A STUDY ON ACYCLIC EDGE COLORING AND DOMINATION NUMBER OF A GRAPH

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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Dedicated To

This thesis is dedicated to my family which is always an inspiration to me.

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **A STUDY ON ACYCLIC EDGE COLORING AND DOMINATION NUMBER OF A GRAPH** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to *certify* that the Research Thesis entitled **A STUDY ON ACYCLIC EDGE COLORING AND DOMINATION NUMBER OF A GRAPH** submitted by **Mr. SHASHANKA KULAMARVA**, (Register Number: 187023MA007) as the record of the research work carried out by him is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

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Abstract

Let $G = (V,E)$ be a graph with n vertices and let C be a given set of colors. A proper edge coloring of the graph G with the colors from the set C is a function $f : E \to C$ such that $f(e_1) \neq f(e_2)$ for any adjacent edges e_1 and e_2.

An acyclic edge coloring of a graph is a proper edge coloring without any bichromatic cycles. The acyclic chromatic index of a graph G denoted by $a'(G)$, is the minimum positive integer k such that G has an acyclic edge coloring with k colors. It was conjectured by Fiamčík (1978) (and independently by Alon, Sudakov and Zaks (2001)) that $a'(G) \leq \Delta + 2$ for any graph G with maximum degree Δ.

Linear arboricity of a graph G, denoted by $la(G)$, is the minimum number of linear forests into which the edges of G can be partitioned. It was conjectured by Akiyama, Exoo and Harary (1980) that for any graph G, $la(G) \leq \lceil \frac{\Delta+1}{2} \rceil$.

A graph is said to be chordless if no cycle in the graph contains a chord. By a result of Basavaraju and Chandran (2010), if G is chordless, then $a'(G) \leq \Delta + 1$. Machado, de Figueiredo and Trotignon (2013) proved that the chromatic index of a chordless graph is Δ when $\Delta \geq 3$. In this thesis, it is proved that for any chordless graph G, $a'(G) = \Delta$, when $\Delta \geq 3$. One can see that this is an improvement over the result of Machado et al. (2013), since any acyclic edge coloring is also a proper edge coloring and we are using the same number of colors. The thesis also provides the sketch of a polynomial-time algorithm that takes a chordless graph G as an input and returns an optimal acyclic edge coloring of G as the output.

As a byproduct, one can prove that $la(G) = \lceil \frac{\Delta}{2} \rceil$, when $\Delta \geq 3$. To obtain the result on the acyclic chromatic index of a chordless graph, an extremal structure in chordless graphs has been proved which is a refinement of the structure given by Machado et al. (2013) in the case of chromatic index. This extremal structure might be of independent interest.

A graph G is said to be k-degenerate if every subgraph of G has a vertex of degree at most k. Basavaraju and Chandran (2010) proved that the acyclic edge coloring conjecture is true for 2-degenerate graphs. In the thesis, it is proved that for a 3-degenerate graph G, $a'(G) \leq \Delta + 5$, thereby bringing the upper bound closer to the conjectured bound.

The thesis also considers the class of k-degenerate graphs with $k \geq 4$ and improves

the existing upper bound given by Fiedorowicz (2011) for the acyclic chromatic index of a k-degenerate graph.

A set of vertices D in graph G is said to be a dominating set if every vertex which is not in D is adjacent to a vertex in D. The size of a minimum dominating set of G is said to be the domination number of G and is denoted by $\gamma(G)$.

A classical upper bound for the domination number of a graph G having no isolated vertices is $\lfloor \frac{n}{2} \rfloor$. However, for several families of graphs, we have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$ which gives a substantially improved upper bound. By the multiplicative version of the Nordhaus-Gaddum type result for the domination number of a graph, for any graph G with n vertices and $\overline{G}$ being the complement of G, we have $\gamma(G)\gamma(\overline{G}) \leq n$. This result will imply that for every graph G, either $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$ or $\gamma(\overline{G}) \leq \lfloor \sqrt{n} \rfloor$ or both.

The thesis presents some conditions necessary for a graph G to have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$, and some conditions sufficient for a graph G to have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$. A characterization of all connected graphs G with $\gamma(G) = \lfloor \sqrt{n} \rfloor$ is also provided. Further, the thesis also provides proof for the statement that if the condition:

$$rad(G) = diam(G) = rad(\overline{G}) = diam(\overline{G}) = 2$$

is not satisfied for a graph G, then the task of deciding whether it is $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$ or $\gamma(\overline{G}) \leq \lfloor \sqrt{n} \rfloor$ can be done in polynomial time. The thesis concludes with a conjecture that a slightly weaker decision problem can be solved in polynomial time for any arbitrary graph.

Along with the above mentioned results, the thesis also introduces a new terminology of t-complementary self-centered graphs and a new concept of freeable colors, thereby contributing to the literature of acyclic edge coloring and the domination in graphs. This concept of freeable colors turns out to be a very useful tool for proving the upper bounds on the acyclic chromatic index.

Keywords: *Acyclic chromatic index; Acyclic edge coloring; Chordless graphs; Linear arboricity; Minimally 2-connected graphs;* 3*-degenerate graphs; k-degenerate graphs; Domination in graphs; Domination number; Private neighbor*

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List of Symbols

$G = (V, E)$	:	Graph G with vertex set V and edge set E
$\overline{G}$	:	Complement of the graph G
K_n	:	Complete graph on n vertices
C_n	:	Cycle on n vertices
P_n	:	Path on n vertices
$K_{m,n}$	:	Complete bipartite graph with m and n vertices in two partitions
$\Delta(G)$	:	Maximum vertex degree in graph G
$\delta(G)$	:	Minimum vertex degree in graph G
$\chi'(G)$	:	Chromatic Index of G
$a'(G)$	:	Acyclic Chromatic Index of G
$G \setminus X$	:	Graph obtained from G by deleting the set X
$la(G)$	:	Linear Arboricity of G
$deg_G(x)$	:	Degree of vertex x in G
$N_G(x)$	:	Set of neighbors of the vertex x in G
$N_G^2(a,b)$	:	Set of degree 2 common neighbors of a and b in G
$\lvert S \rvert$	:	Size (number of elements) of the set S
$G[S]$	:	Subgraph of G induced by the vertex set S
$f(e)$	:	Color assigned to the edge e in the coloring f
$F_x(f)$	:	$\{f(xy) \mid y \in N_G(x)\}$, for a vertex x and coloring f
$F_{ab}(f)$	:	$F_b(f) \setminus \{f(ab)\}$, for an edge ab and a coloring f
I.H.	:	Induction Hypothesis

