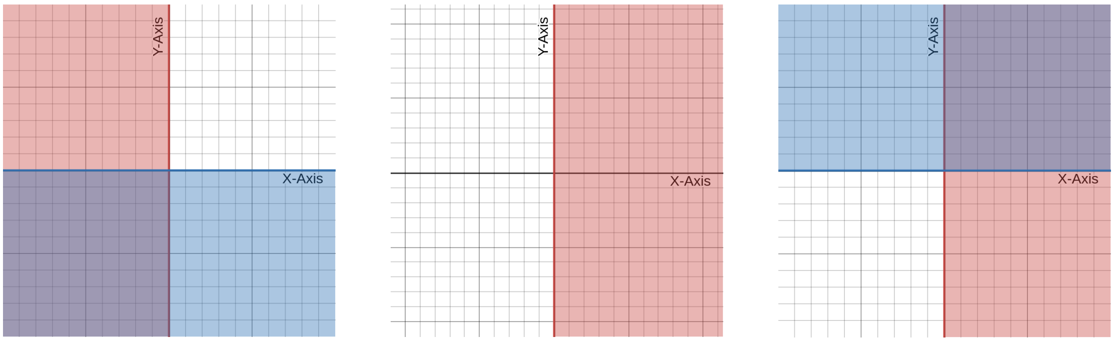
M_1^+



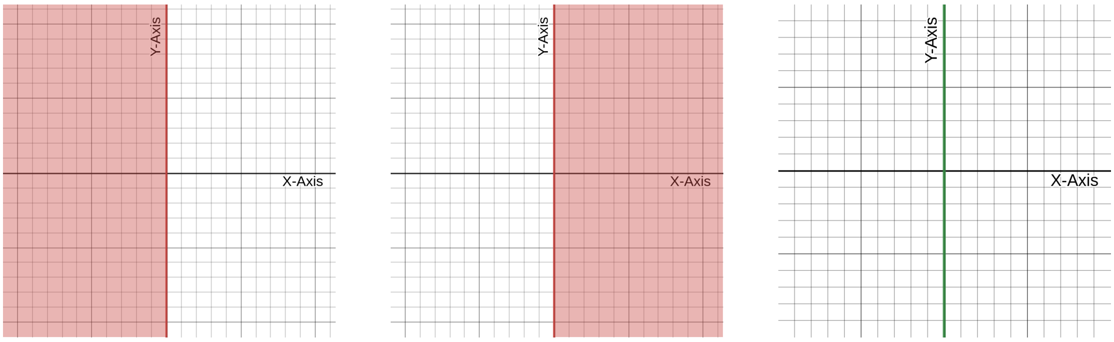
M_1^-



$N_1^- \cup M_1^-$



$N_2^- \cup M_2^-$ $\qquad$ $N_1^+ \cup M_1^+$ $\qquad$ $N_2^+ \cup M_2^+$



D^- $\qquad$ D^+ $\qquad$ $rev(A) = D^- \cap D^+$

Here we can see that $rev(A) = \{(0,y) \in \mathbb{R}^{2\times 2} : y \in \mathbb{R}\}$ is not equal to $\{0\}$. Hence, A is not a P-matrix. We can also see that A in the above example is not P-matrix by determinantal (minor) definition of P-matrix. That is, all principal minors of A are not positive. Next, we give several examples having convex and non-convex sign-reversing sets.

Example 2.6.8. *The following matrices have $rev(A)$ as convex sets:*

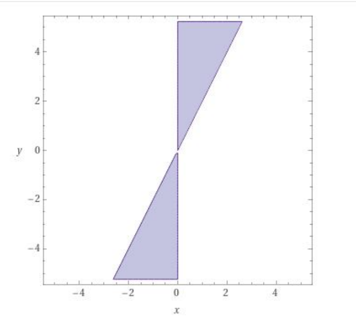
- $A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ *has* $rev(A_1) = \mathbb{R}^{2\times 2}$.
- $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ *has* $rev(A_2) = \{(0,y) \in \mathbb{R}^{2\times 2} : y \in \mathbb{R}\}$.
- $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \in \mathbb{R}^{2\times 2}$ *has* $rev(A_3) = \{0\}$.
- $A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \in \mathbb{R}^{2\times 2}$ *has* $rev(A_4) = \mathbb{R}^2$.
- $A_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ *has* $rev(A_5) = \{(0,y) \in \mathbb{R}^{2\times 2} : y \in \mathbb{R}\}$.

Remark 2.6.9. *It can be observed from the above examples that a matrix having $rev(A)$ as convex sets can have determinant either positive, zero, or negative.*

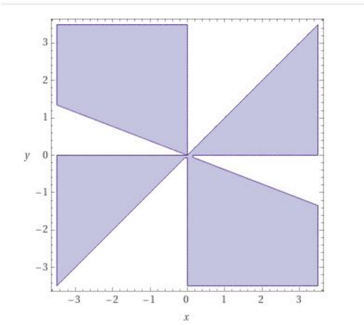
Example 2.6.10. *The following matrices have a non-convex sign-reversing set.*

$$B_1 = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}. \qquad B_2 = \begin{pmatrix} -1 & 1 \\ \frac{-5}{2} & \frac{-13}{2} \end{pmatrix}. \qquad B_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

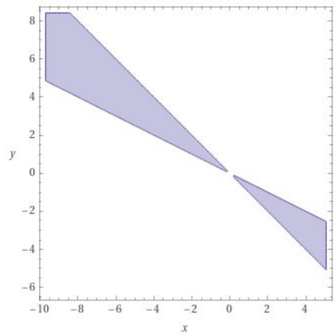
Then the sign-reversing set of these matrices are as follows:



$rev(B_1)$



$rev(B_2)$



$rev(B_3)$

Remark 2.6.11. *It can be observed from the above examples that a matrix having a non-convex sign-reversing set can have determinants either positive, zero, or negative.*

2.6.1 SIGN-REVERSING SET OF 2×2 MATRICES

We have seen the representation of the sign-reversing set with the help of the null space of the matrix $DA - A - D$ in Corollary 2.2.3. Let us consider the matrix $A \in \mathbb{R}^{2\times 2}$ defined by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

One can observe that the determinant $|DA - A - D| = 0$ is a curve on the plane $\mathbb{R}^2$, let us say it as $\mathbf{C}$. Let us observe more about the curve $\mathbf{C}$. The matrix $DA - A - D$ is given by

$$DA - A - D \quad = \quad \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x - a_{11} - x & a_{12}x - a_{12} \\ a_{21}y - a_{21} & a_{22}y - a_{22} - y \end{pmatrix}.$$

Thus, its determinant equal to zero implies:

$$\begin{aligned} |DA - A - D| &= 0 \\ &\Rightarrow (a_{22} - |A|)x + (a_{11} - |A|)y + (1 - trace(A) + |A|)xy + |A| = 0. \end{aligned}$$

This equation is of the form $ax + by + cxy + d = 0$, where $a = a_{22} - |A|$, $b = a_{11} - |A|$, $c = 1 - trace(A) + det(A)$ and $d = |A|$. If we represent the point (x, y) by the complex number $x + \mathbf{i}y$, then we can rotate it 45 degree clockwise by multiplying the complex number $\frac{1-\mathbf{i}}{\sqrt{2}}$ and reading off their x and y coordinates:

$$(x + \mathbf{i}y)\left(\frac{1-\mathbf{i}}{\sqrt{2}}\right) = \frac{x+y}{\sqrt{2}} + \mathbf{i}\frac{y-x}{\sqrt{2}}.$$

Therefore the rotated coordinates of (x, y) are $(\frac{x+y}{\sqrt{2}}, \frac{y-x}{\sqrt{2}}) = (x' + y', y' - x')$. Thus $x = x' + y'$, $y = y' - x'$, and $xy = y'^2 - x'^2$. Hence $ax = ax' + ay'$ and $by = by' - bx'$. Thus, the equation becomes

$$\begin{aligned} ax + by + cxy + d = 0 &\Rightarrow ax' + ay' + by' - bx' + cy'^2 - cx'^2 + d = 0 \\ &\Rightarrow -cx'^2 + (a-b)x' + cy'^2 + (a+b)y' = -d. \end{aligned}$$

Now, completing the squares,

$$\begin{aligned} &\Rightarrow x'^2 - (\frac{a-b}{c})x' + \frac{(a-b)^2}{4c^2} + y'^2 + (\frac{a+b}{c})y' + \frac{(a+b)^2}{4c^2} = -d + \frac{(a-b)^2}{4c^2} + \frac{(a+b)^2}{4c^2} \\ &\Rightarrow (x' - \frac{a-b}{2c})^2 + (y' + \frac{a+b}{2c})^2 = -d + \frac{a^2+b^2}{2c}. \end{aligned}$$

This represents a hyperbola with center at $(-\frac{a-b}{2c}, \frac{a+b}{2c})$ and asymptotes parallel to the x and y-axes. In our case, a hyperbola with center $(-\frac{a_{22}-a_{11}}{2(1-trace(A)+det(A))}, \frac{a_{11}+a_{22}-2|A|}{2(1-trace(A)+det(A))})$ and asymptotes parallel to the x and y-axes.

Corollary 2.2.3 indeed tells us that if the curve $\mathbf{C}$ passes through the region $[0,1] \times [0,1]$, then all the points (x, y) on the curve $\mathbf{C}$ in $[0,1] \times [0,1]$ can take as the diagonal matrix $D = diag(x, y)$ in the Corollary 2.2.3 to get non-zero sign-reversing set. That is, if $M = \{D = diag(x, y) : (x, y) \in \mathbf{C} \subseteq [0,1] \times [0,1]\}$, then

$$rev(A) = \bigcup_{D \in M} ker(DA - A - D).$$

If the curve **C** does not pass through the region $[0,1]\times[0,1]$, then $rev(A)=\{0\}$, that is, A is a P-matrix. This is the situation where all the diagonal matrices D such that $0\le D\le I$, the determinant $|DA-A-D|\neq 0$.

Next, we look at several examples to illustrate the above facts.

Example 2.6.12. *Consider the matrix* $A=\begin{pmatrix}1&1\\2&4\end{pmatrix}$. *Then, for a diagonal matrix* $D=diag(x,y)$, *the determinant* $|DA-A-D|=0$ *implies* $2x-y-2xy+2=0$, *which is a curve on the 2D-plane given by the hyperbola in Figure 2.2.*

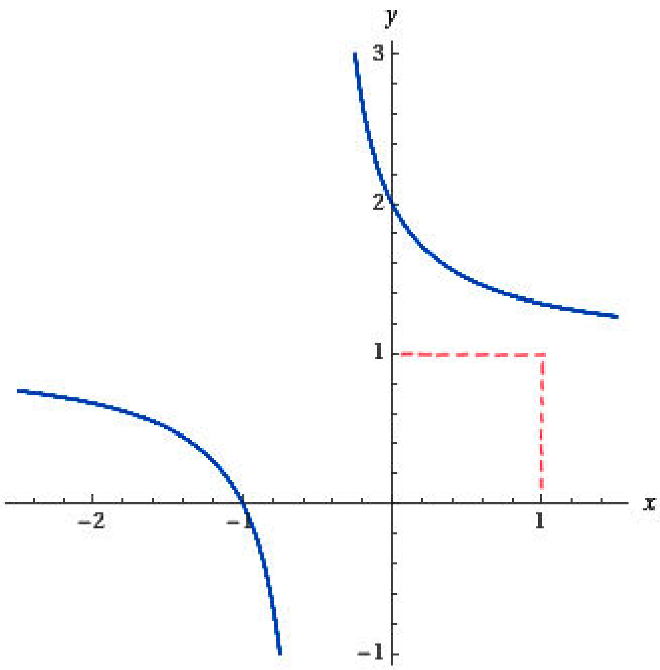


Figure 2.2

Here, we can see that the graph does not pass through the region $[0,1]\times[0,1]$. *Hence for all D with* $0\le D\le I$, $det(DA-A-D)\neq 0$. *Thus by Corollary 2.2.3,* $rev(A)=\{0\}$, *hence A is a P-matrix.*

Example 2.6.13. *Consider the matrix* B_2 *given in Example 2.6.10. Then for a diagonal matrix* $D=diag(x,y)$, *the determinant* $|DA-A-D|=0$ *implies* $-\frac{31}{2}x-10y+\frac{35}{2}xy+9=0$, *which is a curve on the 2D-plane given by the hyperbola in Figure 2.3.*

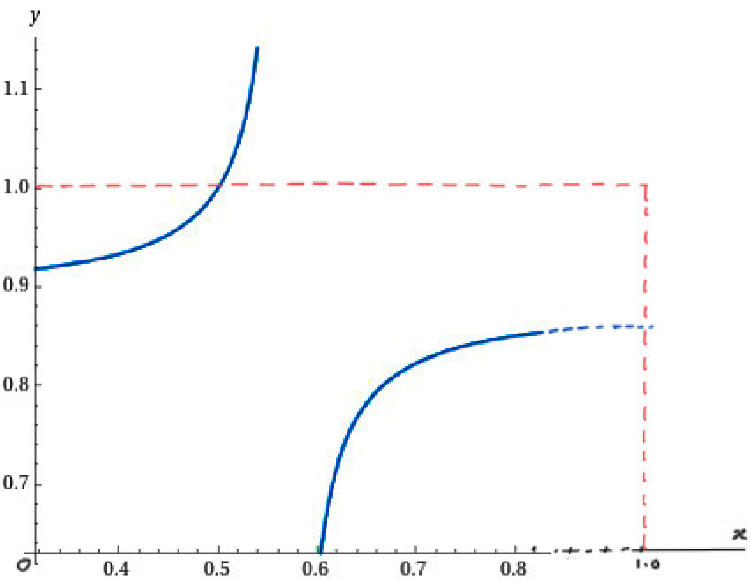


Figure 2.3

Here we can see that the graph passes through the region $[0,1]\times[0,1]$. *Hence for all diagonal matrix* $D = diag(x,y)$, *where* (x,y) *is a point on the curve* $det(DA-A-D)=0$ *in the region* $[0,1]\times[0,1]$, $det(DA-A-D)=0$. *Thus by Corollary 2.2.3,* $rev(A)\neq\{0\}$, *hence A is not a P-matrix.*

CHAPTER 3

P-OPERATORS ON HILBERT SPACES

3.1 INTRODUCTION

P-matrices are characterized by the sign non-reversal property of matrices. With the help of the sign non-reversal feature of matrices, the notion of P-matrix has been extended to the idea of P-operator to infinite dimensional Banach spaces having a Schauder basis by Kannan and Sivakumar (2016). In this chapter, we define the P-operator on separable real Hilbert spaces relative to an orthonormal basis with the help of the inner product structure of the Hilbert spaces.