$\sqrt{n}$	:	Square root of n
$\lfloor n \rfloor$	:	Largest integer value less than or equal to n
$\lceil n \rceil$	:	Smallest integer value greater than or equal to n
$d(x,y)$	:	Length of a shortest path between the vertices x and y in G
$e_G(x)$	:	Eccentricity of the vertex x in G
$rad(G)$	:	Radius of G
$diam(G)$	:	Diameter of G
$\gamma(G)$	:	Domination Number of G
$\gamma_t(G)$	:	Total Domination Number of G
$\gamma_c(G)$	:	Connected Domination Number of G
$pn[v,S]$	:	Set of private neighbors of the vertex v w.r.t. $S \subseteq V$
$epn[v,S]$	:	Set of external private neighbors of the vertex v w.r.t. $S \subseteq V$

Chapter 1

Introduction

Mathematics is a discipline that every one of us requires in daily life. Without it, the world would be a dangerous place to live. Would the modern world of inventions be the same without mathematics? The answer is a big No! Mathematics has played a big part in shaping the world into what it is today with all its advancements. Mathematics is considered the foundation for all fields of science and technology. Any field becomes void without the existence of mathematics. In essence, the field of mathematics is a building block for the world of technology. Adding a mathematical model to any subject increases its significance by a tremendous margin. Therefore, a mathematician's job is always associated with high expectations from everyone.

The elegant tree of mathematics looks beautiful because of the existence of many branches in it. Every branch of mathematics has its significance when it comes to applications. Mathematical analysis has a variety of applications in physical and biological sciences. Many concepts in vector spaces and differential geometry are incomplete without abstract algebra. Linear algebra is useful in electrical circuits, calculation of speed, distance and time, medical diagnosis, etc. Differential equations are necessary to analyze the flow of electricity and understand the concepts of fluid dynamics and thermodynamics. Operations Research is the building block of the concepts of optimization and linear programming. Finally, the field of graph theory and combinatorics which mainly constitute theoretical computer science, is useful in solving several problems of the real-world related to networking and communication.

This thesis is completely about the study of a couple of concepts in the area of graph theory, namely the acyclic edge coloring and the domination number of the considered graph. These concepts are widely applied to several real-world problems. The entire discussion will revolve around these two concepts. Graph coloring is a specific area of graph theory that has gained significance among researchers over a while. People across the world are working on solving several graph coloring problems. To begin with, let us try to have a glance at the origin with an overview of the history of graph coloring; which roots down to a famous problem.

1.1 Four Color Theorem

The problem of graph coloring began with a question on the possibility of a particular type of coloring: Is it possible to color all the regions in a map with four colors so that any two regions that share a common boundary don't receive the same color? Initially, the question was proposed as a conjecture; but now, it is a theorem.

It all began when Guthrie (1854) came up with the four color conjecture. The two brothers, Francis Guthrie and Frederick Guthrie were jointly responsible for the raise of the conjecture. In the beginning, it was nothing other than a simple observation by Francis; but he was not aware of the possible impact of this observation on the field of graph theory in the future. Francis communicated the problem to his brother Frederick, who in turn discussed it with his teacher Augustus De Morgan. Later, De Morgan (1860) found this problem to be very interesting; therefore, he gave a formal structure to this problem and published it. The following is a graph-theoretical formulation of the four color theorem:

Theorem 1.1. *A graph is said to be planar if it can be drawn on a plane without any edges crossing one another. Let G be any such planar graph. The vertices of G vertices can be colored with a maximum of four colors, ensuring that any pair of neighboring vertices has a unique color.*

After many failed attempts and wrong proofs, Appel and Haken (1976) were the pioneers in successfully proving the four color theorem using a computer. Later, Robertson, Sanders, Seymour and Thomas (1997) came up with another proof, still using a computer, but simpler than the proof of Appel and Haken in several aspects. The theorem was also proved by Georges Gonthier in 2005 with the general-purpose theorem-proving software. Astonishingly, there is no mathematical proof (without using a computer) for the theorem even today which indicates the level of difficulty of the problem.

The field of graph coloring gained significance since the introduction of four color theorem and some other types of coloring questions and problems were raised in the literature gradually over a while. Significant research is being carried out all over the world on graph coloring problems. If the objects being colored are the vertices of a graph, then we get the vertex coloring; and if they are the edges of a graph, then we get the edge coloring. This thesis has an inclination towards the edge coloring of a graph.

1.2 Basics: Definitions and Notations

If an ordered pair of sets (V,E) is such that the set E is a collection of unordered 2-element subsets of the set V, then the pair is said to be a *graph*, denoted by $G = (V,E)$. *Vertices* of the graph are the elements of V. *Edges* of the graph are the elements of E. The convention is to represent the vertices as points and draw a line between a pair of points whenever there is an edge between the corresponding vertices. $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively.

Two vertices u and v in $V(G)$ are said to be *adjacent* or *neighbors* if $uv \in E(G)$, i.e., if there is an edge between the vertices u and v. Further, if $uv \in E(G)$, then the edge uv is said to be *incident* on both u and v. If a pair of edges e_1 and e_2 have a common end vertex, then they are said to be *incident* edges.

In the thesis, only simple, finite and undirected graphs are considered; hence, the multi-edges and loops do not arise in the discussions. The *order* of graph G is the size of set V (i.e., number of vertices in G). The *size* of graph G is the size of set E (i.e., number of edges in G). Unless otherwise specified, it is a usual practice to denote the size and the order of G to be m and n respectively, i.e., $|E(G)| = m$ and $|V(G)| = n$.

The *degree* of any vertex x in G represented as $deg_G(x)$ or simply $deg(x)$ (if G is understood from the context), is the number of edges in G which are incident on the vertex x. An *isolated vertex* in a graph is a vertex having its degree zero. The maximum among the numerical values of the degrees of the vertices in G is denoted by $\Delta(G)$ (or Δ when there is no ambiguity). Similarly, the minimum among the numerical values of the degrees of the vertices in G is denoted by $\delta(G)$ (or δ when there is no ambiguity). For any vertex $x \in V$, $N_G(x)$ is the set of all neighbors of x in G. Throughout the thesis, $N_G(x)$ is written as $N(x)$ for simplicity.

If $deg(x) = n-1$ for every vertex x in $V(G)$, i.e., if there is an edge between every pair of vertices in G, then graph G is said to be *complete*. A complete graph having t vertices is denoted as K_t. If $deg(x) = 0$ for every vertex x in $V(G)$, i.e., if there is no edge between any pair of vertices in G, then graph G is said to be *totally disconnected*.

If the degrees of all the vertices in a graph are equal to k, then the graph is said to be *k-regular*. If a graph is 3-regular, then it is also referred to as *cubic*. A graph is said to be *subcubic* if the degree of every vertex in the graph is at most 3.

Let $H = (V(H), E(H))$ be another graph. H is said to be a *subgraph* of G if the vertex set of H is a subset of the vertex set of G, i.e., $V(H) \subseteq V(G)$, and the edge set of H is a subset of the edge set of G, i.e., $E(H) \subseteq E(G)$.

Let R be a subset of the edge set E of G. Then $G \setminus R$ is the subgraph of G obtained by the vertex set V and the edge set $E \setminus R$. Let S be a subset of the vertex set V of G.

Then $G \setminus S$ is the subgraph of G obtained by the vertex set $V \setminus S$ and the edge set:

$$E \setminus \{e \in E : \exists v \in S \text{ such that } e \text{ is incident to } v\}$$

Note that if either R or S is a singleton set $\{x\}$ (a set with exactly one element is said to be a *singleton*), then writing $G \setminus x$ is preferred over $G \setminus \{x\}$, omitting the brackets.

The set S is considered to be a *clique* if there is an edge between every pair of its vertices. The set S is considered to be an *independent set* if there are no edges between any two of its vertices.

An *induced subgraph* of G with respect to S is that subgraph of G induced by the vertices in the set S and is denoted by the symbol $G[S]$, i.e.,

$$G[S] = (V', E') \qquad \text{where} \qquad V' = S \qquad \text{and} \qquad E' = \{ab \in E : a \in S, b \in S\}$$

In other words, $G[S]$ is a collection of all vertices in the set S together with the edges in G which have both of its end vertices in S.

An *induced subgraph* of G with respect to R is that subgraph of G induced by the edges in the set R and is denoted by the symbol $G[R]$, i.e.,

$$G[R] = (V'', E'') \qquad \text{where} \qquad V'' = \{v \in V : u \in V, uv \in R\} \qquad \text{and} \qquad E'' = R$$

In other words, $G[R]$ is a collection of all edges in the set R together with the union of their end vertices.

In a graph G, a *path* is a sequence of distinct vertices of G where the subsequent vertices in the sequence are adjacent. A path starting from a vertex x ending at a vertex y is denoted as an (x,y)-path. If we draw an edge between the starting vertex and the ending vertex of a path, the resulting structure is called a *cycle*.

The *length* of a path P is the number of edges in P, and is denoted by $|P|$. An (x,y)-path P in G is said to be the *shortest* if there does not exist another (x,y)-path Q in G with $|Q| < |P|$. The number of edges in a cycle C is said to be the *length* of the cycle and is denoted by $|C|$. A cycle C in G is said to be the *shortest* if there does not exist another cycle C' in G with $|C'| < |C|$. The length of a shortest cycle in a graph G is said to be the *girth* of G.

The *distance* between any two vertices x and y in G is the number of edges in the shortest (x,y)-path. The *eccentricity* of a vertex x in G, denoted by $e_G(x)$, is the maximum distance from x to any vertex in G. The *radius* of G, denoted by $rad(G)$, is the minimum among the vertex eccentricities of G. The *diameter* of G, denoted by $diam(G)$, is the maximum among the vertex eccentricities of G. A graph G is said to be *self-centered* if all the vertices in G have the same eccentricities, i.e., for a self-centered graph G, we have $rad(G) = diam(G)$.

A graph is said to be *connected* if there exists a path between any pair of vertices in the graph. A graph is said to be disconnected if it is not connected. A graph is said to be a *tree* if it is connected and acyclic (does not contain any cycles).

A vertex $x \in V$ is said to be a *cut vertex* in G if the graph $G \setminus x$ is disconnected. Graph G is said to be *k-connected* if the removal of less than k vertices from G does not result in a disconnected graph. An edge $e \in E$ is said to be a *cut edge* or a *bridge* in G if the graph $G \setminus e$ is disconnected. Graph G is said to be *k-edge-connected* if the removal of less than k edges from G does not result in a disconnected graph.

An odd cycle in a graph is a cycle with an odd number of edges in it. Graph G is said to be *bipartite* if the vertex set V can be partitioned into two sets X and Y such that every edge in G has one end in X and the other end in Y. It also implies that X and Y are independent sets. It is proved that a graph is bipartite if and only if it does not have an odd cycle in it (proof can be found in the book by West (2001)). A bipartite graph with a bipartition X and Y is said to be a *complete bipartite graph* if there is an edge between any vertex in X and any vertex in Y. If $|X| = a$ and $|Y| = b$, then the complete bipartite graph is denoted by $K_{a,b}$.

Let $e = xy$ be an edge in G. Then the process of deleting the vertices x and y from G and replacing it with a new vertex z such that the condition:

$$N(z) = (N(x) \setminus y) \cup (N(y) \setminus x)$$

holds is called the *edge contraction* corresponding to the edge e. If the operation of edge contraction results in a multi-edge, then only one edge among them is considered to keep the resultant graph simple.

A *matching* is a set of edges in a graph such that any pair of edges in the set don't share a common end vertex. We say that the matching M *saturates* a vertex v in G if there exists an edge e in M such that e is incident on the vertex v. A matching that saturates every vertex in the graph is said to be a *perfect matching*.

The graph with the following vertex set (edges defined subsequently) is the *cartesian product* of graphs G and H, represented by $G \square H$.

$$V(G \square H) = V(G) \times V(H) = \{(g,h) : g \in V(G) \text{ and } h \in V(H)\}$$

and there exists an edge between the vertices (g_1,h_1) and (g_2,h_2) if and only if either $(g_1,g_2) \in E(G)$ and $h_1 = h_2$, or $(h_1,h_2) \in E(H)$ and $g_1 = g_2$. The graphs which can be expressed as the cartesian product of d graphs each of which is an induced path or a cycle is said to be a *d-dimensional partial tori*. These graphs are also termed as *grid-like* graphs.

A graph is said to be a *series-parallel* graph if it has a source vertex s and a sink vertex t (together called terminal vertices) and can be turned into a complete graph K_2 by a sequence of the following operations:

- A pair of parallel edges is replaced with a single edge that is incident on their common endpoints.
- A pair of edges incident on a vertex of degree 2 other than s or t is replaced with a single edge by contracting one of those edges.

A graph that can be drawn on a plane such that no two edges cross each other is said to be a *planar* graph. A drawing of a planar graph on a plane such that no two edges cross each other is said to be a *planar embedding* of the graph. If a graph has a planar embedding with all the vertices belonging to the outer face of the embedding, then the graph is said to be *outerplanar* .

A *chord* in a cycle is an edge between a non-consecutive pair of vertices in the cycle of a graph. A graph is said to be *chordless* if it does not contain a chord.

Definition 1.2. *A graph G is said to be k-degenerate if every subgraph of G has a vertex of degree at most k. For a given k-degenerate graph G, k-degeneracy ordering is defined as an ordering σ of the vertices of G such that every vertex in σ will have at most k neighbors before it in the ordering σ. It is easy to see that any k-degenerate graph has a k-degeneracy ordering.*

A k-connected graph G is said to be *minimally k-connected* if $G \setminus e$ is no longer k-connected for any edge e in G. The following lemma is useful in various places in the subsequent chapters. A short proof is also included for the sake of completion.