We know that a countable orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ exists for every separable Hilbert space $\mathscr{H}$ such that for any element x in the Hilbert space $\mathscr{H}$, we have $x = \sum_i \langle x, e_i \rangle e_i$.

Definition 3.1.1. *(cf. Limaye (1996), p.460) Let $\mathscr{H}$ be a Hilbert space. A bounded linear operator U on $\mathscr{H}$ is said to be a unitary operator if $U^*U = I = UU^*$, where I is the identity operator on $\mathscr{H}$.*

If an orthonormal basis is known, say $\{e_i\}_{i=1}^{\infty}$, then any orthonormal basis of $\mathscr{H}$ is of the form $\{Ue_i\}_{i=1}^{\infty}$ for some unitary operator U on $\mathscr{H}$.

Proposition 3.1.2. *(Sunder (2015)) Every unitary operator on a Hilbert space $\mathscr{H}$ maps an orthonormal basis to an orthonormal basis.*

Definition 3.1.3. *(cf. Gowda (1986)) Let $\mathscr{H}$ be a real Hilbert space and let T be a bounded linear operator on H. Let K denote a non-empty closed convex cone in H. We say that T is positive definite on K if $\langle Tx, x \rangle > 0$ for all $x \in K \backslash \{0\}$.*

A linear operator T is called positive-semidefinite (or non-negative) if for every $x \in Dom(T)$, $\langle Tx, x \rangle \in \mathbb{R}$ and $\langle Tx, x \rangle \geq 0$, where $Dom(T)$ is the domain of T. Positive-semidefinite operators are denoted as $T \geq 0$. The operator is said to be positive-definite if $\langle Tx, x \rangle \geq 0$ for all $x \in Dom(T) \backslash \{0\}$ and it is denoted by $T > 0$.

Definition 3.1.4. *Let $\mathscr{H}$ be a real Hilbert space and $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$. A bounded linear operator T on $\mathscr{H}$ is said to be a P-operator relative to the given orthonormal basis $\mathscr{B}$ if for $x \in \mathscr{H}$, the inequalities*

$$\langle x, e_i \rangle \langle Tx, e_i \rangle \leq 0$$

for all i imply that $x = 0$.

Example 3.1.5. *Let ℓ^2 denote the square summable sequence space of real numbers. Let $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of ℓ^2, where e_i denotes the vector whose i^{th} entry is one, and all other entries are zero. Define $T : \ell^2 \to \ell^2$ by*

$$T(x_1, x_2, x_3, \ldots) = (\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots),$$

for any $(x_1, x_2, x_3, \ldots) \in \ell^2$ with $\alpha_i > 0$ for all i and $\sup_i |\alpha_i| < \infty$. Then T is a bounded linear operator, and it is a P-operator relative to $\mathscr{B}$.

Example 3.1.6. *The right shift operator T_R and the left shift operator T_L on ℓ^2 are defined by*

$$T_R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$$

and

$$T_L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

respectively. The operators T_R and T_L are not P-operators relative to the standard orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ of ℓ^2. Indeed, the non-zero element $x = (1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots) \in \ell^2$ satisfies the inequalities $\langle x, e_i \rangle \langle T_R(x), e_i \rangle \leq 0$ and $\langle x, e_i \rangle \langle T_L(x), e_i \rangle \leq 0$, for all i.

Example 3.1.7. *The operators $I + T_R$ and $I + T_L$ on ℓ^2 are P-operators relative to the standard orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ of ℓ^2, where I is the identity operator on ℓ^2. Note that $I + T_R$ is a bounded linear operator. Suppose for $x = (x_1, x_2, x_3, \ldots) \in \ell^2$, the inequalities $\langle x, e_i \rangle \langle (I + T_R)x, e_i \rangle \leq 0$ for all i. This leads to the inequalities $x_1^2 \leq 0$, $x_{i-1}x_i + x_i^2 \leq 0$, for all $i \geq 2$. From these inequalities, we get that $x_i = 0$, for all i, hence $x = 0$.*

Next, to see that $I + T_L$ is a P-operator relative to $\mathscr{B}$, it is noted that $I + T_L$ is a bounded linear operator. Suppose for $x = (x_1, x_2, x_3, \ldots) \neq 0 \in \ell^2$ the inequalities

$$\langle x, e_i \rangle \langle (I + T_L)x, e_i \rangle \leq 0$$

hold for all i. This implies that $x_i^2 + x_i x_{i+1} \leq 0$ for all i. Suppose $x_j = 0$ for some index j, then we get that $x_{j-1}^2 + x_{j-1}x_j \leq 0$, hence $x_i = 0$ for all $i \leq j$. Thus x is of the form

$x = (0, 0, \ldots, 0, 0, x_{i+1}, \ldots)$ with $x_i \neq 0$ for all $i \geq j+1$ satisfying $x_i^2 + x_i x_{i+1} \leq 0$. Thus $|x_i^2| \leq |x_i|\,|x_{i+1}|$ for all $i \geq j+1$.

Now as $x_i \neq 0$ for all $i \geq j+1$, we have $|x_i| > 0$ for all $i \geq j+1$. Thus by dividing the inequalities by $|x_i|$, we get $|x_i| \leq |x_{i+1}|$ for all $i \geq j+1$. This shows that the absolute values of the components of x are increasing, and hence i^{th} term of x will not converge to 0 when i tends to infinity, hence $x \notin \ell^2$. Thus if $\langle x, e_i\rangle\langle (I+T_L)x, e_i\rangle \leq 0$ for all i, then x must be equal to 0.

Example 3.1.8. *Let $\mathscr{H} = L^2[-\pi, \pi]$ be the square-integrable function space, with the inner product,*

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f\bar{g}\,dm.$$

The space $\mathscr{H}$ is a separable Hilbert space with an orthonormal basis given by $\{u_n(t) = \frac{e^{int}}{\sqrt{2\pi}}, t \in [-\pi, \pi], n = 0, \pm 1, \pm 2, \ldots\}$. Then any element $x(t) \in \mathscr{H}$ can be written in the form

$$x(t) = \sum_{-\infty}^{\infty} \hat{x}(n) e^{int}.$$

where $\hat{x}(n)$ is called Fourier coefficient of $x(t)$ (cf. Limaye (1996)) and it is given by

$$\hat{x}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t) e^{-int}\,dm(t).$$

Define an operator T on $\mathscr{H}$ by

$$T(x(t)) = f(t)x(t).$$

where $f(t) = d > 0$, a constant function, which is a measurable function on $[-\pi, \pi]$. Then T is a P-operator relative to the orthonormal basis $\{u_n(t) : t \in [-\pi, \pi], n = 0, \pm 1, \pm 2, \ldots\}$, Since the inequalities $\langle x(t), u_n(t)\rangle\langle T(x(t)), u_n(t)\rangle \leq 0$ for all n imply that $2\pi d(\hat{x}(n))^2 \leq 0$ for all n imply that $\hat{x}(n)^2 \leq 0$ for all n imply that $x = 0$. Hence T is a P-operator relative to $\{u_n(t) : t \in [-\pi, \pi], n = 0, \pm 1, \pm 2, \ldots\}$.

Example 3.1.9. *Let $\mathscr{H} = \ell^2$ be the square summable sequence space and $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of $\mathscr{H}$. Define $T : \mathscr{H} \to \mathscr{H}$ by*

$$T(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots).$$

Suppose $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i implies $\frac{x_i^2}{i} \leq 0$ for all i, implies $x_i = 0$ for all i, that is, $x = 0$. Hence, T is a P-operator.

3.2 BASIC PROPERTIES OF P-OPERATORS ON HILBERT SPACES

Theorem 3.2.1. *Let T be a P-operator on $\mathscr{H}$ relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. If D and E are diagonal operators on $\mathscr{H}$ such that $De_i = d_{ii}e_i$ with $d_{ii} > 0$ for all i and $Ee_i = \lambda e_i$ with $\lambda > 0$, then*

1. *$T+D$ is a P-operator relative to $\mathscr{B}$.*

2. *DT is a P-operator relative to $\mathscr{B}$.*

3. *TE is a P-operator relative to $\mathscr{B}$.*

4. *DTE is a P-operator relative to $\mathscr{B}$.*

Proof. Let $x \in \mathscr{H}$. Then we can write $x = \sum_i x_i e_i$ so that $Dx = \sum_i x_i d_{ii} e_i$ and $Ex = \sum_i x_i \lambda e_i$. Also let $Te_i = \sum_j t_{ji} e_j$, then $Tx = \sum_i x_i Te_i = \sum_i x_i \sum_j t_{ji} e_j$. As T is a P-operator, we have, if $\langle x, e_i\rangle\langle Tx, e_i\rangle \le 0$ for all i implies $x = 0$. The inequalities are equivalent to $\langle \sum_i x_i e_i, e_i\rangle\langle \sum_i x_i \sum_j t_{ji} e_j, e_i\rangle \le 0$ for all i implies $x = 0$. That is,

$$x_i(x_1 t_{i1} + x_2 t_{i2} + x_3 t_{i3} + \cdots) \le 0 \text{ for all } i \text{ implies } x = 0. \tag{3.2.1}$$

1. Suppose $\langle x, e_i\rangle\langle (T+D)x, e_i\rangle \le 0$ for all i.

$$\Rightarrow \langle x, e_i\rangle\langle Tx, e_i\rangle + \langle x, e_i\rangle\langle Dx, e_i\rangle \le 0 \text{ for all i.}$$
$$\Rightarrow x_i(Tx)_i + x_i d_i x_i \le 0 \text{ for all i.}$$

As $d_i > 0$ and $x_i d_i x_i = x_i^2 d_i > 0$, the later inequalities are equivalent to

$$\langle x, e_i\rangle\langle Tx, e_i\rangle \le 0 \text{ for all i.}$$

As T is P-operator relative to $\mathscr{B}$, we get $x = 0$, so that $T+D$ is P-operator relative to $\mathscr{B}$.

2. Suppose $\langle x, e_i\rangle\langle (DT)x, e_i\rangle \le 0$ for all i.

$$\Rightarrow \langle \sum_i x_i e_i, e_i\rangle\langle DT(\sum_i x_i e_i), e_i\rangle \le 0 \text{ for all i.}$$
$$\Rightarrow \langle \sum_i x_i e_i, e_i\rangle\langle (\sum_i d_i e_i \sum_j x_j t_{ij}), e_i\rangle \le 0 \text{ for all i.}$$

As $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis, the inequalities are equivalent to

$$\Rightarrow x_i(x_1t_{i1}+x_2t_{i2}+x_3t_{i3}+\cdots)d_{ii} \leq 0 \text{ for all } i.$$
$$\Rightarrow x_i(x_1t_{i1}+x_2t_{i2}+x_3t_{i3}+\cdots) \leq 0 \text{ for all } i \text{ (as } d_{ii} > 0).$$

Therefore equation (3.2.1) gives $x = 0$. Hence, DT is the P-operator relative to $\mathscr{B}$.

3. Suppose $\langle x, e_i\rangle\langle (TE)x, e_i\rangle \leq 0$ for all i.

$$\Rightarrow \langle \sum_i x_ie_i, e_i\rangle\langle TE(\sum_i x_ie_i), e_i\rangle \leq 0 \text{ for all i.}$$
$$\Rightarrow \langle \sum_n x_ie_i, e_i\rangle\langle (\sum_i \lambda e_i \sum_j x_jt_{ij}), e_i\rangle \leq 0 \text{ for all i.}$$

As $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis, the inequalities are equivalent to

$$\Rightarrow x_i(x_1t_{i1}+x_2t_{i2}+x_3t_{i3}+\cdots)\lambda \leq 0 \text{ for all i.}$$
$$\Rightarrow x_i(x_1t_{i1}+x_2t_{i2}+x_3t_{i3}+\cdots) \leq 0 \text{ for all i (as } \lambda > 0).$$

Therefore equation (3.2.1) gives $x = 0$. Hence, TE is the P-operator relative to $\mathscr{B}$.

4. In (3), we proved that for a P-operator T relative to $\mathscr{B}$ and a diagonal operator E as given in the hypothesis of the theorem, TE is P-operator relative to $\mathscr{B}$. As DT is a P-operator relative to $\mathscr{B}$ from (2), using (3) for DT and E, we see that DTE is also a P-operator relative to $\mathscr{B}$. □

Theorem 3.2.2. *Let T be a P-operator on $\mathscr{H}$ relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. Then $DI+(I-D)T$, where D is a diagonal operator on $\mathscr{H}$ with diagonal elements from $[0,1]$, is also a P-operator relative to the orthonormal basis $\mathscr{B}$.*

Proof. Let $x = \sum_i x_ie_i \in \mathscr{H}$ and D be a diagonal operator such that $De_i = d_{ii}e_i$, $d_{ii} \in [0,1]$. If $\langle x, e_i\rangle\langle (DI+(I-D)T)x, e_i\rangle \leq 0$ for all i, then

$$\langle x, e_i\rangle\langle Dx, e_i\rangle + \langle x, e_i\rangle\langle Tx, e_i\rangle - \langle x, e_i\rangle\langle DTx, e_i\rangle \leq 0 \text{ for all i.}$$
$$\Rightarrow x_id_{ii}x_i + x_i(Tx)_i - x_id_{ii}(Tx)_i \leq 0 \text{ for all i.}$$
$$\Rightarrow x_i^2d_{ii} + (1-d_{ii})x_i(Tx)_i \leq 0 \text{ for all i.}$$

For all index i, for which $d_{ii} = 1$, these inequalities become $x_i^2 \leq 0$ which implies $x_i = 0$. Now for $0 \leq d_{ii} < 1$, as $x_i^2d_{ii} \geq 0$, from the last inequalities we get $(1-d_{ii})x_i(Tx)_i \leq 0$ for all i. As $(1-d_{ii}) \geq 0$, we get $x_i(Tx)_i \leq 0$ for all i, that is, $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i. But, as T is P-operator relative to $\mathscr{B}$, we get $x = 0$. Hence $DI+(I-D)T$ is a P-operator relative to $\mathscr{B}$. □

Remark 3.2.3. *The sum of two P-operators relative to an orthonormal basis may not be a P-operator relative to the orthonormal basis.*

Example 3.2.4. *Consider the Hilbert space $\mathscr{H} = \ell^2$, the square summable sequence space and define two operators T and S on $\mathscr{H}$ by*

$$T(x_1, x_2, x_3, \ldots) = (x_1, 2x_1 + x_2, 0, \ldots) \text{ and}$$

$$S(x_1, x_2, x_3, \ldots) = (x_1 + 2x_2, x_2, 0, \ldots).$$

Then relative to the standard orthonormal basis of $\mathscr{H}$, the operators S and T are P-operators, but the sum $S+T$ given by

$$(S+T)(x_1, x_2, x_3, \ldots) = (2x_1 + 2x_2, 2x_1 + 2x_2, 0, \ldots).$$

is not a P-operator relative to the standard orthonormal basis of $\mathscr{H}$ because for the non-zero element $x = (1, -1, 0, \ldots) \in \mathscr{H}$, it satisfies the condition $\langle x, e_i \rangle \langle (S+T)x, e_i \rangle \leq 0$, for all i.