Lemma 1.3. *Let G be a 2-connected graph. Then $\delta(G) \geq 2$.*

Proof. By way of contradiction, assume that $\delta(G) \leq 1$. If $\delta(G) = 0$, then we have an isolated vertex in G, a contradiction since G being 2-connected implies G is connected.

Thus we have $\delta(G) = 1$. Let x be a vertex of degree 1 in G with x' being its unique neighbor. Let y be a vertex in G not in the set $\{x, x'\}$. Such a vertex y exists because any 2-connected graph should have at least three vertices. Now, obtain $G' = G \setminus x'$. Since x' is the unique neighbor of x in G, there is no path between x and y in G', implying that G' is a disconnected graph. Observe that we have obtained a disconnected graph from G by deleting only one vertex from it, a contradiction to the fact that G is 2-connected. Hence, our assumption that $\delta(G) \leq 1$ is wrong and the lemma is valid. ■

1.3 Proper Vertex Coloring

Let C be a given set of colors. A *proper vertex coloring* of a graph $G = (V, E)$, with the colors from C, is a function $f : V \to C$ such that $f(v_1) \neq f(v_2)$ for any adjacent vertices v_1 and v_2. In other words, if we consider any two vertices in the graph that have an edge between them, they should see two different colors in a proper vertex coloring. The minimum number of colors required to perform a proper vertex coloring of a given graph G is called the *chromatic number* of G and is denoted by $\chi(G)$.

One can order the vertices of a graph G arbitrarily and color the vertices as per the order (to perform a greedy coloring). Every vertex has a restriction of at most Δ colors. Thus we can color the vertices of any graph with $\Delta + 1$ colors implying that $\Delta + 1$ is an upper bound for the chromatic index of a graph.

A couple of examples will help in giving more clarity on the definition. Consider the graph that is a cycle on three vertices, i.e., C_3. In Figure 1.1, the vertices of C_3 are assigned some colors; therefore, it is a vertex coloring of C_3. Observe that in graph C_3, two vertices x and z are adjacent; but they are colored with the same color 1. Hence, this vertex coloring is not proper.

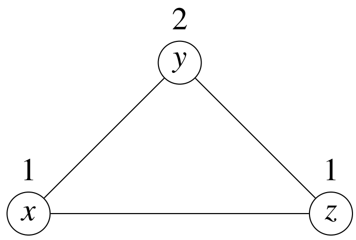

Figure 1.1: A vertex coloring of C_3 that is not proper

A proper vertex coloring of C_3 is given in Figure 1.2. Observe that the coloring uses $\Delta = 3$ colors. One can verify that 3 colors are necessary for any proper vertex coloring of C_3. Hence, we are sure that the chromatic number of C_3 is 3.

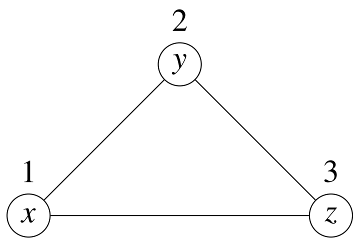

Figure 1.2: A proper vertex coloring of C_3

1.4 Proper Edge Coloring

Let C be a given set of colors. A *proper edge coloring* of a graph $G = (V, E)$, with the colors from C, is a function $f : E \to C$ such that $f(e_1) \neq f(e_2)$ for any incident edges e_1 and e_2. In other words, if we consider any two edges in the graph that are incident on a common vertex, they should see two different colors in a proper edge coloring. The minimum number of colors required to perform a proper edge coloring of a given graph G is called the *chromatic index* of G and is denoted by $\chi'(G)$.

By Vizing's theorem, the proof of which is provided in Diestel (2017), we have that $\Delta \leq \chi'(G) \leq \Delta + 1$. Therefore, for any given graph G, we are sure that the value of the chromatic index $\chi'(G)$ is either Δ or $\Delta + 1$.

Again, a couple of examples will help in giving more clarity on the definition. An edge coloring of C_3 is given in Figure 1.3. Observe that in graph C_3, the two edges xy and yz are incident on a common vertex y; but both the edges xy and yz are colored with the same color 1. Hence, this edge coloring is not proper.

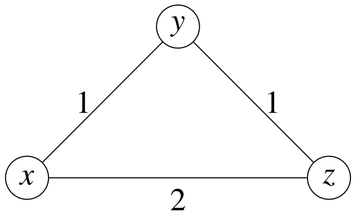

Figure 1.3: An edge coloring of C_3 that is not proper

A proper edge coloring of the complete graph on 4 vertices, i.e., K_4, is given in Figure 1.4. Observe that the coloring uses $\Delta = 3$ colors. Since Δ is a lower bound for the chromatic index, we are sure that the chromatic index of K_4 is 3.

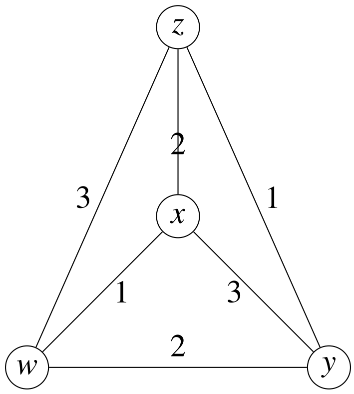

Figure 1.4: A proper edge coloring of K_4

A proper edge coloring of graph C_5 which is a cycle on five vertices is given in Figure 1.5. Observe that the coloring uses $\Delta + 1 = 3$ colors. One can verify that it is not possible to color with $\Delta = 2$ colors, implying that the chromatic index of C_5 is 3.

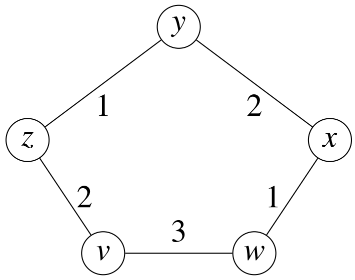

Figure 1.5: A proper edge coloring of C_5

So, some graphs require $\Delta + 1$ colors for the proper edge coloring and some graphs can be colored with Δ colors. Hence, both the upper and the lower bounds for the chromatic index are tight.

It is known that finding the chromatic index of a graph is NP-hard even for special classes of graphs. As proved by Holyer (1981), it is NP-hard even for the class of cubic graphs (3-regular graphs). Leven and Galil (1983) proved that it is NP-hard to determine whether it is possible to color the edges of a regular graph of degree k with k colors for any $k \geq 3$.

Therefore, it is interesting to find out the classes of graphs where the problem is solvable in polynomial time. There are polynomial-time algorithms to find the chromatic index of bipartite graphs (König, 1916), series-parallel graphs (Zhou, Suzuki and Nishizeki, 1996) and so on. Machado et al. (2013) studied the problem in the case of chordless graphs and obtained a polynomial-time algorithm for determining the chromatic index of the same.