Remark 3.2.5. *The composition of two P-operators relative to an orthonormal basis may not be P-operator relative to the orthonormal basis.*

Example 3.2.6. *Consider the Hilbert space $\mathscr{H} = \ell^2$, the square summable sequence space and define two operators T and S on $\mathscr{H}$ by*

$$T(x_1, x_2, x_3, \ldots) = (x_1 - x_2, x_2, 0, \ldots) \text{ and}$$

$$S(x_1, x_2, x_3, \ldots) = (x_1, x_1 + x_2, 0, \ldots).$$

Then relative to the standard orthonormal basis of $\mathscr{H}$, the operators S and T are P, but the composition ST on $\mathscr{H}$ given by,

$$ST(x_1, x_2, x_3, \ldots) = (x_1 - x_2, x_1, 0, \ldots).$$

is not a P-operator relative to the standard orthonormal basis of $\mathscr{H}$ because for the non-zero element $x = (0, 1, 0, \ldots) \in \mathscr{H}$, it satisfies the condition $\langle x, e_i \rangle \langle (ST)x, e_i \rangle \leq 0$ for all i.

3.3 P-OPERATORS RELATIVE TO VARIOUS ORTHONORMAL BASES ON HILBERT SPACES

Next, we give a result that says that an operator can be a P-operator relative to several orthonormal bases.

Theorem 3.3.1. *Let T be a bounded linear operator on $\mathscr{H}$ satisfying $TU = UT$ for a unitary operator U on $\mathscr{H}$. Then T is a P-operator relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ of $\mathscr{H}$ if and only if T is a P-operator relative to the orthonormal basis $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$ of $\mathscr{H}$.*

Proof. Let T be a P-operator relative to the orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ of $\mathscr{H}$ satisfying $TU = UT$. Suppose $\langle x, Ue_i\rangle\langle Tx, Ue_i\rangle \le 0$ for all i. Then $\langle U^*x, e_i\rangle\langle U^*Tx, e_i\rangle \le 0$ for all i, hence $\langle U^*x, e_i\rangle\langle TU^*x, e_i\rangle \le 0$ for all i, because $TU = UT$. As T is a P-operator relative to the orthonormal basis $\mathscr{B}$, we get $U^*x = 0$, hence $x = 0$. Therefore T is a P-operator relative to the orthonormal basis $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$.

On the other hand, assume that T is a P-operator relative to the orthonormal basis $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$ of $\mathscr{H}$ satisfying $TU = UT$. Suppose $\langle x, e_i\rangle\langle Tx, e_i\rangle \le 0$ for all i. As U is a unitary operator, we get $\langle x, U^*Ue_i\rangle\langle Tx, U^*Ue_i\rangle \le 0$ for all i, so $\langle Ux, Ue_i\rangle\langle UTx, Ue_i\rangle \le 0$ for all i. Hence $\langle Ux, Ue_i\rangle\langle TUx, Ue_i\rangle \le 0$ for all i. Since T is a P-operator relative to $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$, we get $Ux = 0$, hence $x = 0$. Therefore T is a P-operator relative to the orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. □

The condition $TU = UT$ in Theorem 3.3.1 cannot be dropped, which is illustrated in the example given below. The example also tells that an operator T can be a P-operator relative to one orthonormal basis, whereas the same operator relative to another orthonormal basis may not be a P-operator.

Example 3.3.2. *Define $T : \ell^2 \to \ell^2$ by*

$$T(x_1, x_2, x_3, \dots) = (x_1, 2x_1 + x_2, 2x_2 + x_3, \dots)$$

for $x = (x_1, x_2, x_3, \dots) \in \ell^2$. Then T is a bounded linear operator, and it is a P-operator relative to the standard orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ of ℓ^2.
Now consider the unitary operator U on ℓ^2 given by

$$U(x_1, x_2, x_3, \dots) = \left(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}, \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} + \frac{x_4}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} - \frac{x_4}{\sqrt{2}}, \dots\right),$$

then $U^* = U$ and $UT \neq TU$. The operator T is not a P-operator relative to the orthonor-

mal basis $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$ *of* $\mathscr{H}$*, because the non-zero element* $x = (1,-1,0,0,\ldots) \in \ell^2$ *satisfies the inequalities* $\langle x, Ue_i\rangle\langle Tx, Ue_i\rangle \leq 0$*, for all i.*

Remark 3.3.3. *Theorem 3.3.1 tells us that the condition* $TU = UT$ *is sufficient for the operator* T *to be a P-operator relative to the orthonormal bases* $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ *and* $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$*. But it is not a necessary condition. That is, an operator* T *can be P-operator relative to two orthonormal bases* $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ *and* $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$*, but it may not satisfy the relation* $TU = UT$*. The following example shows this fact.*

Example 3.3.4. *Define* $T : \ell^2 \to \ell^2$ *by*

$$T(x_1,x_2,x_3,\ldots) = (x_1 - x_2, x_1 + x_2, x_3 - x_4, x_3 + x_4, \ldots)$$

for $x = (x_1,x_2,x_3,\ldots) \in \ell^2$*. Then* T *is a P-operator relative to the standard orthonormal basis* $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ *of* $\mathscr{H}$*. To see this, here the operator* T *is bounded linear. Suppose* $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ *for all i. Then* $x_i(x_i - x_{i+1}) \leq 0$ *for odd i and* $x_i(x_i + x_{i-1}) \leq 0$ *for even i. Solving these inequalities together will lead to* $x = 0$*. Hence,* T *is a P-operator relative to* $\mathscr{B}$*.*

Now consider the unitary operator U *on* $\mathscr{H}$ *given by*

$$U(x_1,x_2,x_3,\ldots) = \left(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}, \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} + \frac{x_4}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} - \frac{x_4}{\sqrt{2}}, \ldots\right),$$

then $U^* = U$ *and* $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$ *is the another orthonormal basis of* $\mathscr{H}$*. Then* T *is also a P-operator relative to* $\mathscr{B}'$*. To see this, suppose* $\langle x, Ue_i\rangle\langle Tx, Ue_i\rangle \leq 0$ *for all i. Then* $x_i(x_i - x_{i-1}) \leq 0$ *for odd i and* $x_i(x_i + x_{i+1}) \leq 0$ *for even i. Solving these inequalities together will lead to* $x = 0$*. Hence,* T *is a P-operator relative to* $\mathscr{B}'$*. Note that*

$$UT(x) = (\sqrt{2}x_1, -\sqrt{2}x_2, \sqrt{2}x_3, \ldots)$$

and

$$TU(x) = (x_1, x_2, x_3, \ldots)$$

for any $x = (x_1,x_2,x_3,\ldots) \in \ell^2$*, thus* $TU \neq UT$*.*

Theorem 3.3.5. *Let* $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ *be an orthonormal basis of* $\mathscr{H}$*. Then the following statements hold good:*

(a) T *is a P-operator on* $\mathscr{H}$ *relative to* $\mathscr{B}$ *if and only if the operator* UTU^* *is a P-operator relative to* $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$*, for any unitary operator U.*

(b) T *is a P-operator on* $\mathscr{H}$ *relative to* $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$ *of* $\mathscr{H}$ *where U is any unitary operator on* $\mathscr{H}$ *if and only if the operator* U^*TU *is a P-operator on* $\mathscr{H}$ *relative to* $\mathscr{B}$*.*

Proof. (a) Assume that T is a P-operator relative to $\mathscr{B}$. Then we have, if $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i imply that $x = 0$. Suppose $\langle x, Ue_i\rangle\langle UTU^*x, Ue_i\rangle \leq 0$ for all i. Then

$$\langle U^*x, e_i\rangle\langle TU^*x, e_i\rangle \leq 0$$

for all i. As T is a P-operator relative to $\mathscr{B}$, we get here $U^*x = 0$ and hence $x = 0$. Therefore UTU^* is a P-operator relative to $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$.

Conversely, assume that UTU^* is a P-operator relative to the orthonormal basis $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$. Suppose $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i. Then $\langle Ux, Ue_i\rangle\langle UTU^*Ux, Ue_i\rangle \leq 0$ for all i. As UTU^* is a P-operator relative to $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$, we get that $x = 0$. Therefore, T is a P-operator relative to $\mathscr{B}$.

(b) Assume that T is a P-operator relative to the orthonormal basis $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$. Suppose $\langle x, e_i\rangle\langle U^*TUx, e_i\rangle \leq 0$ for all i. Then $\langle Ux, Ue_i\rangle\langle TUx, Ue_i\rangle \leq 0$ for all i. As T is a P-operator relative to $\mathscr{B}'$, we get that $Ux = 0$ and hence $x = 0$. Therefore, U^*TU is a P-operator relative to $\mathscr{B}$.

Conversely, assume that U^*TU is a P-operator relative to $\mathscr{B}$. Suppose that

$$\langle x, Ue_i\rangle\langle Tx, Ue_i\rangle \leq 0$$

for all i. Then $\langle U^*x, e_i\rangle\langle U^*TUU^*x, e_i\rangle \leq 0$ for all i. As U^*TU is a P-operator relative to $\mathscr{B}$, we get that $x = 0$. Therefore T is a P-operator relative to $\mathscr{B}' = \{Ue_i\}_{i=1}^{\infty}$. □

Remark 3.3.6. *A bounded linear operator T on $\mathscr{H}$ is called invertible if there is a bounded linear operator S on $\mathscr{H}$ so that ST and TS are the identity operators. We say that S is the inverse of T in this case and it is denoted by T^{-1}. It is observed in Kannan and Sivakumar (2016) that every P-matrix is invertible, and its inverse is also a P-matrix. However, the P-operator does not guarantee its invertibility in infinite-dimensional spaces, as shown in the following example. Moreover, if we have an invertible P-operator, then its inverse is also a P-operator.*

Example 3.3.7. *Consider the linear operator $T : \ell^2 \to \ell^2$ defined by*

$$T(x_1, x_2, x_3, \dots) = \big(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots),$$

for $x = (x_1, x_2, x_3, \dots\big) \in \ell^2$. Then T is a P-operator relative to the standard orthonormal basis of ℓ^2, but T is not invertible.

Theorem 3.3.8. *Let T be an invertible P-operator on $\mathscr{H}$ relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. Then the inverse of T is also a P-operator relative to $\mathscr{B}$.*

Proof. Since T is a P-operator on $\mathscr{H}$ relative to $\mathscr{B}$, we have, if the inequalities $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i imply $x = 0$. Suppose $\langle y, e_i\rangle\langle T^{-1}y, e_i\rangle \leq 0$ for all i. Hence $\langle Tx, e_i\rangle\langle x, e_i\rangle \leq 0$ for all i, where $x = T^{-1}y$. As T is a P-operator, we get $x = 0$, hence $y = 0$. Thus, T^{-1} is a P-operator relative to the orthonormal basis $\mathscr{B}$. □

Theorem 3.3.9. *Let T be a positive definite operator on $\mathscr{H}$. Then T is a P-operator on $\mathscr{H}$ relative to any orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ of $\mathscr{H}$.*

Proof. Assume that T is a positive definite operator on $\mathscr{H}$. Then for every $0 \neq x \in \mathscr{H}$, $\langle Tx, x\rangle > 0$. Let x be a non-zero element of $\mathscr{H}$. Then $x = \sum_i \langle x, e_i\rangle e_i$, $Tx = \sum_i \langle Tx, e_i\rangle e_i$ and $\langle Tx, x\rangle = \sum_i \langle Tx, e_i\rangle\langle x, e_i\rangle > 0$. Therefore there exists some j, for which $\langle x, e_j\rangle\langle Tx, e_j\rangle > 0$. Hence, T is a P-operator. □

The converse of the above theorem need not be true which is shown in the following example.

Example 3.3.10. *Let $T : \ell^2 \to \ell^2$ be defined by*

$$T(x_1, x_2, x_3, \ldots) = (x_1 - 7x_2, x_2, x_3, \ldots),$$

for $x = (x_1, x_2, x_3, \ldots) \in \ell^2$. Then T is a P-operator on ℓ^2 relative to the standard basis $\mathscr{B}$ of ℓ^2 as $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i imply that $x = 0$. But T is not a positive definite operator because $\langle Tx, x\rangle = x_1^2 - 7x_1x_2 + x_2^2$ is negative for $x = (1, 1, 0, \ldots) \neq 0$.

CHAPTER 4

SIGN-REVERSING SET OF OPERATORS ON HILBERT SPACES

4.1 INTRODUCTION

We start this chapter by defining the sign-reversing property for operators on a Hilbert space $\mathscr{H}$.

Definition 4.1.1. *Let $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$ and $T \in B(\mathscr{H})$. We say that T reverses the sign of the vector $x \in \mathscr{H}$ relative to $\mathscr{B}$ if*

$$\langle x, e_i \rangle \langle Tx, e_i \rangle \leq 0, \quad \textit{for all } i.$$

Example 4.1.2. *Let us consider the left shift operator T_L on $\mathscr{H} = \ell^2$ defined by*

$$T_L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

and let $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of $\mathscr{H}$. Then T reverses the sign of the vector $x = (1, -1, 1, 0, 0, \ldots)$ relative to $\mathscr{B}$. But T does not reverse the sign of the vector $y = (1, 1, 1, 0, 0, \ldots)$ relative to $\mathscr{B}$ because $\langle x, e_1 \rangle \langle Tx, e_1 \rangle = 1 > 0$.

Remark 4.1.3. *An operator T reverses the sign of the vector x relative to an orthonormal basis $\mathscr{B}_1$ does not imply that the operator T reverses the sign of the vector x relative to another orthonormal basis $\mathscr{B}_2$. The following example illustrates this fact.*

Example 4.1.4. *Let us consider the right shift operator T_R on $\mathscr{H} = \ell^2$ defined by*

$$T_R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$$

and let $\mathscr{B}_1 = \{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of $\mathscr{H}$. Then T_R reverses the sign of the vector $x = (1, 0, 0, \ldots)$ relative to $\mathscr{B}_1$. Let us now consider a unitary operator

U on ℓ^2 given by

$$U(x_1,x_2,x_3,x_4,\dots) = \Big(\frac{x_1}{\sqrt{2}}+\frac{x_2}{\sqrt{2}}, \frac{x_1}{\sqrt{2}}-\frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}}+\frac{x_4}{\sqrt{2}}, \frac{x_3}{\sqrt{2}}-\frac{x_4}{\sqrt{2}},\dots\Big).$$

Then $\mathscr{B}_2 = \{Ue_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathscr{H}$. Here $\langle x, Ue_1\rangle\langle Tx, Ue_1\rangle = \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} = \frac{1}{2} > 0$, so T does not reverse the sign of the vector x relative to $\mathscr{B}_2$.