Now, if some more conditions are imposed on the existing proper coloring of a graph, then various types of colorings like acyclic coloring, strong coloring, and star coloring are obtained. This thesis concentrates mainly on the concept of acyclic coloring (especially acyclic edge coloring).

1.5 Acyclic Vertex Coloring

A *linear forest* is a graph without cycles, in which every vertex has its degree at most two. In other words, a linear forest is a disjoint union of paths. One can see that every component in a linear forest is a path.

Definition 1.4. *A proper vertex coloring of a graph G is said to be acyclic if there are no bichromatic cycles in G or equivalently if the union of any two color classes (set of all vertices colored with one of those two colors) induces a linear forest in G.*

Definition 1.5. *The acyclic chromatic number of a graph G is the minimum number of colors required to perform an acyclic vertex coloring of G and is denoted by* $A(G)$.

Note that the upper bound of $\Delta+1$ that we have for the chromatic number of a graph is not valid here since the greedy coloring (performed for the proper coloring of the vertices of G with $\Delta+1$ colors) may result in bichromatic cycles.

Again, a couple of examples will help us in better visualization. Now, consider the proper vertex coloring of graph C_4 provided in Figure 1.6. In the coloring, there exists a bichromatic cycle $wxyz$ that is colored only with the colors 1 and 2. Alternatively, the union of the color classes 1 and 2 induces a cycle in C_4 (the entire C_4 itself), and hence, does not induce a linear forest. Thus we can conclude that the coloring of C_4 provided in Figure 1.6 is not an acyclic vertex coloring.

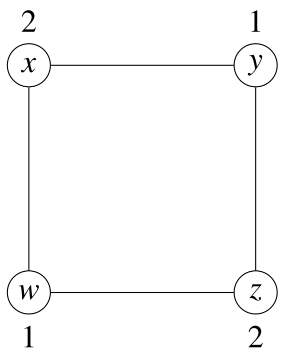

Figure 1.6: A proper vertex coloring of C_4 that is not acyclic

An acyclic vertex coloring of C_4 with $\Delta+1=3$ colors is given in Figure 1.7.

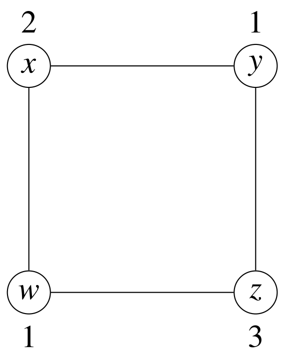

Figure 1.7: An acyclic vertex coloring of C_4

One can verify that in this coloring, there are no bichromatic cycles. Alternatively, one can also verify the following:

- The subgraph of C_4 induced by the color classes 1 and 2 is a path wxy, as shown in Figure 1.8.
- The subgraph of C_4 induced by the color classes 1 and 3 is a path wzy, as shown in Figure 1.9.
- The subgraph of C_4 induced by the color classes 2 and 3 is a totally disconnected graph, as shown in Figure 1.10.

Since the list is exhaustive, we can conclude that any two color classes induce a linear forest in C_4, implying that the coloring in Figure 1.7 is an acyclic vertex coloring of C_4.

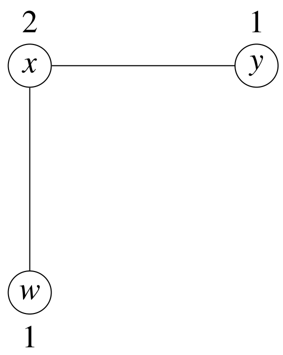

Figure 1.8: Subgraph induced by colors 1 and 2

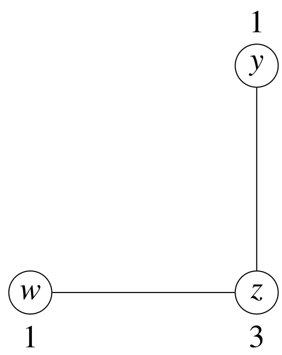

Figure 1.9: Subgraph induced by colors 1 and 3

Figure 1.10: Subgraph induced by colors 2 and 3

1.6 Acyclic Edge Coloring

To begin with, let us take a look at the definitions of acyclic edge coloring and acyclic chromatic index which will be the topic to be discussed in most of the subsequent chapters.

1.6.1 Definition

Definition 1.6. *A proper edge coloring of a graph G is said to be acyclic if there are no bichromatic cycles in G or equivalently if the union of any two color classes (set of all edges colored with one of those two colors) induces a linear forest in G.*

Definition 1.7. *The acyclic chromatic index (also called acyclic edge chromatic number) of a graph G is the minimum number of colors required to perform an acyclic edge coloring of G and is denoted by $a'(G)$.*

Since any acyclic edge coloring of a graph is also a proper edge coloring by definition, we have $a'(G) \geq \chi'(G) \geq \Delta$. Therefore, Δ is also a lower bound for the acyclic chromatic index.

Again, a couple of examples will help us in better visualization. Now, consider the proper edge coloring of K_4 provided in Figure 1.4. Observe that in the coloring, there exists a bichromatic cycle *wxzy* which is colored only with the colors 1 and 2. Alternatively, if we consider the union of the color classes 1 and 2, they induce a cycle *wxzy* in K_4, as shown in Figure 1.11. Hence, the condition of any pair of color classes inducing a linear forest in K_4 fails in this coloring. Thus we can conclude that the coloring of K_4 provided in Figure 1.4 is not an acyclic edge coloring.

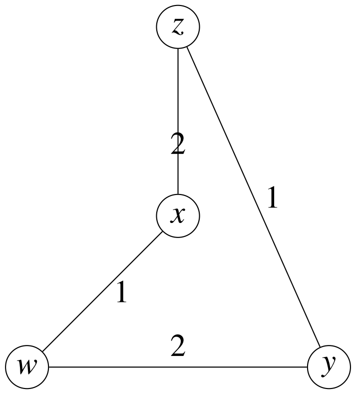

Figure 1.11: A subgraph of K_4 induced by the colors 1 and 2

An acyclic edge coloring of K_4 is given in Figure 1.12. One can easily verify that the coloring is acyclic since the color of only one edge appears more than once. Observe that this coloring uses $\Delta + 2 = 5$ colors for the acyclic edge coloring.

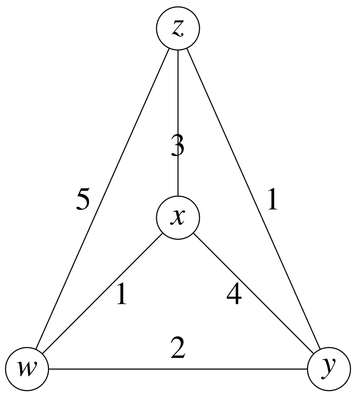

Figure 1.12: An acyclic edge coloring of K_4

Further, by using the definition of acyclic edge coloring, one can see that the proper edge coloring of C_5 given in Figure 1.5 is also an acyclic edge coloring with $\Delta+1=3$ colors.