Definition 4.1.5. *Limaye (1996) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathscr{H}$. Then for every $x \in \mathscr{H}$, we have*

$$x = \sum_i \langle x, e_i\rangle e_i.$$

4.2 SIGN-REVERSING SETS IN HILBERT SPACES

The sign-reversing set of an operator on a Hilbert space is defined by

Definition 4.2.1. *Let $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$ and $T \in B(\mathscr{H})$. We denote the sign-reversing set of T relative to $\mathscr{B}$ by $rev_{\mathscr{B}}(T)$ which is defined by*

$$rev_{\mathscr{B}}(T) = \{x : \langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0, \ \textit{for all } i\}.$$

Example 4.2.2. *The sign-reversing set of the right shift operator T_R realative to the standard orthonormal basis of $\mathscr{H}$ is*

$$rev_{\mathscr{B}}(T) = \{x \in \ell^2 : x_i x_{i+1} \leq 0, \ \textit{for all } i\}.$$

Proposition 4.2.3. *Let $T \in B(\mathscr{H})$ and $\mathscr{B}_1 = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$. Let U be a unitary operator on $\mathscr{H}$ satisfying $TU = UT$ and let $\mathscr{B}_2 = \{Ue_i\}_{i=1}^{\infty}$. Then $U(rev_{\mathscr{B}_1}(T)) = rev_{\mathscr{B}_2}(T)$.*

Proof. Let $x \in rev_{\mathscr{B}_1}(T)$. Then for each i, we have

$$\begin{aligned}
&\Longrightarrow \quad \langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0,\\
&\Longrightarrow \quad \langle U^*Ux, e_i\rangle\langle TU^*Ux, e_i\rangle \leq 0,\\
&\Longrightarrow \quad \langle U^*Ux, e_i\rangle\langle U^*TUx, e_i\rangle \leq 0,\\
&\Longrightarrow \quad \langle Ux, Ue_i\rangle\langle TUx, Ue_i\rangle \leq 0,\\
&\Longrightarrow \quad Ux \in rev_{\mathscr{B}_2}(T).
\end{aligned}$$

This shows that $U(rev_{\mathscr{B}_1}(T)) \subseteq rev_{\mathscr{B}_2}(T)$. On other hand, let $y \in rev_{\mathscr{B}_2}(T)$. Then for each i, we have

$$\begin{aligned}
&\implies \langle y, Ue_i\rangle\langle Ty, Ue_i\rangle \leq 0,\\
&\implies \langle U^*y, e_i\rangle\langle U^*Ty, e_i\rangle \leq 0,\\
&\implies \langle U^*y, e_i\rangle\langle TU^*y, e_i\rangle \leq 0,\\
&\implies U^*y \in rev_{\mathscr{B}_1}(T).
\end{aligned}$$

This shows that $U^*(rev_{\mathscr{B}_2}(T)) \subseteq rev_{\mathscr{B}_1}(T)$ and thus $rev_{\mathscr{B}_2}(T) \subseteq U(rev_{\mathscr{B}_2}(T))$. Both directions of inclusion will give equality relation. □

Proposition 4.2.4. *Let $T \in B(\mathscr{H})$ and $\mathscr{B}_1 = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$. Let U be a unitary operator on $\mathscr{H}$ satisfying $TU = UT$. Let $\mathscr{B}_2 = \{Ue_i\}_{i=1}^{\infty}$ and $U^*(rev_{\mathscr{B}_1}(T)) \subseteq rev_{\mathscr{B}_1}(T)$, $U(rev_{\mathscr{B}_2}(T)) \subseteq rev_{\mathscr{B}_2}(T)$. Then $rev_{\mathscr{B}_1}(T) = rev_{\mathscr{B}_2}(T)$.*

Proof. Let $x \in rev_{\mathscr{B}_1}(T)$. Then $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i. Thus $\langle x, Ue_i\rangle\langle Tx, Ue_i\rangle = \langle U^*x, e_i\rangle\langle U^*Tx, e_i\rangle = \langle U^*x, e_i\rangle\langle TU^*x, e_i\rangle \leq 0$ for all i, as $U^*(rev_{\mathscr{B}_1}(T)) \subseteq rev_{\mathscr{B}_1}(T)$. This implies that $x \in rev_{\mathscr{B}_2}(T)$. That is, $rev_{\mathscr{B}_1}(T) \subseteq rev_{\mathscr{B}_2}(T)$.

On other hand, suppose that $y \in rev_{\mathscr{B}_2}(T)$. Then $\langle y, Ue_i\rangle\langle Ty, Ue_i\rangle \leq 0$ for all i. As $U(rev_{\mathscr{B}_2}(T)) \subseteq rev_{\mathscr{B}_2}(T)$, there exists $x \in rev_{\mathscr{B}_2}(T)$ such that $Uy = x$, that is, $y = U^*x$. Thus $\langle y, e_i\rangle\langle Ty, e_i\rangle = \langle U^*x, e_i\rangle\langle TU^*x, e_i\rangle = \langle x, Ue_i\rangle\langle Tx, Ue_i\rangle \leq 0$ for all i, hence $y \in rev_{\mathscr{B}_1}(T)$. Therefore $rev_{\mathscr{B}_2}(T) \subseteq rev_{\mathscr{B}_1}(T)$. Thus $rev_{\mathscr{B}_1}(T) = rev_{\mathscr{B}_2}(T)$. □

Remark 4.2.5. *Let $T \in B(\mathscr{H})$ and $\mathscr{B}$ be an orthonormal basis of $\mathscr{H}$. Then $rev_{\mathscr{B}}(T)$ need not be a subspace of $\mathscr{H}$. The following example shows this fact.*

Example 4.2.6. *Consider the operator T_R and the orthonormal basis $\mathscr{B}$ as given in Example 4.2.2. Then $x = (-1, 1, -1, 0, 0, 0, \dots)$ and $y = (2, \frac{-1}{2}, 2, -1, 0, 0, 0, \dots) \in rev_{\mathscr{B}}(T_R)$. But $x + y = (1, \frac{1}{2}, 1, -1, 0, 0, 0, \dots) \notin rev_{\mathscr{B}}(T_R)$.*

Proposition 4.2.7. *Let $T \in B(\mathscr{H})$. If there exists an orthonormal basis $\mathscr{B}$ such that $rev_{\mathscr{B}}(T) = \mathscr{H}$, then T is negative semi-definite on $\mathscr{H}$.*

Proof. As $rev_{\mathscr{B}}(T) = \mathscr{H}$, for any $x \in \mathscr{H}$, we have $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$, for all i. This implies that for any $x \in \mathscr{H}$, $\sum_{i=1}^{\infty}\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ which is equivalent to $\langle Tx, x\rangle \leq 0$. □

Example 4.2.8. *Let us consider any diagonal operator D in ℓ^2 given by $De_i = d_{ii}e_i$ with $d_{ii} \leq 0$, where $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$. Then $rev_{\mathscr{B}}(D) = \mathscr{H}$.*

Remark 4.2.9. *An operator T which is negative semi-definite does not imply that $rev_{\mathscr{B}}(T) = \mathscr{H}$, for any orthonormal basis $\mathscr{B}$. The following example shows this fact.*

Example 4.2.10. *Let us consider the Hilbert space $\mathscr{H} = \ell^2$ and let $T \in B(\mathscr{H})$ be defined by*

$$T(x_1,x_2,x_3,x_4,\ldots) = (-x_1-x_2,-x_1-x_2,-x_3-x_4,-x_3-x_4,\ldots).$$

Let $\mathscr{B}$ be the standard orthonormal basis of $\mathscr{H}$. Note that the operator T is a negative semi-definite operator. Here T does not reverse the sign of the vector $(-1,2,0,0,\ldots)$.

Remark 4.2.11. *The sign-reversing set of an operator T on a Hilbert space $\mathscr{H}$ relative to an orthonormal basis $\mathscr{B}_1$ need not be equal to the sign reversing set of the operator relative to another orthonormal basis $\mathscr{B}_2$. The following example illustrates this fact.*

Example 4.2.12. *Define $T : \ell^2 \to \ell^2$ by*

$$T(x_1,x_2,x_3,\ldots) = (x_1, 2x_1+x_2, 2x_2+x_3,\ldots)$$

for $x = (x_1,x_2,x_3,\ldots) \in \ell^2$. Then, T is a bounded linear operator. The sign-reversing set of T relative to the standard orthonormal basis $\mathscr{B}_1 = \{e_i\}_{i=1}^{\infty}$ of ℓ^2 is given by $rev_{\mathscr{B}_1}(T) = \{0\}$. To see this, suppose $\langle x,e_i\rangle\langle Tx,e_i\rangle \leq 0$ for all i. Then, for the case $i = 1$ we get, $x_1^2 \leq 0$, implies that $x_1 = 0$ and for $i \geq 2$ we have $x_i(2x_{i-1}+x_i) \leq 0$, by solving these inequalities together will lead to $x = 0$. Hence $rev_{\mathscr{B}_1}(T) = \{0\}$. Now consider the unitary operator U on ℓ^2 given by

$$U(x_1,x_2,x_3,\ldots) = \Big(\frac{x_1}{\sqrt{2}}+\frac{x_2}{\sqrt{2}},\frac{x_1}{\sqrt{2}}-\frac{x_2}{\sqrt{2}},\frac{x_3}{\sqrt{2}}+\frac{x_4}{\sqrt{2}},\frac{x_3}{\sqrt{2}}-\frac{x_4}{\sqrt{2}},\ldots\Big),$$

then $\mathscr{B}_2 = \{Ue_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathscr{H}$, and $rev(T)_{\mathscr{B}_2} \neq \{0\}$ because the non-zero element $x = (-2,\sqrt{2},0,0,\ldots) \in \ell_2$ satisfies the inequalities $\langle x,Ue_i\rangle\langle Tx,Ue_i\rangle \leq 0$, for all i. This shows that $rev_{\mathscr{B}_1}(T) \neq rev_{\mathscr{B}_2}(T)$.

Proposition 4.2.13. *Let $T_1,T_2 \in B(\mathscr{H})$ and $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$. Then the following are true:*

1. *$rev_{\mathscr{B}}(T_1-T_2)\cap rev_{\mathscr{B}}(T_2) \subseteq rev_{\mathscr{B}}(T_1)$.*
2. *$rev_{\mathscr{B}}(T_1)\cap rev_{\mathscr{B}}(T_2) \subseteq rev_{\mathscr{B}}(T_1+T_2)$.*

Proof. (1) Let $x \in rev_{\mathscr{B}}\cap(T_1-T_2)rev_{\mathscr{B}}(T_2)$. Then, for each i, we have

$$\langle x,e_i\rangle\langle T_2x,e_i\rangle \leq 0 \quad \text{and} \quad \langle x,e_i\rangle\langle (T_1-T_2)x,e_i\rangle \leq 0.$$

Therefore

$$\langle x,e_i\rangle\langle (T_1-T_2)x,e_i\rangle \leq 0 \implies \langle x,e_i\rangle\langle T_1x,e_i\rangle - \langle x,e_i\rangle\langle T_2x,e_i\rangle \leq 0$$

$$\begin{aligned} &\implies \langle x,e_i\rangle\langle T_1x,e_i\rangle \le \langle x,e_i\rangle\langle T_2x,e_i\rangle \\ &\implies \langle x,e_i\rangle\langle T_1x,e_i\rangle \le 0 \;\; \forall i \text{ as } \langle x,e_i\rangle\langle T_2x,e_i\rangle \le 0. \end{aligned}$$

This shows that $x \in rev_{\mathscr{B}}(T_1)$. Hence $rev_{\mathscr{B}}(T_1 - T_2) \cap rev_{\mathscr{B}}(T_2) \subseteq rev_{\mathscr{B}}(T_1)$.
(2) Let $x \in rev_{\mathscr{B}}(T_1) \cap rev_{\mathscr{B}}(T_2)$. Then for each i, we have

$$\langle x,e_i\rangle\langle T_1x,e_i\rangle \le 0 \quad \text{and} \quad \langle x,e_i\rangle\langle T_2x,e_i\rangle \le 0.$$

Therefore $\langle x,e_i\rangle\langle T_1x,e_i\rangle + \langle x,e_i\rangle\langle T_2x,e_i\rangle \le 0$. This implies that $\langle x,e_i\rangle\langle (T_1+T_2)x,e_i\rangle \le 0$. Hence $x \in rev_{\mathscr{B}}(T_1+T_2)$.

□

4.3 CHARACTERIZATION OF THE SIGN-REVERSING SET

In this section, we give a characterization for the sign-reversing set of an operator $T \in B(\mathscr{H})$ relative to an orthonormal basis $\mathscr{B}$ of $\mathscr{H}$. First, we prove a lemma.

Lemma 4.3.1. *Let $T \in B(\mathscr{H})$ and $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$. Let $x \in \mathscr{H}$ be a non-zero element. If $x \in rev_{\mathscr{B}}(T)$, then there exists a diagonal operator D so that $De_i = d_{ii}e_i$, with $0 \le d_{ii} \le 1$ such that $x \in ker(DT - T - D)$.*

Proof. To prove this, we choose the diagonal elements d_{ii} of the diagonal operator D as follows :

1. Suppose the case that $\langle x,e_i\rangle\langle Tx,e_i\rangle = 0$. If both $\langle x,e_i\rangle = 0$ and $\langle Tx,e_i\rangle = 0$ are zero, then we can choose any $d_{ii} \in [0,1]$. Because, by Definition 4.1.5, we have $(DT-T-D)x = \sum_j \langle (DT-T-D)x, e_j\rangle e_j$, when $j = i$, we see that

$$\begin{aligned} \langle DTx - Tx - Dx, e_i\rangle &= \langle DTx,e_i\rangle - \langle Tx,e_i\rangle - \langle Dx,e_i\rangle \\ &= \langle Tx,De_i\rangle - \langle Tx,e_i\rangle - \langle x,De_i\rangle \\ &= \langle Tx,d_{ii}e_i\rangle - \langle Tx,e_i\rangle - \langle x,d_{ii}e_i\rangle \\ &= d_{ii}\langle Tx,e_i\rangle - \langle Tx,e_i\rangle - d_{ii}\langle x,e_i\rangle \\ &= 0 \end{aligned}$$

irrespective of the choices of d_{ii}, as $\langle x,e_i\rangle = 0$ and $\langle Tx,e_i\rangle = 0$.

Otherwise, there are two possibilities.

(a) If $\langle x, e_i \rangle = 0$, then we choose $d_{ii} = 1$. In this case, we can also see that $\langle (DT - T - D)x, e_i \rangle = 0$.

(b) If $\langle x, e_i \rangle \neq 0$ and $\langle Tx, e_i \rangle = 0$, then we choose $d_{ii} = 0$. Thus we have $\langle (DT - T - D)x, e_i \rangle = 0$.

2. Suppose that $\langle x, e_i \rangle \langle Tx, e_i \rangle < 0$. Then, $\langle x, e_i \rangle$ and $\langle Tx, e_i \rangle$ are non-zero and of opposite sign. Take $d_{ii} = \frac{-\langle Tx, e_i \rangle}{\langle x, e_i \rangle - \langle Tx, e_i \rangle} = \frac{-\langle x, e_i \rangle \langle Tx, e_i \rangle}{\langle x, e_i \rangle^2 - \langle x, e_i \rangle \langle Tx, e_i \rangle}$. In this case, we have

$$\begin{aligned}
\langle DTx - Tx - Dx, e_i \rangle &= \langle DTx, e_i \rangle - \langle Tx, e_i \rangle - \langle Dx, e_i \rangle \\
&= \langle Tx, De_i \rangle - \langle Tx, e_i \rangle - \langle x, De_i \rangle \\
&= \langle Tx, d_{ii} e_i \rangle - \langle Tx, e_i \rangle - \langle x, d_{ii} e_i \rangle \\
&= d_{ii} \langle Tx, e_i \rangle - \langle Tx, e_i \rangle - d_{ii} \langle x, e_i \rangle .
\end{aligned}$$

$$\begin{aligned}
&= \frac{-\langle x, e_i \rangle \langle Tx, e_i \rangle^2}{\langle x, e_i \rangle^2 - \langle x, e_i \rangle \langle Tx, e_i \rangle} - \langle Tx, e_i \rangle + \frac{\langle x, e_i \rangle^2 \langle Tx, e_i \rangle}{\langle x, e_i \rangle^2 - \langle x, e_i \rangle \langle Tx, e_i \rangle} \\
&= \frac{-\langle x, e_i \rangle \langle Tx, e_i \rangle^2 - \langle x, e_i \rangle^2 \langle Tx, e_i \rangle + \langle x, e_i \rangle \langle Tx, e_i \rangle^2 + \langle x, e_i \rangle^2 \langle Tx, e_i \rangle}{\langle x, e_i \rangle^2 - \langle x, e_i \rangle \langle Tx, e_i \rangle} \\
&= 0.
\end{aligned}$$

Thus for all i, we have $\langle (DT - T - D)x, e_i \rangle = 0$. Hence by Definition 4.1.5, we get $(DT - T - D)x = 0$, which implies that $x \in ker(DT - T - D)$. This completes the proof.

□

The next theorem gives a characterization for the sign-reversing set of an operator T in a Hilbert space $\mathscr{H}$ relative to an orthonormal basis $\mathscr{B}$ in terms of kernel of the operator $DT - T - D$ for a diagonal operator $D = (d_{ii})$, with $0 \leq d_{ii} \leq 1$, we denote such diagonal operators as $0 \leq D \leq I$.

Theorem 4.3.2. *Let $T \in B(\mathscr{H})$ and $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$. Then*

$$rev_{\mathscr{B}}(T) = \bigcup_{0 \leq D \leq I} ker(DT - T - D).$$

Proof. By Lemma 4.3.1, we have $rev_{\mathscr{B}}(T) \subseteq \bigcup_{0 \leq D \leq I} ker(DT - T - D)$.

Conversely, let $D = (d_{ii})$ be a diagonal operator such that $0 \leq d_{ii} \leq 1$. Let $x \in ker(DT - T - D)$. Then $(DT - T - D)x = 0$, hence $DTx - Tx - Dx = 0$. Thus for each i, we have $\langle DTx, e_i \rangle - \langle Tx, e_i \rangle - \langle Dx, e_i \rangle = 0$ and consequently $\langle x, e_i \rangle \langle DTx, e_i \rangle - \langle x, e_i \rangle \langle Tx, e_i \rangle - \langle x, e_i \rangle \langle Dx, e_i \rangle = 0$, this implies that $\langle x, e_i \rangle \langle Tx, De_i \rangle - \langle x, e_i \rangle \langle Tx, e_i \rangle -$

$\langle x, e_i \rangle \langle x, De_i \rangle = 0$, that is, $d_{ii}\langle x, e_i \rangle \langle Tx, e_i \rangle - \langle x, e_i \rangle \langle Tx, e_i \rangle - d_{ii}\langle x, e_i \rangle \langle x, e_i \rangle = 0$. Hence, we deduce the following:

(i) If $d_{ii} = 0$, then $-\langle x, e_i \rangle \langle Tx, e_i \rangle = 0$. Hence $\langle x, e_i \rangle \langle Tx, e_i \rangle = 0$.

(ii) If $d_{ii} = 1$, then $-\langle x, e_i \rangle^2 = 0$. Thus $\langle x, e_i \rangle \langle Tx, e_i \rangle = 0$.

(iii) If $0 < d_{ii} < 1$, then $d_{ii}\langle x, e_i \rangle \langle Tx, e_i \rangle - \langle x, e_i \rangle \langle Tx, e_i \rangle - d_{ii}\langle x, e_i \rangle \langle x, e_i \rangle = 0$ which implies that $\langle x, e_i \rangle \langle Tx, e_i \rangle = \frac{d_{ii}\langle x, e_i \rangle^2}{(d_{ii}-1)} < 0$.

Hence $x \in rev_{\mathscr{B}}(T)$. □

4.4 SUFFICIENT OPERATORS ON HILBERT SPACES

Next, we generalize the concept given in Definition 1.4.24 in Hilbert space settings.

Definition 4.4.1. *Let $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$. Then $T \in B(\mathscr{H})$ is said to be:*

1. *A C-sufficient operator relative to the given orthonormal basis $\mathscr{B}$ if for $x \in \mathscr{H}$, the inequalities $\langle x, e_i \rangle \langle Tx, e_i \rangle \leq 0$ for all i imply that $\langle x, e_i \rangle \langle Tx, e_i \rangle = 0$ for all i.*
2. *A R-sufficient operator relative to the given orthonormal basis $\mathscr{B}$ if the adjoint of T is a C-sufficient operator relative to $\mathscr{B}$.*
3. *A sufficient operator if it is both a C-sufficient and R-sufficient operator relative to the given orthonormal basis $\mathscr{B}$.*

Remark 4.4.2. *An operator can be a C-sufficient operator relative to an orthonormal basis $\mathscr{B}_1$, whereas the same operator need not be a C-sufficient operator relative to another orthonormal basis $\mathscr{B}_2$.*

Example 4.4.3. *Consider the Hilbert space $\mathscr{H} = \ell^2$ and the standard orthonormal basis $\mathscr{B}_1 = \{e_i\}_{i=1}^{\infty}$ of $\mathscr{H}$. Let T be an operator on $\mathscr{H}$ defined by*

$$T(x_1, x_2, x_3, \ldots) = (0, x_1 + x_2, 0, x_3 + x_4, 0, x_5 + x_6, 0, \ldots).$$

Then T is a bounded linear operator on $\mathscr{H}$. The operator T is not a C-sufficient operator relative to the orthonormal basis $\mathscr{B}_1$. Because the inequalities $\langle x, e_i \rangle \langle Tx, e_i \rangle \leq 0$, for all i, imply that $x_1 x_2 + x_2^2 \leq 0$. Thus we can observe that the element $(-5, \sqrt{2}, 0, 0, \ldots)$ satisfies $\langle x, e_i \rangle \langle Tx, e_i \rangle \leq 0$ for all i, but $\langle x, e_2 \rangle \langle Tx, e_2 \rangle \neq 0$. Thus, T is not a C-sufficient operator relative to $\mathscr{B}_1$.

Now consider another orthonormal basis $\mathscr{B}_2 = \{Ue_i\}_{i=1}^{\infty}$ of $\mathscr{H}$, where U is the unitary operator on $\mathscr{H}$ given by

$$U(x_1,x_2,x_3,\ldots) = \Big(\frac{x_1}{\sqrt{2}}+\frac{x_2}{\sqrt{2}},\frac{x_1}{\sqrt{2}}-\frac{x_2}{\sqrt{2}},\frac{x_3}{\sqrt{2}}+\frac{x_4}{\sqrt{2}},\frac{x_3}{\sqrt{2}}-\frac{x_4}{\sqrt{2}},\ldots\Big).$$

Then we see that the operator T is a C-sufficient operator relative to the orthonormal basis $\mathscr{B}_2$. Because the inequalities $\langle x,Ue_i\rangle\langle Tx,Ue_i\rangle \leq 0$ for all i imply that $(\frac{x_i}{\sqrt{2}}+\frac{x_{i+1}}{\sqrt{2}})^2 \leq 0$ and $(\frac{x_{i+1}^2}{2}-\frac{x_i^2}{2}) \leq 0$. Solving these inequalities, we get $x_i = -x_{i+1}$, but in that case $T(x_1,x_2,x_3,\ldots) = (0,0,0,\ldots)$, thus we get $\langle x,Ue_i\rangle\langle Tx,Ue_i\rangle = 0$ for all i. This shows that T is a C-sufficient operator relative to $\mathscr{B}_2$.

Remark 4.4.4. *A C-sufficient operator relative to an orthonormal basis $\mathscr{B}$ need not be an R-sufficient operator relative to the orthonormal basis $\mathscr{B}$. The following example illustrates this fact.*

Example 4.4.5. *Let us consider the Hilbert space $\mathscr{H} = \ell^2$ and the standard orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ of $\mathscr{H}$. Let $T \in B(\mathscr{H})$ be defined by*

$$T(x_1,x_2,x_3,\ldots) = (x_1,x_1,x_3,x_3,x_5,x_5,\ldots).$$

Then, the adjoint of the operator T is given by

$$T^*(y_1,y_2,y_3,\ldots) = (y_1+y_2,y_3+y_4,y_5+y_6,\ldots).$$

The operator T is a C-sufficient operator relative to the given orthonormal basis $\mathscr{B}$. Because the inequalities $\langle x,e_i\rangle\langle Tx,e_i\rangle \leq 0$ for all i imply that $\langle x,e_i\rangle\langle Tx,e_i\rangle = 0$ for all i. But T is not R-sufficient operator, because the non-zero element $(2,-3,0,0,0,\ldots)$, satisfies $\langle x,e_i\rangle\langle T^*x,e_i\rangle \leq 0$ for all i. Thus $\langle x,e_i\rangle\langle T^*x,e_i\rangle \neq 0$ when $i = 2$.

Remark 4.4.6. *A R-sufficient operator relative to an orthonormal basis $\mathscr{B}$ need not be a C-sufficient operator relative to the orthonormal basis $\mathscr{B}$. The following example illustrates this fact.*

Example 4.4.7. *Let us consider the Example 4.4.5. Let S be the operator on $\mathscr{H}$ given by $S = T^*$. Then $S^* = T$. As S^* is C-sufficient relative to the orthonormal basis $\mathscr{B}$, the operator S is an R-sufficient operator relative to the orthonormal basis $\mathscr{B}$. Also, we have seen that $S = T^*$ is not C-sufficient relative to the orthonormal basis $\mathscr{B}$.*

Next, we give an example of a sufficient operator relative to an orthonormal basis.

Example 4.4.8. *Let us consider the Hilbert space $\mathscr{H} = \ell^2$ and the standard orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ of $\mathscr{H}$. Let T be an operator on $\mathscr{H}$ defined by*

$$T(x_1, x_2, x_3, x_4, x_5, \dots) = (x_1, x_1 + x_2, x_3, x_3 + x_4, x_5, \dots).$$

Then T is a bounded linear operator on $\mathscr{H}$ and the adjoint of the operator T is given by

$$T^*(y_1, y_2, y_3, y_4, y_5, \dots) = (y_1 + y_2, y_2, y_3 + y_4, y_4, \dots).$$

Then T is a C-sufficient operator relative to the given orthonormal basis $\mathscr{B}$. Because $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i implies that $\langle x, e_i\rangle\langle Tx, e_i\rangle = 0$ for all i.

Here T is also R-sufficient operator relative to the given orthonormal basis $\mathscr{B}$. Because T^* is a C-sufficient operator relative to the given orthonormal basis $\mathscr{B}$. Thus, T is a sufficient operator relative to the given orthonormal basis $\mathscr{B}$.

Proposition 4.4.9. *If T is a P-operator relative to a given orthonormal basis $\mathscr{B}$, then T is a C-sufficient operator relative to the orthonormal basis $\mathscr{B}$.*

Proof. The proof follows directly from the definition of these two classes of operators. □

Remark 4.4.10. *An operator can be a C-sufficient operator relative to an orthonormal basis $\mathscr{B}$, whereas the same operator need not be a P-operator relative to the given orthonormal basis $\mathscr{B}$. The following example shows this fact.*

Example 4.4.11. *Let us consider the Hilbert space $\mathscr{H} = \ell^2$ and the standard orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ of $\mathscr{H}$. Let T be an operator on $\mathscr{H}$ defined by*

$$T(x_1, x_2, x_3, x_4, \dots) = (x_1, 0, x_3, 0, x_5, \dots).$$

Then T is a bounded linear operator on $\mathscr{H}$, and T is a C-sufficient operator relative to the given orthonormal basis $\mathscr{B}$. Because the inequalities $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i imply that $\langle x, e_i\rangle\langle Tx, e_i\rangle = 0$ for all i.

Here the operator T is not a P-operator relative to the given orthonormal basis $\mathscr{B}$. Because the non-zero element $(0, \frac{1}{2}, 0, \frac{1}{2^2}, 0, \frac{1}{2^3}, \dots) \in \mathscr{H}$, satisfies $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i.

4.5 CHARACTERIZATIONS OF SUFFICIENT OPERATORS

Theorem 4.5.1. *Let $T \in B(\mathscr{H})$. The operator T is a C-sufficient operator relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ if and only if the operator $I - D + DT$, where D is a diagonal operator relative to $\mathscr{B}$ such that $De_i = d_{ii}e_i$ with $0 \le d_{ii} \le 1$, is a C-sufficient operator relative to the orthonormal basis $\mathscr{B}$.*

Proof. First, suppose that T is a C-sufficient operator relative to an orthonormal basis $\mathscr{B}$. Then we have, if for $y \in \mathscr{H}$, $\langle y, e_i\rangle\langle Ty, e_i\rangle \le 0$ for all i, then $\langle y, e_i\rangle\langle Ty, e_i\rangle = 0$ for all i. Now, let $x \in \mathscr{H}$ be such that $\langle x, e_i\rangle\langle (I - D + DT)x, e_i\rangle \le 0$ for all i. This is equivalent to $\langle x, e_i\rangle\langle x, e_i\rangle - d_{ii}\langle x, e_i\rangle\langle x, e_i\rangle + d_{ii}\langle x, e_i\rangle\langle Tx, e_i\rangle \le 0$ for all i. Hence

$$(1 - d_{ii})\langle x, e_i\rangle^2 + d_{ii}\langle x, e_i\rangle\langle Tx, e_i\rangle \le 0 \text{ for all } i. \tag{4.5.1}$$

Thus $\langle x, e_i\rangle\langle Tx, e_i\rangle \le 0$ for all i. As T is a C-sufficient operator relative to an orthonormal basis $\mathscr{B}$, we have $\langle x, e_i\rangle\langle Tx, e_i\rangle = 0$ for all i. Thus by Equation (4.5.1), we have $\langle x, e_i\rangle\langle (I - D + DT)x, e_i\rangle = 0$ for all i. Hence, $I - D + DT$ is a C-sufficient operator relative to the orthonormal basis $\mathscr{B}$.