1.6.2 Overview

Grünbaum (1973) was the one who first proposed the idea of acyclic coloring. Later, it was extended to edge coloring by Fiamčík (1978) and independently by Alon, McDiarmid and Reed (1991). The bounds can be obtained for the graph parameters like star chromatic number (Fertin, Raspaud and Reed, 2004) and oriented chromatic number (Kostochka, Sopena and Zhu, 1997) using the acyclic chromatic index and the acyclic chromatic number. These two parameters have several practical applications which also include wavelength routing in optical networks (Amar, Raspaud and Togni, 2001).

The vertex analog of the acyclic chromatic index is called the acyclic chromatic number. In the introductory paper, Grünbaum (1973) studied the acyclic chromatic number of planar graphs and proved that the acyclic chromatic number of planar graphs is at most 9 and conjectured that it is at most 5. This Conjecture was settled by Borodin (1979) who proved that the acyclic chromatic number of planar graphs is at most 5. Later, Fiamčík (1978) and independently Alon et al. (2001) studied the edge analog of acyclic coloring and conjectured as below:

Conjecture 1.8 (Fiamčík, 1978)**.** *If G is a graph with Δ being its maximum degree, then $a'(G) \leq \Delta+2$.*

Once the conjecture was formulated researchers across the globe are attempting on progressing towards this conjecture. However, the conjecture is open for an arbitrary graph and has been proved only for some special families of graphs.

Using probabilistic arguments, Alon et al. (1991) proved that for an arbitrary graph G with maximum degree Δ, $a'(G) \leq 60\Delta$. Later, Molloy and Reed (1999) improved this bound by proving that $a'(G) \leq 16\Delta$ for an arbitrary graph G. Muthu, Narayanan and Subramanian (2007b) proved that $a'(G) \leq 4.52\Delta$ for any graph G of girth at least 220.

Ndreca, Procacci and Scoppola (2012) proved that for an arbitrary graph G, $a'(G) \leq \lceil 9.62(\Delta-1)\rceil$. Again, this bound was improved by Esperet and Parreau (2013) by proving that $a'(G) \leq 4(\Delta-1)$ for an arbitrary graph G. Later the bound was once again improved by Giotis, Kirousis, Psaromiligkos and Thilikos (2017) by proving that $a'(G) \leq \lceil 3.74(\Delta-1)\rceil + 1$. However, the best-known upper bound for the acyclic chromatic index of an arbitrary graph till date is $3.569(\Delta-1)$ given by Fialho, de Lima and Procacci (2020). Most of these authors use probabilistic methods to come up with the upper bound.

Even though this bound is far from the conjectured bound, the conjecture is already proved for some special families of graphs. Alon et al. (2001) came up with the following theorem:

Theorem 1.9 (Alon et al., 2001)**.** *There exists a constant k such that* $a'(G) \leq \Delta + 2$ *for any graph G whose girth is at least* $k\Delta log(\Delta)$.

Alon et al. (2001) also demonstrated that for almost all Δ-regular graphs, we have $a'(G) \leq \Delta + 2$. Later, Nešetřil and Wormald (2005) strengthened this result by showing that $a'(G) \leq \Delta + 1$ for a random Δ-regular graph. Muthu, Narayanan and Subramanian (2010) considered grid-like graphs and proved the conjecture by proving that $a'(G) \leq \Delta + 1$ for this class of graphs which is better than the conjectured bound.

Burnstein (1979) showed that if Δ is at most 4, then it takes at most fives colors to color the vertices G acyclically. Note that the line graph of a subcubic graph (i.e., a graph with Δ at most 3) satisfies this condition of Δ being at most 4. Since coloring the edges of a graph acyclically is the same as coloring the vertices of its line graph acyclically, we are sure that a subcubic graph requires at most $\Delta + 2 = 5$ colors for acyclic edge coloring. A polynomial-time algorithm for coloring a subcubic graph using 5 colors was given by Skulrattanakulchai (2004). Later, Basavaraju and Chandran (2008) proved that any non-regular subcubic graph requires at most four colors for acyclic edge coloring with the result being tight. Caro and Roditty (1994) considered 2-degenerate graphs in order to study the acyclic edge coloring.

Theorem 1.10 (Caro and Roditty, 1994)**.** *For any 2-degenerate graph G with maximum degree Δ,* $a'(G) \leq \Delta + k - 1$*, where k is the maximum edge-connectivity defined as* $k = max_{(u,v) \in V(G)} \lambda(u,v)$*, where* $\lambda(u,v)$ *is the edge-connectivity of the vertex pair* (u,v).

Note that in the above theorem, k can go up to Δ. Muthu, Narayanan and Subramanian (2007a) proved the conjecture for a subclass of 2-degenerate graphs namely outerplanar graphs, by proving that $a'(G) \leq \Delta + 1$. They also suggested an open problem to prove the conjecture for 2-degenerate graphs. Later, Basavaraju and Chandran (2010) solved this open problem by settling the conjecture for 2-degenerate graphs.

The acyclic chromatic index has been exactly determined for some classes of graphs like outerplanar graphs when $\Delta \neq 4$ (Hou and Wu, 2013), (Hou, Wu, Liu and Liu, 2010), series-parallel graphs when $\Delta \neq 4$ (Wang and Shu, 2011), planar graphs with girth at least 5 and $\Delta \geq 19$ (Basavaraju, Chandran, Cohen, Havet and Müller, 2011), and planar graphs with $\Delta \geq 4.2 \times 10^{14}$ (Cranston, 2019). In the case of outerplanar and series-parallel graphs, $a'(G) = \Delta$ if $\Delta \geq 5$ and when $\Delta = 3$, they characterize the graphs that require 4 colors.

Note that in the case of acyclic edge coloring, at most one color class can be a perfect matching since otherwise, the two color classes which are perfect matchings will give rise to at least one cycle when we take their union. This implies that for any k-regular graph G with $k \geq 2$, we have $a'(G) \geq \Delta(G) + 1$. Thus for any cubic graph G, we have $a'(G) \geq 4$.

By a result of Burnstein (1979), we also have $a'(G) \leq 5$ for a cubic graph G. Andersen, Máčajová and Mazák (2012) proved that $a'(G) = 4$ for a connected cubic graph G other than K_4 and $K_{3,3}$. This exactly determines the acyclic chromatic index of cubic graphs and we can conclude that if the cubic graph G has either K_4 or $K_{3,3}$ as its subgraph, then $a'(G) = 5$ and if the cubic graph has neither K_4 nor $K_{3,3}$ as a subgraph, then $a'(G) = 4$.