Conversely, suppose that $I - D + DT$ is a C-sufficient operator relative to the orthonormal basis $\mathscr{B}$. Then we have if for $y \in \mathscr{H}$, $\langle y, e_i\rangle\langle (I - D + DT)y, e_i\rangle \le 0$ for all i, then $\langle y, e_i\rangle\langle (I - D + DT)y, e_i\rangle = 0$ for all i. By taking the diagonal operator as the identity operator, we get T is a C-sufficient operator relative to the orthonormal basis $\mathscr{B}$. □

Theorem 4.5.2. *Let $T \in B(\mathscr{H})$. Then T is a R-sufficient operator relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ if and only if the operator $I - D + DT$, where D is a diagonal operator relative to $\mathscr{B}$ such that $De_i = d_{ii}e_i$ with $0 \le d_{ii} \le 1$, is a R-sufficient operator relative to the orthonormal basis $\mathscr{B}$.*

Proof. The proof is similar to that of Theorem 4.5.1, by replacing T with its adjoint T^*. □

Corollary 4.5.3. *Let $T \in B(\mathscr{H})$. Then T is a sufficient operator relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ if and only if the operator $I - D + DT$, where D is a diagonal operator relative to $\mathscr{B}$ such that $De_i = d_{ii}e_i$ with $0 \le d_{ii} \le 1$, is a sufficient operator relative to the orthonormal basis $\mathscr{B}$.*

Proof. The proof follows from Theorem 4.5.1 and Theorem 4.5.2. □

Theorem 4.5.4. *Let $T \in B(\mathscr{H})$ and $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$. Then the following statements are equivalent:*

1. *T is a C-sufficient operator relative to the orthonormal basis $\mathscr{B}$.*

2. *For each $x \in ker(DT - T - D)$ for the diagonal operator D, so that $De_i = d_{ii}e_i$ with $0 \leq d_{ii} \leq 1$, if $0 < d_{ii} < 1$, then $\langle x, e_i \rangle = 0$.*

Proof. Suppose that T is a C-sufficient operator relative to the orthonormal basis $\mathscr{B}$ and there exists diagonal operator D, so that $De_i = d_{ii}e_i$ with $0 \leq d_{ii} \leq 1$ and $x \in ker(DA - A - D)$, such that $\langle x, e_i \rangle \neq 0$ when $0 < d_{ii} < 1$.

Since $d_{ii}\langle x, e_i\rangle\langle Tx, e_i\rangle - \langle x, e_i\rangle\langle Tx, e_i\rangle - d_{ii}\langle x, e_i\rangle^2 = 0$ implies $(d_{ii} - 1)\langle x, e_i\rangle\langle Tx, e_i\rangle = d_{ii}\langle x, e_i\rangle^2$ implies $\langle x, e_i\rangle\langle Tx, e_i\rangle = \frac{d_{ii}\langle x, e_i\rangle^2}{(d_{ii}-1)} < 0$, which is not possible as T is C-sufficient operator relative to the orthonormal basis $\mathscr{B}$. Thus, $\langle x, e_i \rangle$ must be equal to 0.

Conversely, suppose that T is not a C-sufficient operator relative to the orthonormal basis $\mathscr{B}$. Then there exists an $x \in rev_{\mathscr{B}}(T)$ and an index set α such that $\overline{\alpha} \neq \phi$ and

$$\langle x, e_i\rangle\langle Tx, e_i\rangle = 0, i \in \alpha, \quad \text{and}$$
$$\langle x, e_i\rangle\langle Tx, e_i\rangle < 0, i \in \overline{\alpha}.$$

Define the diagonal operator D as $De_i = d_{ii}e_i$ so that

$$d_{ii} = \begin{cases} 0 & \text{if } i \in \alpha \text{ and } \langle Tx, e_i\rangle = 0 \\ 1 & \text{if } i \in \alpha \text{ and } \langle Tx, e_i\rangle \neq 0,\ \langle x, e_i\rangle = 0 \\ \frac{-\langle x, e_i\rangle\langle Tx, e_i\rangle}{\langle x, e_i\rangle^2 - \langle x, e_i\rangle\langle Tx, e_i\rangle} & \text{if } i \in \overline{\alpha}. \end{cases}$$

Then $0 \leq d_{ii} \leq 1$ and $(DT - T - D)x = 0$ imply $x \in ker(DT - T - D)$. However $0 < d_{ii} < 1$ and $\langle x, e_i \rangle \neq 0$ for each $i \in \overline{\alpha}$, so the result follows by method of contrapositive. $\square$

CHAPTER 5

SPECTRAL THEORY OF CERTAIN POSITIVE OPERATORS

5.1 INTRODUCTION

Eigenvalues and eigenvectors, key concepts in linear algebra, have wide-ranging applications in both science and engineering. Certain properties of a matrix can be analyzed if the eigenvalues of the matrix are known to us. For example, if a matrix has no non-zero eigenvalue, then it is invertible. The spectrum is the infinite-dimensional analogue of the set of matrix eigenvalues. Several core results of matrix theory are extended to linear operators on a Hilbert space, where the proofs are typically quite different and one often needs additional assumptions on the operators. In this chapter, we discuss results which characterize P-matrices and we focus on elucidating spectral properties associated with certain classes of positive operators under consideration. Understanding their spectral behavior is essential for analyzing the dynamics and stability of systems governed by such operators. We start with a few known spectral characterizations of P-matrices.

Theorem 5.1.1. *(Fiedler and Pták (1962)) Let $A \in \mathbb{R}^{n \times n}$. The following statements are equivalent :*

1. *A is a P-matrix.*
2. *The real eigenvalues of the principal submatrices of A are positive.*

Theorem 5.1.2. *(Fiedler and Pták (1962)) Let A be a real square matrix with all non-positive off-diagonal elements. The following statements are equivalent :*

1. *A is a P-matrix.*
2. *Each real eigenvalue of A is positive.*

3. *The real part of each eigenvalue of A is positive.*

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n real numbers or complex numbers having both conjugate pairs in the collection. The k^{th} elementary symmetric function is defined by

$$\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{t=1}^{k} \lambda_{i_t}.$$

The elementary symmetric functions characterize the spectra of P-matrices and P_0-matrices as shown in the following result.

Theorem 5.1.3. *(Hershkowitz (1983)) The set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a spectrum of some P-matrix iff*

$$\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) > 0, \; k = 1, 2, \ldots, n.$$

In the above result, if $\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) \geq 0, k = 1, 2, \ldots, n$, it characterizes P_0-matrices.

Example 5.1.4. *Let $A = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$ be a matrix whose spectrum is $\{1, 1\}$. Though the symmetric functions $\sigma_k(\{1, 1\})$ are positive, for $k = 1, 2$, the matrix A is not a P-matrix. However, by Theorem (5.1.3), the set $\{1, 1\}$ is a spectrum of the P-matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.*

If $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of a matrix having positive sums of principal minor, then it satisfies

$$\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) > 0, k = 1, 2, \ldots, n.$$

Moreover, the converse is also true as shown in Theorem (5.1.3) : $\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is positive for $k = 1, 2, \ldots, n$ if and only if it is a spectrum of some P-matrix. Such a set S is called a P-set.

Kellogg (1972) proved that elements of a P-set cannot lie in the given wedge around the negative axis. More precisely he proved the following result.

Theorem 5.1.5. *(Kellogg (1972))*

1. *If $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a P-set, then*

$$|arg \; \lambda_i| < \frac{n-1}{n}\pi, \quad i = 1, 2, \ldots, n. \tag{5.1.1}$$

2. *If $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_i \neq 0$, $i = 1, 2, \ldots, n$, is a P_0 set, then*

$$|arg \; \lambda_i| \leq \frac{n-1}{n}\pi, \quad i = 1, 2, \ldots, n.$$

Equality holds in the above inequality if and only if

$$\begin{aligned}\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) &= 0, \qquad k = 1, 2, \ldots, n-1, \\ \sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) &> 0.\end{aligned}$$

Hershkowitz and Berman (1983) showed that if the number of elements in $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ in the right half plane, or in the left half plane, is given, then the bound 5.1.1 can be improved, namely, there exists α such that

$$|arg\ \lambda_i| < \alpha < \frac{n-1}{n}\pi,\ i = 1, 2, \ldots, n.$$

It is also proved that if S has exactly one element in the right half plane, then

$$|arg\ \lambda_i| < \frac{2}{3}\pi,\ i = 1, 2, \ldots, n.$$

The above inequality is independent of n. It was conjectured in Hershkowitz and Berman (1983) that if a P-matrix A has exactly k eigenvalues in the left half plane (k is an even integer), then

$$|arg\ \lambda| < \frac{2k+1}{2k+2}\pi, \quad \text{for all } \lambda \in \sigma(A).$$

The following result has given a negative answer to the conjecture.

Theorem 5.1.6. *(Hershkowitz (1983)) Let S be a finite set of complex numbers, consisting of positive numbers and non-real conjugate pairs. Then one can obtain a set with all elementary symmetric functions positive by adding positive numbers to S.*

Every P-matrix can have almost all of its eigenvalues in the left half of the complex plane as shown below.

Theorem 5.1.7. *(Hershkowitz (1983)) There exists a P-matrix all of those eigenvalues, except one when n is even or two when n is odd, have negative real parts.*

A matrix A all of whose powers are P-matrices is denoted $A \in \mathscr{P}\mathscr{M}$ (respectively, $A \in \mathscr{P}_0\mathscr{M}$ if only $A^k \in \mathscr{P}_0$). If we denote the set of eigenvalues with multiple appearances corresponding to algebraic multiplicities of A by $\sigma(A)$, it is then natural to raise the following question :

$$\text{if } A \in \mathscr{P}M, \text{ does } \lambda \in \sigma(A) \text{ imply } \lambda > 0?$$

The above question has been answered affirmatively by Hershkowitz (1983) for $n \leq 4$ and it is not resolved for $n \geq 5$.

5.2 FACTORIZATION OF P-MATRICES USING EIGENVALUES

Proposition 5.2.1. *Let $A \in \mathbb{R}^{n\times n}$ such that $I+A$ is invertible. We define*

$$U(A) = (I+A)^{-1}(I-A).$$

Then the following statements hold:

1. $A = U(U(A)) = (I+U(A))^{-1}(I-U(A))$.

2. $I+U(A) = 2(I+A)^{-1}$.

3. *If A is invertible, then* $I-U(A) = 2(I+A^{-1})^{-1}$.

Proof. 1. We have $(I+A)U(A) = I-A$. So, $U(A)+AU(A) = I-U(A)$. Hence

$$A(I+U(A)) = I-U(A) \tag{5.2.1}$$

It is noted that if $(I+U(A))x = 0$, then $x = 0$. Therefore $-1 \notin \sigma(U(A))$. Since $(I+U(A))^{-1}$ and $I+U(A)$ commute, by the equation 5.2.1, we get that $A = U(U(A))$.

2. We have

$$\begin{aligned} I+U(A) &= I+(I+A)^{-1}(I-A) \\ &= (I+A)^{-1}[I+A+I-A] \\ &= 2(I+A)^{-1}. \end{aligned}$$

3. We have

$$\begin{aligned} I-U(A) &= I-(I+A)^{-1}(I-A) \\ &= (I+A)^{-1}[I+A-I+A] \\ &= 2(I+A)^{-1}A. \end{aligned}$$

If A is invertible then we have

$$\begin{aligned} I - U(A) &= 2(I+A)^{-1}A \\ &= 2(I+A)^{-1}(A^{-1})^{-1} \\ &= 2[A^{-1}(I+A)]^{-1} \\ &= 2(I+A^{-1})^{-1}. \end{aligned}$$

□

Theorem 5.2.2. *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix. Then A is a product of P-matrices.*

Proof. By Theorem 5.1.2 (1), each real eigenvalue of A is positive. Hence A has no negative real eigenvalues, So $(I+A)^{-1}$ and $(I-A)$ exist. Therefore $U(A) = (I+A)^{-1}(I-A)$ is well defined. By Proposition 5.2.1, we have

$$\begin{aligned} I + U(A) &= 2(I+A)^{-1} \\ I - U(A) &= 2(I+A^{-1})^{-1}. \end{aligned}$$

By Theorem 1.4.5, $I+U(A)$ and $I-U(A)$ are P-matrices. Thus $A = (I-U(A))^{-1}(I-U(A))$ is a factorization of the P-matrix A. □

Definition 5.2.3. *A matrix A is called positive stable if all its eigenvalues have positive real parts. It is easy to see that a P-matrix may not be positive stable.*

Proposition 5.2.4. *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix and let $D \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix, then AD is Positive stable.*

Proof. Since A is P-matrix, it has no negative real eigenvalues. As D is a diagonal matrix with positive diagonal entries, every eigenvalues of AD has positive real parts.

□

The following result describes the factorization of a P-matrix into a product of positive stable matrices which are P-matrices as well.

Theorem 5.2.5. *Let $A \in \mathbb{R}^{n\times n}$. If A is a P-matrix, then there exists positive diagonal matrices S and T such that $S^{-1}AT$ is a product of positive stable P-matrices.*

Proof. By Theorem 1.4.5, $I+U(A)$ and $I-U(A)$ are P-matrices. Thus by Proposition 5.2.4, $(I+U(A))S$ and $(I-U(A))T$ are positive stable matrices. Thus,

$$\begin{aligned} S^{-1}AT &= S^{-1}(I+U(A))^{-1}(I-U(A))T \\ &= [(I+U(A))S]^{-1}(I-U(A))T. \end{aligned}$$

This completes the proof. □

5.3 SPECTRAL THEORY OF P-OPERATORS

We start this section by re-calling definition of different spectrum of operators.

Definition 5.3.1. Limaye (1996) *Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. A Scalar k is called an eigenvalue of T if there is a non-zero $x \in \mathscr{H}$ such that $T(x) = kx$. In this case x is called an eigenvector of T corresponding to k.*

The Set of all eigenvalue of T constitute the eigenspectrum of T. We denote eigenspectrum of T by $\sigma_e(T)$. Note that if x is an eigenvector corresponding to an eigenvalue k of T, then same hold for the unit vector $\frac{x}{||x||}$. Hence

$$\sigma_e(T) = \{k \in \mathbb{K} : T(x) = kx \text{ for some } x \in \mathscr{H} \text{ with } ||x|| = 1\}.$$

Note that a scalar k is an eigenvalue of T if and only if the operator $T - kI$ is not injective. In that case the closed subspace $Z(T - kI)$-the null space of $T - kI$ - of $\mathscr{H}$ is non-zero. It is called the eigenspace of T corresponding to the eigenvalue k.