Further, all the above-mentioned results are constructive in nature and hence, they also yield a polynomial-time algorithm to output an optimal coloring.

It is difficult to determine $a'(G)$ for any given graph G, both theoretically and algorithmically. Complete graphs are a straightforward and highly structured class of graphs; however, the exact value of $a'(G)$ is still unknown for complete graphs. Moreover, Alon and Zaks (2002) proved that it is NP-hard to determine $a'(G)$ for every given graph G, even when $\Delta(G) = 3$. One can easily observe that the graph constructed by Alon and Zaks (2002) for the reduction is a 2-degenerate graph. Thus it is NP-hard to determine $a'(G)$ even for 2-degenerate graphs.

1.6.3 Preliminaries

This section contains some definitions and lemmas given by Basavaraju and Chandran (2010) that are useful for the discussion in the subsequent chapters on acyclic edge coloring. These are mentioned here for the sake of completeness.

Definition 1.11 (Basavaraju and Chandran, 2010)**.** *An edge coloring f of a subgraph H of a graph G is called a partial edge coloring of G.*

An edge coloring of G is also a partial edge coloring of G because G is also a subgraph of itself. A partial edge coloring f of G corresponding to a subgraph H is said to be proper if it is proper in the subgraph H. Similarly, f is said to be acyclic if it is acyclic in the subgraph H.

Note that with respect to a partial coloring f, for an edge e, $f(e)$ may or may not be defined. So, whenever there is a mention of $f(e)$ for some edge e, it is implicitly assumed that $f(e)$ is defined.

Let f be a partial edge coloring of the graph G. For any vertex $x \in V$, $F_x(f)$ is the set of all colors seen on the edges incident to the vertex x, i.e.,

$$F_x(f) = \{f(xy) : y \in N_G(x)\}$$

For any edge $uv \in E$, $F_{uv}(f)$ is the set of all colors seen on the edges incident to the vertex v excluding the color $f(uv)$, i.e.,

$$F_{uv}(f) = F_v(f) \setminus \{f(uv)\}$$

Further, whenever the partial coloring f is understood from the context, F_x and F_{uv} are preferred over $F_x(f)$ and $F_{uv}(f)$ for simplicity. Observe that F_{uv} is different from F_{vu}.

Definition 1.12 (Basavaraju and Chandran, 2010). *An (α, β)-maximal bichromatic path with respect to a partial coloring f of G is a maximal path in G consisting of edges that are colored using the colors α and β alternatingly. An (α, β, u, v)-maximal bichromatic path is an (α, β)-maximal bichromatic path which starts at the vertex u with an edge colored with α and ends at the vertex v.*

The following lemma is mentioned as a fact by Basavaraju and Chandran (2010). It follows from the definition of acyclic edge coloring. This lemma is used at various places throughout the thesis either implicitly or explicitly.

Lemma 1.13 (Basavaraju and Chandran, 2010). *Given a pair of colors α and β in a proper coloring f of G, there is most one (α, β)-maximal bichromatic path containing a particular vertex v in G, with respect to f.*

Definition 1.14 (Basavaraju and Chandran, 2010). *If the vertices u and v are adjacent in the graph G, then an (α, β, u, v)-maximal bichromatic path in G, which ends at v with an edge colored α, is said to be an (α, β, uv)-critical path in G.*

Definition 1.15 (Basavaraju and Chandran, 2010). *Let f be a partial edge coloring of $G = (V, E)$. Let $u, a, b \in V$ and $ua, ub \in E$. A color exchange with respect to the edges ua and ub is defined as the process of obtaining a new partial coloring g from the current partial coloring f by exchanging the colors of the edges ua and ub. The color exchange defines g as follows. $g(ua) = f(ub)$, $g(ub) = f(ua)$, and for all other edges e in G, $g(e) = f(e)$.*

The *proper* color exchange with respect to the edges ua and ub is a color exchange that does not violate the condition of proper coloring. Similarly, the *valid* color exchange with respect to the edges ua and ub is a color exchange that does not violate the condition of acyclic coloring.

With respect to a partial coloring f, a color γ is said to be a *candidate* color for an edge e in G if the any of adjacent edges of e are not colored γ. If assigning a candidate color γ to the edge e does not result in any new bichromatic cycle in G, then the candidate color γ is said to be *valid* for the edge e. Basavaraju and Chandran (2010) mentioned the following lemma as a fact, since it is obvious.

Lemma 1.16 (Basavaraju and Chandran, 2010)**.** *Let f be a partial coloring of G. A candidate color γ is not valid for an edge $e = uv$ if and only if there exists a color $\eta \in F_{uv} \cap F_{vu}$ such that the graph G has an (η, γ, uv)-critical path with respect to the coloring f.*

1.7 Domination Number

To begin with, let us look at the definitions of a dominating set and the domination number of a graph. For the basic graph theoretical terminologies used in the thesis, one can also refer to the book by Chartrand, Lesniak and Zhang (2015). Specifically for domination-related concepts, one can also refer to the book by Haynes, Hedetniemi and Slater (1998).

1.7.1 Definition

A vertex u is said to be *dominated* by some vertex v in G if u and v are adjacent in the graph G.

Definition 1.17. *A subset D of the vertex set of the graph $G = (V,E)$ is said to be a dominating set of G if every vertex v in the set $V \setminus D$ is adjacent to a vertex in D.*

For a dominating set D of G, one can easily see that every vertex of G which is not in D is dominated by some vertex in D. In other words, if D is a dominating set of G, then if we arbitrarily consider a vertex v from $V(G)$, then either v is in D or it is adjacent to some vertex in D.

Definition 1.18. *The domination number of a graph G is the minimum cardinality of a dominating set of G. It is denoted by $\gamma(G)$.*

We omit G from the notations whenever the graph G is understood from the context. A dominating set D of G with $|D| = \gamma$ is called a γ-set of G. A minimum dominating set of the cycle C_6 is shown in Figure 1.13. Vertices marked with a box are in the dominating set. Observe that the domination number of C_6 is 2.

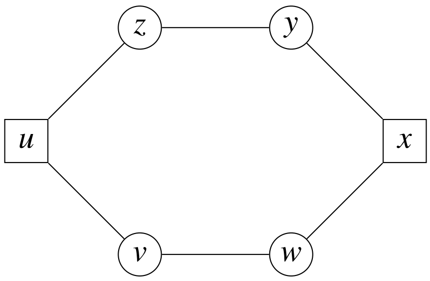

Figure 1.13: A minimum dominating set of C_6

A complete bipartite graph $K_{1,6}$ is an example of a graph having a dominating set of size 1. It is depicted in Figure 1.14 with the vertex marked inside a box being in the dominating set. Note that a graph of the form $K_{1,n}$ for any $n \geq 1$ is called a *star*. Thus any star has a dominating set of size 1. Notice that the necessary and sufficient condition for a graph to have a dominating set of size 1 is the existence of a vertex of degree $n-1$. Therefore, we can infer that in graph G of order n, there exists a dominating set of size 1 if and only if there exists a vertex v in $V(G)$ with $deg(v) = n-1$.