A scalar k is called an approximate eigenvalue of T if the operator $T - kI$ is not bounded below, that is, for every $\beta > 0$, there is some $x \in \mathscr{H}$ with $||x|| = 1$ and $||T(x) - kx|| < \beta$.The set of all approximate eigenvalues of T constitutes the approximate eigenspectrum of T, we denote it by $\sigma_a(T)$. It is then clear that

$$\sigma_a(T) = \{k \in \mathbb{K} : T(x_n) - kx_n \longrightarrow 0 \text{ for some } (x_n) \in \mathscr{H} \text{ with } ||x_n|| = 1 \ \forall n\}$$

A scalar k is called spectral value of T if the bounded operator $T - kI$ is not invertible in $\mathscr{B}(\mathscr{H})$. The set of all spectral values of T constitute the spectrum of T. We denote it by $\sigma(T)$.

$$\sigma(T) = \{k \in \mathbb{K} : T - kI \text{ is either not injective or not surjective } \}.$$

Theorem 5.3.2. Kowalski (2009) *(Spectral theorem for compact operators). Let $\mathscr{H}$ be an infinite dimen- sional Hilbert space, and let $T \in K(\mathscr{H})$ be a compact operator.*

1. *Except for the possible value 0, the spectrum of T is entirely point spectrum; in other words $\sigma(T) - \{0\} = \sigma_p(T) - \{0\}$.*

2. *We have $0 \in \sigma(T)$, and $0 \in \sigma_p(T)$ if and only if T is not injective.*

3. *The point spectrum outside of 0 is countable and has finite multiplicity: for each $\lambda \in \sigma_p(T) - \{0\}$, we have $dim(\lambda I - T) < +\infty$.*

4. *Assume T is normal. Let* $\mathscr{H}_0 = ker(T)$, *and* $\mathscr{H}_1 = ker(T)^{\perp}$. *Then T maps* $\mathscr{H}_0$ *to* $\mathscr{H}_0$ *and* $\mathscr{H}_1$ *to* $\mathscr{H}_1$; *on* $\mathscr{H}_1$, *which is separable, there exists an orthonormal basis* $(e_1, \ldots, e_n, \ldots)$ *and* $\lambda_n \in \sigma_p(T) - \{0\}$ *such that*

$$lim_{n \to 0} \lambda_n = \{0\}$$

and $T(e_n) = \lambda_n e_n$ *for all* $n > 1$.

in particular, if $(f_i)_{i \in I}$ *is an arbitrary orthonormal basis of* $\mathscr{H}_0$, *which may not be separable, we have*

$$T(\sum_{i \in I} \alpha_i f_i + \sum_{n \geq 1} \alpha_n e_n) = \sum_{n \geq 1} \lambda_n \alpha_n e_n$$

for all scalars α_n, $\alpha_i \in \mathbb{C}$ *for which the vector on the left-hand side lies in* $\mathscr{H}$, *and the series on the right converges in* $\mathscr{H}$. *This can be expressed also as*

$$T(v) = \sum_{n \geq 1} \lambda_n \langle v, e_n \rangle e_n. \tag{5.3.1}$$

Proposition 5.3.3. *Let* $T \in \mathscr{B}(\mathscr{H})$ *be a P-operator relative to an orthonormal basis* $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. *Then any real eigen value of T is positive.*

Proof. Let $\lambda \in \sigma_e(T)$, then there exists $x \neq 0$ such that $T(x) = \lambda x$. As $T \in \mathscr{B}(\mathscr{H})$ be a P-operator relative to the orthonormal basis $\mathscr{B}$, there exist some index j for which,

$$\begin{aligned} & \langle x, e_j \rangle \langle Tx, e_j \rangle > 0, \\ \Longrightarrow \quad & \langle x, e_j \rangle \langle \lambda x, e_j \rangle > 0, \\ \Longrightarrow \quad & \langle x, e_j \rangle \lambda \langle x, e_j \rangle > 0, \\ \Longrightarrow \quad & \lambda \langle x, e_j \rangle^2 > 0, \\ \Longrightarrow \quad & \lambda > 0. \end{aligned}$$

□

Proposition 5.3.4. *Let* $T \in \mathscr{B}(\mathscr{H})$ *be a P-operator relative to an orthonormal basis* $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. *Then any real element in* $\sigma_a(T)$ *is positive.*

Proof. Let $\lambda \in \sigma_a(T)$ be real, then there exists a sequence $\{x_n\} \in \mathscr{H}$ with $||x_n|| = 1$ and $Tx_n - \lambda x_n \longrightarrow 0$ as $n \longrightarrow \infty$, that is $(T - \lambda I)x_n \longrightarrow 0$ as $n \longrightarrow \infty$, This implies that $Tx_n \longrightarrow \lambda x$ as $n \longrightarrow \infty$, where $x_n \longrightarrow x$ as $n \longrightarrow \infty$. As T is a P-operator relative to the orthonormal basis $\mathscr{B}$ and $||x_n|| = 1$ there exist j_n such that $\langle x_n, e_{j_n} \rangle \langle Tx_n, e_{j_n} \rangle > 0$, that is $\langle x, e_{j_n} \rangle \lambda \langle x, e_{j_n} \rangle > 0$ an $n \longrightarrow \infty$. Thus $\lambda \langle x, e_{j_n} \rangle^2 > 0$, this implies $\lambda > 0$. □

Remark 5.3.5. *In general for any $T \in \mathscr{B}(\mathscr{H})$ we have $\sigma_e(T) \subseteq \sigma_a(T)$. Thus in fact inspite of Proposition 5.3.4 the Proposition 5.3.3 is redundunt.*

Theorem 5.3.6. *Let $T \in \mathscr{B}(\mathscr{H})$ be a normal operator and let $\mathscr{H}_0 = ker(T)$ and $\mathscr{H}_1 = ker(T)^{\perp}$. If T', the restriction of T to $\mathscr{H}_1$, is a P-operator relative to an orthonormal basis $\mathscr{B}' = \{e_i\}_{i=1}^{\infty}$, where $\mathscr{B}'$ made of eigenvectors of T with $T(e_i) = \lambda_i e_i$, Then there exists a positive operator $R \in K(\mathscr{H})$ such that $R^2 = T$, which is denoted $\sqrt{T'}$ or $(T')^{1/2}$. It is unique among positive bounded operators.*

Proof. We have $T(e_i) = \lambda_i e_i$, where $\lambda_i \in \sigma_e(T) - \{0\}$. As T' is a P-operator relative to $\mathscr{B}'$, we have

$$\begin{aligned} \lambda_i &= \lambda_i \langle e_i, e_i \rangle, \\ &= \langle \lambda_i e_i, e_i \rangle, \\ &= \langle T e_i, e_i \rangle, \\ &= \langle e_i, e_i \rangle \langle T e_i, e_i \rangle > 0 \ \forall i, \end{aligned}$$

Now we define,

$$R(u_0 + u_1) = \sum_{i \geq 1} \sqrt{\lambda_i} \alpha_i e_i.$$

For any $u_0 \in \mathscr{H}_0$ and $u_1 = \sum_{i \geq 1} \alpha_i e_i \in \mathscr{H}_1$. Then we can see that R is a diagonal operator with coefficients $\sqrt{\lambda_i}$. Now by Theorem 5.3.2 we have $\sqrt{\lambda_i} \longrightarrow 0$ as $i \longrightarrow \infty$. This shows that R is well defined compact operator. Thus for every $u \in \mathscr{H}_1$, we have

$$\begin{aligned} R^2(u) &= R(\sum_{i \geq 1} \sqrt{\lambda_i} \alpha_i e_i), \\ &= \sum_{i \geq 1} \lambda_i \alpha_i e_i, \\ &= T'(u) \ \ \forall u \in \mathscr{H}_1. \end{aligned}$$

Therefor $R^2 = T'$. We next show the uniqueness. Let $R \in B(\mathscr{H})$ be such that $R^2 = T'$, then $RT' = RR^2 = R^2 R = T'R$, that is R and T' commute. It follows that any non-zero eigenvalue λ_i of T', R induces operator

$$R_i : ker(T' - \lambda_i I) \longrightarrow ker(T' - \lambda_i I)$$

given by $R_i = \sqrt{\lambda_i} I$ on the finite-dimensional λ_i - eigenspace of T', we can see that this R_i are P-operators relative to the orthonormal basis of $ker(T' - \lambda_i I)$ and it satisfies

$R_i^2 = \lambda_i iI$. Also note that,

$$\begin{aligned}
||R(u)||^2 &= \langle R(u), R(u)\rangle, \\
&= \langle R^2(u), u\rangle, \\
&= \langle T'(u), u\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
ker(R) &= \{u \in \mathscr{H} : R(u) = 0\}, \\
&= \{u \in \mathscr{H} : ||R(u)||^2 = 0\}, \\
&= \{u \in \mathscr{H} : \langle T'(u), u\rangle = 0\}.
\end{aligned}$$

Now by the expression 5.3.1 of the Theorem 5.3.2, we have

$$T'(u) = \sum_{i\geq 1} \lambda_i \langle u, e_i\rangle e_i.$$

Thus

$$\begin{aligned}
\langle T'(u), u\rangle &= \langle \sum_{i\geq 1} \lambda_i \langle u, e_i\rangle e_i, u\rangle \\
&= \sum_{i\geq 1} \lambda_i |\langle u, e_i\rangle|^2.
\end{aligned}$$

In terms of orthonormal basis of eigenvectors $\{e_i\}_{i=1}^{\infty}$ and the positivity $\lambda_i > 0$ of eigenvalues we have $\langle T'(u), u\rangle = 0$ if and only if u is perpendicular to span of $\{e_i\}_{i=1}^{\infty}$, that is by the construction $u \in ker(T')$. Thus the P-operator R is uniquely determined on each eigenspace of T' and $ker(T')$. By the Spectral theorem, this implies that R is unique. □

Next we define the concept of kth elementary symmetric function in infinite dimensional cases as follows:

Definition 5.3.7. *Let $\sigma_k(\lambda_1, \lambda_2, \lambda_3, \ldots)$ denote the kth symmetric function of the numbers $\lambda_1, \lambda_2, \lambda_3 \ldots$, given by*

$$\sigma_k(\lambda_1, \lambda_2, \lambda_3, \ldots) = \sum_{1\leq i_1 < i_2 < \ldots < i_k} \lambda_{i_1} \cdot \lambda_{i_2} \cdot \ldots \cdot \lambda_{i_k}.$$

Theorem 5.3.8. *The set $\{\lambda_1, \lambda_2, \lambda_3, \ldots,\}$ is an eigen spectrum of some P-operator $T \in \mathscr{B}(\mathscr{H})$ relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ if and only if*

$$\sigma_k(\lambda_1, \lambda_2, \lambda_3, \dots) > 0, \; k = 1, 2, 3, \dots$$

Proof. First Suppose that the set $\{\lambda_1, \lambda_2, \lambda_3, \dots, \}$ is an eigen spectrum of some P-operator relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. As $\lambda_i \in \sigma_e(T)$, we have, for each i, $\lambda_i > 0$ by Proposition 5.3.3. As $\sigma_k(\lambda_1, \lambda_2, \lambda_3, \dots)$ is the result of sums of the products of λ_i, we have $\sigma_k(\lambda_1, \lambda_2, \lambda_3, \dots) > 0$ for each k.

Conversely suppose that the set $\{\lambda_1, \lambda_2, \lambda_3, \dots, \}$ satisfies $\sigma_k(\lambda_1, \lambda_2, \lambda_3, \dots) > 0$ for each k. Let us define the operator $T \in \mathscr{B}(\mathscr{H})$ by

$$T(x) = \sum_{i=1}^{\infty} \alpha \sigma_i \langle x, e_i \rangle e_i,$$

where $\alpha > 0$ so small, so that T is bounded and σ_i denotes $\sigma_i = \sigma_i(\lambda_1, \lambda_2, \lambda_3, \dots)$. Then T is the required P-operator relative to the orthonormal basis $\mathscr{B}$. $\square$

Theorem 5.3.9. *Let $T \in \mathscr{B}(\mathscr{H})$ be a P-operator relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. Then T satisfies*

$$\inf_{x \in \mathscr{H}^+} \sup_{e_i \in \mathscr{B}} \frac{\langle Tx, e_i \rangle}{\langle x, e_i \rangle} \geq 0$$

where $\mathscr{H}^+ = \{x \in \mathscr{H} : \langle x, e_i \rangle \neq 0 \, for \, all \, i\}$ moreover if T satisfies $Tu = \rho(T)u$ for some $u \in \mathscr{H}^+$, then

$$\inf_{x \in \mathscr{H}^+} \sup_{e_i \in \mathscr{B}} \frac{\langle Tx, e_i \rangle}{\langle x, e_i \rangle} = \sup_{e_i \in \mathscr{B}} \inf_{x \in \mathscr{H}^+} \frac{\langle Tx, e_i \rangle}{\langle x, e_i \rangle} = \rho(T)$$

Proof. As T is a P-operator relative to an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ there exist an index j such that $\langle x, e_j \rangle \langle Tx, e_j \rangle > 0$, thus $\frac{\langle x, e_j \rangle \langle Tx, e_j \rangle}{\langle x, e_j \rangle^2} > 0$, that is $\frac{\langle Tx, e_j \rangle}{\langle x, e_j \rangle} > 0$, so for any $x \in \mathscr{H}^+$, $\sup_{e_i \in \mathscr{H}} \frac{\langle Tx, e_j \rangle}{\langle x, e_j \rangle} > 0$. Thus if we take infimum of these positive numbers we get result again non-negative, so this implies that $\inf_{x \in \mathscr{H}^+} \sup_{e_i \in \mathscr{B}} \frac{\langle Tx, e_i \rangle}{\langle x, e_i \rangle} \geq 0$. Now

$$\inf_{x \in \mathscr{H}^+} \sup_{e_i \in \mathscr{B}} \frac{\langle Tx, e_i \rangle}{\langle x, e_i \rangle} \leq \sup_{e_i \in \mathscr{B}} \frac{\langle Tu, e_i \rangle}{\langle u, e_i \rangle} = \sup_{e_i \in \mathscr{B}} \frac{\rho(T) \langle u, e_i \rangle}{\langle u, e_i \rangle} = \rho(T) \tag{5.3.2}$$

also we have,

$$\sup_{e_i \in \mathscr{B}} \inf_{x \in \mathscr{H}^+} \frac{\langle Tx, e_i \rangle}{\langle x, e_i \rangle} \geq \frac{\langle Tu, e_i \rangle}{\langle u, e_i \rangle} = \frac{\rho(T) \langle u, e_i \rangle}{\langle u, e_i \rangle} = \rho(T). \tag{5.3.3}$$

Now by the well known result that for a function $\psi : X \times Y \longrightarrow X \times Y$, where X and

Y are any two sets, we have

$$\inf_{x\in X}\sup_{y\in Y}\psi \geq \sup_{y\in Y}\inf_{x\in X}\psi$$

and From equation (5.3.2) and (5.3.3) we get

$$\rho(T) \leq \sup_{e_i\in\mathscr{B}}\inf_{x\in\mathscr{H}^+}\frac{\langle Tx, e_i\rangle}{\langle x, e_i\rangle} \leq \inf_{x\in\mathscr{H}^+}\sup_{e_i\in\mathscr{B}}\frac{\langle Tx, e_i\rangle}{\langle x, e_i\rangle} \leq \rho(T) \tag{5.3.4}$$

thus we get

$$\inf_{x\in\mathscr{H}^+}\sup_{e_i\in\mathscr{B}}\frac{\langle Tx, e_i\rangle}{\langle x, e_i\rangle} = \sup_{e_i\in\mathscr{B}}\inf_{x\in\mathscr{H}^+}\frac{\langle Tx, e_i\rangle}{\langle x, e_i\rangle} = \rho(T).$$

□

Theorem 5.3.10. *Let $\mathscr{H}$ be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$, $S, T \in \mathscr{B}(\mathscr{H})$ and D be a diagonal operator such that $De_i = d_{ii}e_i$, where $0 \leq d_{ii} \leq 1$. then the following are true*

1. *If ST^{-1} is a P-operator relative to the orthonormal basis $\mathscr{B}$ then $0 \notin \sigma_e(DT + (I-D)S)$.*
2. *If $S^{-1}T$ is a P-operator relative to the orthonormal basis $\mathscr{B}$ then $0 \notin \sigma_e(TD + S(I-D))$.*

Proof.

1. Let ST^{-1} be a P-operator relative to $\mathscr{B}$. Suppose that $0 \in \sigma_e(DT + (I-D)S)$. Then there exists some $0 \neq x \in \mathscr{H}$ such that $0 = (DT + (I-D)S)x = (D + (I-D)ST^{-1})Tx$. Set $y = Tx$. Then $y \neq 0$, since T is invertible. Now, $Dy + (I-D)ST^{-1}y = 0$ implies $\langle De_i, e_i\rangle\langle y, e_i\rangle = -\langle (I-D)e_i, e_i\rangle\langle ST^{-1}y, e_i\rangle$. If $\langle y, e_i\rangle \geq 0$, then $\langle (ST^{-1}y), e_i\rangle \leq 0$ so that $\langle y, e_i\rangle\langle (ST^{-1}y), e_i\rangle \leq 0$. If $\langle y, e_i\rangle \leq 0$, then $\langle (ST^{-1}y), e_i\rangle \geq 0$ so that $\langle y, e_i\rangle\langle (ST^{-1}y), e_i\rangle \leq 0$. That is, $\langle y, e_i\rangle\langle (ST^{-1}y), e_i\rangle \leq 0$ for all i, a contradiction to ST^{-1} being a P-operator relative to $\mathscr{B}$. Thus $0 \notin \sigma_e(DT + (I-D)S)$.

2. Let $S^{-1}T$ be a P-operator relative to $\mathscr{B}$. Suppose that $0 \in \sigma_e(TD + S(I-D))$. Then there exists some $0 \neq x \in \mathscr{H}$ such that $0 = (TD + S(I-D))x = S(S^{-1}TD + (I-D))x$. As S is invertible, we have $S^{-1}TDx + (I-D)x = 0$. Now, $\langle (I-D)e_i, e_i\rangle\langle x, e_i\rangle = -\langle De_i, e_i\rangle\langle S^{-1}Tx, e_i\rangle$. If $\langle x, e_i\rangle \geq 0$, then $\langle (S^{-1}Tx), e_i\rangle \leq 0$ so that $\langle x, e_i\rangle\langle (S^{-1}Tx), e_i\rangle \leq 0$. If $\langle x, e_i\rangle \leq 0$, then we get $\langle (S^{-1}Tx), e_i\rangle \geq 0$ so that $\langle x, e_i\rangle\langle (S^{-1}Tx), e_i\rangle \leq 0$. That is, $\langle x, e_i\rangle\langle (S^{-1}Tx), e_i\rangle \leq 0$ for all i, a contradiction to $S^{-1}T$ being a P-operator relative to $\mathscr{B}$. Thus $0 \notin \sigma_e(TD + S(I-D))$.

□

5.4 SPECTRAL THEORY OF SUFFICIENT OPERATORS

In this section we discuss some spectral results for sufficient operators defined in Chapter 4. Let us begin with the following definition.

Definition 5.4.1. *Xu (1993b) Let M be an $n \times n (n > 2)$ real matrix. The term λ-eigenvector denotes an eigenvector u corresponding to the eigenvalue λ of M, i.e., $Mu = \lambda u$. If such an eigenvector $u = (u_1, \ldots, u_n)$ has $u_i \neq 0$ for all $i = 1, \ldots, n$, then u is called a strictly non-zero eigenvector. On the other hand, if $u_i = 0$ for at least one i, then u is called a partly zero eigenvector*

We define this concept of partly zero eigenvector in Hilbert space settings as follows:

Definition 5.4.2. *Let $\mathscr{H}$ be separable Hilbert space with an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. Let $T \in \mathscr{B}(\mathscr{H})$. The term λ-eigenvector denotes an eigenvector u corresponding to the eigenvalue λ of T, i.e., $Tu = \lambda u$. If such an eigenvector u is such that $\langle u, e_i \rangle \neq 0$ for all i, then u is called a strictly non-zero eigenvector. On the other hand, if $\langle u, e_i \rangle = 0$ for at least one i, then u is called a partly zero eigenvector.*

Remark 5.4.3. *Let $\mathscr{H}$ be separable Hilbert space with an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$ and $T \in \mathscr{B}(\mathscr{H})$. Let us denote T_α for $\alpha \subseteq \mathbb{N}$, is the restriction of T to $H' = span\{e_i : i \in \alpha\}$. Any vector in H' we denote by x_α.*

Theorem 5.4.4. *Let $\mathscr{H}$ be a separable Hilbert space with an orthonormal basis $\mathscr{B} = \{e_i\}_{i=1}^{\infty}$. Let $T \in \mathscr{B}(\mathscr{H})$, then the following are equivalent.*

1. *T is C-sufficient operator relative to an orthonormal basis $\mathscr{B}$.*
2. *For any index set $\alpha \subseteq \mathbb{N}$ and non-negative diagonal operator $D_\alpha \neq 0$, if $T_\alpha + D_\alpha$ is not invertible then every zero eigenvector of $T_\alpha + D_\alpha$ is partly zero eigenvector.*

Proof. (1) $\Rightarrow$ (2): Suppose there exists an index set $\alpha \subseteq \mathbb{N}$, a non-negative diagonal operator $D_\alpha \neq 0$, and a strictly non-zero 0-eigenvector x_α, such that $(T_\alpha + D_\alpha)x_\alpha = 0$. Define a vector y such that $y_\alpha = x_\alpha$, and $y_{\overline{\alpha}} = 0$. Then $\langle y, e_i \rangle \langle Ty, e_i \rangle \leq 0$ for all i. It follows from the hypothesis that $\langle y, e_i \rangle \langle Ty, e_i \rangle = 0$ for all i. Since x_α, is a strictly non-zero vector, we have $-Dx_\alpha = T_\alpha x_\alpha = 0$; hence $D_\alpha = 0$, which is a contradiction. Thus the zero eigenvector of $T_\alpha + D_\alpha$ is partly zero eigenvector.

(2) $\Rightarrow$ (1): Suppose $\langle y, e_i \rangle \langle Ty, e_i \rangle \leq 0$ for any i. Let $\alpha = \{i : \langle x, e_i \rangle \neq 0\}$. If $T_\alpha x_\alpha \neq 0$, then there exists a non-negative diagonal matrix $D \neq 0$ such that $T_a \alpha = -Dx$. It follows that there exists a strictly non-zero vector x, satisfying $(T_\alpha + D)x_\alpha = 0$, which is a

contradiction. Hence $T_\alpha x_\alpha = 0$; therefore, $\langle y, e_i\rangle\langle Ty, e_i\rangle \leq 0$ for any i. This terminates the proof of the theorem. □

Theorem 5.4.5. *Let $\mathscr{H}$ be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ and T be a C-sufficient operator relative to $\mathscr{B}$. Then any eigenvector corresponding to a non-zero eigenvalue is not an element of $rev_{\mathscr{B}}(T)$.*

Proof. Let $\lambda \neq 0 \in \sigma_e(T)$. Then there exist $x \neq 0 \in \mathscr{H}$ such that $Tx = \lambda x$. If $x \in rev_{\mathscr{B}}(T)$, then $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all i. As T is C-sufficient operator we have $\langle x, e_i\rangle\langle Tx, e_i\rangle = 0$ for all i, that is $\langle x, e_i\rangle\langle \lambda x, e_i\rangle = 0$ for all i, Thus $\lambda \langle x, e_i\rangle^2 = 0$ for all i, this implies that $\langle x, e_i\rangle = 0$ for all i, thus $x = 0$. This is a contradiction. Hence $x \notin rev_{\mathscr{B}}(T)$. □

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 CONCLUSION

The sign-reversing property of matrices is one of the important properties of the matrices which has applications in linear complementarity problems, game theory, and many more. The sign-reversing property of matrices can be used to discuss certain positivity classes of matrices such as P-matrices, row sufficient matrices, column sufficient matrices, sufficient matrices, row adequate matrices, column adequate matrices, totally positive /negative matrices, N-matrices and many more.

The sign-reversing set of a matrix is denoted by rev(A) and it is defined as $\{x \in \mathbb{R}^n : x_i(Ax)_i \leq 0, \ \forall i\}$. It is observed that rev(A) is a non-convex cone in general. In the majority of mathematical optimization problems that require convex cone structure. In chapter two we mainly discussed the following points:

1. We have obtained a characterization of the sign-reversing set of a matrix A in terms of the null spaces of the operators $DA - A - D$, where D ranges over all diagonal matrices such that $0 \leq D \leq I$.

2. We have given a sub-class of matrices having convex sign-reversing set and we observed that the collection contains some of the notable classes of matrices such as P-matrices, adequate matrices and sufficient matrices.

3. We explored more on the sign-reversing set of some of the special classes of matrices such as P-matrices and sufficient matrices.

4. Some of the geometrical aspects of the sign-reversing set of matrices have been discussed.

The class of P-matrices has more significance in linear complementarity problems, such as discussing the existence and uniqueness of the solution of linear complementarity problems. The concept of P-matrices has extended as P-operators on Banach spaces having a Schauder basis by Kannan and Sivakumar (2016). In the third chapter we mainly discussed the following points:

1. Inspired by the work of Kannan and Sivakumar (2016), we defined P-operators on Hilbert spaces relative to an orthonormal basis with the help of inner product structure of the Hilbert space.

2. We observed that an operator can be P-operator relative to several orthonormal bases under certain conditions.

3. We explored fundamental characteristics of P-operators in Hilbert spaces, including their summation, composition, and certain derived P-properties.

In the fourth chapter, we have generalized the concept of the sign-reversing set for general operators in infinite dimensional separable real Hilbert spaces. Precisely,

1. We defined the sign-reversing set of an operator relative to an orthonormal basis in a Hilbert space, and we observed how different orthonormal bases influence the sign-reversing set of an operator.

2. We have obtained a similar characterization for the sign-reversing set, as that of matrices, of an operator relative to an orthonormal basis in terms of the null spaces of the operators $DA - A - D$, where D ranges over all diagonal operators such that $0 \leq D \leq I$.

3. We have generalized the concept of column sufficient matrices and row sufficient matrices in separable Hilbert space with the names C-sufficient operator relative to an orthonormal basis and R-sufficient operator relative to an orthonormal basis and also we discussed some results for these novel operators.

In chapter five, our focus delved into exploring various spectral properties exhibited by P-operators within Hilbert spaces, alongside we examined the spectral characteristics of sufficient operators within the same context. This investigation likely involved dissecting the behaviors and traits of these operators concerning their spectral decomposition, eigenvalues, and eigenvectors, among other pertinent properties. By delving into these spectral properties, we likely gained a deeper understanding of the underlying structure and behavior of these operators within the realm of Hilbert spaces.

6.2 FUTURE WORK

Chapter 2 unveiled that the sign-reversing set of a matrix may lack convexity. Although we have identified a subclass of matrices boasting a convex sign-reversing set, it doesn't encompass all matrices with such properties. Our forthcoming focus is on delineating further specialized subclasses of matrices/operators leveraging the sign-reversing property. We aim to ascertain their impact on the convexity of the sign-reversing set, and ultimately, to characterize comprehensively all matrices exhibiting this convex property. Additionally, within this chapter, we've initiated an exploration into the geometrical facets of matrix sign-reversing sets—a realm we intend to delve deeper into.

In chapter three we defined P-operators relative to an orthonormal basis on Hilbert spaces and we obtained some fundamental properties of such operators. one of the future works is to explore further P-operators in Hibert spaces.

Chapter four opens the following future works:

1. We would like further exploration into the properties and behaviors of sign-reversing sets of operators in different contexts.

2. Investigating specific classes of operators where the sign-reversing set may have unique characteristics such as convexity and compactness.

3. We also would like to explore how different orthonormal bases influence these sets. Whether there exists a collection of orthonormal bases so that an operator has the same sign-reversing set relative to any orthonormal basis from this collection.

4. We also like to explore more special classes of operators such as C-sufficient operators and R-sufficient operators with the help of sign-reversing set of operators.

As we know that, one of the notable properties of the spectral radius of a positive operator is its equivalence to its operator norm. This significant attribute emerges as a direct consequence of the spectral theorem for bounded self-adjoint operators on Hilbert spaces. However, there's an intriguing avenue for exploration in extending such spectral radius results to newly defined operators, such as P-operators on Hilbert spaces and the diverse sufficient operators introduced in Chapter Four. These operators, which may possess distinct properties and operating principles compared to traditional self-adjoint operators, offer a rich terrain for investigation. By probing the spectral radius properties of these novel operators, we aim to unveil unique insights into their behavior, spectral decomposition, and norm characteristics, thereby broadening our understanding of these operators within the context of Hilbert spaces.

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3. Rashid A. and P. Sam Johnson, *Sign-reversing Set of Operators on Hilbert Spaces* (Communicated).

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