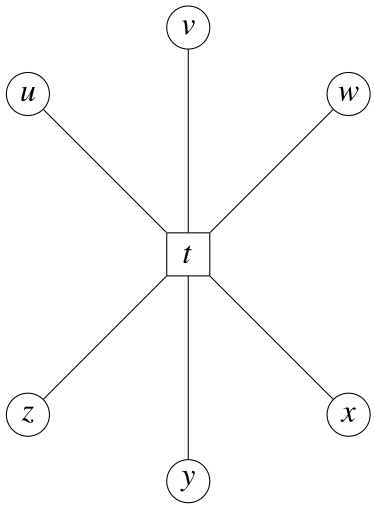

Figure 1.14: $K_{1,6}$: A graph with a dominating set of size 1

In K_n, a complete graph on n vertices, since any vertex is adjacent to all the remaining vertices, any single vertex is a dominating set of K_n. Hence, the domination number of K_n is 1. Now, consider the graph $\overline{K_n}$, the complement of K_n where there are n isolated vertices. Since there are no edges in $\overline{K_n}$, any dominating set should include all the vertices in it. Hence, the domination number of $\overline{K_n}$ is n. Further, for any other graph, the domination number lies between these two values by definition. Therefore, the domination number of a graph of order n ranges from 1 to n.

1.7.2 History

The concept of domination number basically originated from chessboard problems like mutually attacking queens etc. The problem of mutually attacking queens translates well into a domination number problem. Apart from the chessboard problems, domination in graphs has several other applications like the problem of the closest facility determination, where there are a fixed number of facilities (say, hospitals or fire stations) and a person wants to go to the closest facility by minimizing the travel distance to save time.

Although the rigorous study of domination in graphs started in the year 1958, there are some references to the problems related to graph domination about 100 years before that. The first reference was when De Jaenisch (1862) attempted to solve a problem with the chessboard. His attempt aimed at determining the "minimum number of queens required to cover an $n \times n$ chess board" which turns out to be a problem of domination in graphs.

Later, Rouse Ball (1892) pointed out three fundamental problems that the chess players were interested in during that time:

(i). Covering: This problem aims at determining the minimum number of chess pieces of a given type that are necessary to attack or cover (or dominate) every square of an $n \times n$ chess board.

(ii). Independent Covering: This problem aims at determining the minimum number of mutually non-attacking chess pieces of a given type that are necessary to attack or cover (or dominate) every square of an $n \times n$ chess board.

(iii). Independence: This problem aims at determining the maximum number of chess pieces of a given type that can be placed on an $n \times n$ chess board such that no two pieces attack each other. Further, note that if the chess piece being considered is the queen, then this type of problem is famously known as the N-queens Problem in the literature.

It is interesting to see that all these problems are direct applications of the domination number of a graph. But after this point, there are no references to this problem of domination in graphs until Berge (1958) re-initiated the study and from thereon, the concept of domination in graphs was further developed.

In the book by Berge (1958), a new concept called the *coefficient of external stability* was introduced. This was the starting point of the conceptual study of domination because the same concept introduced by Berge (1958) is currently known as the domination number of a graph. Later, Ore (1962) published a book on graph theory, in which he coined the terms *dominating set* and *domination number* for the first time, and from thereon, these terms are widely used in the literature.

The problems listed by Rouse Ball (1892) were studied in detail by two brothers Yaglom and Yaglom (1964). They were able to come up with solutions to some of these problems by considering some specific chess pieces. To obtain a solution, they considered knights, bishops, rooks and kings.

The notation for the domination number of a graph, i.e., $\gamma(G)$ was first used in a survey paper published by Cockayne and Hedetniemi (1977). Since this paper was published, the concept of domination in graphs has been extensively studied and several research papers have been published on this topic. This thesis also contributes to the literature on domination in graphs.

1.8 Organization of the Thesis

This thesis comprises seven chapters. The chapter-wise organization of the content of the thesis is provided here.

Chapter 1 deals with the necessary introduction to acyclic edge coloring and the domination number of a graph together with the literature review and the applications of the same. It also includes some preliminaries that are necessary to understand the subsequent chapters.

Chapter 2 consists of some special structural results on chordless graphs that are modifications to the existing ones in the literature. These structural results are helpful in determining the acyclic chromatic index of chordless graphs and are also of independent interest with respect to research.

Chapter 3 is about exactly determining the acyclic chromatic index of chordless graphs. To achieve this, some preliminary results and the structural results in Chapter 2 are combined while obtaining the proof.

Chapter 4 introduces k-degenerate graphs together with a theorem on the existence of a special edge in them. An improved upper bound on the acyclic chromatic index of k-degenerate graphs is also obtained.

Chapter 5 deals with the study of 3-degenerate graphs. An upper bound is obtained for the acyclic chromatic index of 3-degenerate graphs. This upper bound is a constant addition to the value in Conjecture 1.8.

Chapter 6 emphasizes improving the existing upper bound on the domination number of a graph. A new graph class is introduced and used to characterize the graphs based on their domination number.

Further, chapter-wise concluding discussions and summaries are provided at the end of each chapter. However, the conclusion of the thesis and the possible scope for future work are provided in Chapter 7.

1.9 Conclusion

All the topics under discussion in the thesis are appropriately introduced and explained with examples whenever necessary. The origin, history, and much-needed literature review on all the topics are the highlights of the chapter. Some definitions and preliminaries on acyclic edge coloring are discussed and these are useful in the subsequent chapters on the same. The concept of domination number is introduced with an overview and a couple of examples for the same so that the main chapter on domination gets a solid base. Finally, the outline of the thesis is presented to get a better picture of the chapters. In essence, this chapter in the thesis is like a strong foundation for a building. The first one being solid is extremely important for the existence of the latter in both scenarios.

Chapter 2

Structural Study of Chordless Graphs

This is an introductory chapter on the family of chordless graphs. There are some existing structural results in the literature for chordless graphs which will be discussed. The goal of this chapter is to improve those results and come up with the existence of a new structure in chordless graphs. This new structure might be extremely useful in solving several coloring problems on chordless graphs. The next chapter provides evidence for the same by utilizing this new structure for studying the acyclic edge coloring of a chordless graph.

2.1 Definition

The class of chordless graphs is an important family in graph theory. Let us begin the study of chordless graphs with the definition of the same.

Definition 2.1. *An edge $e = uv$ of a graph G is said to be a chord if the vertices u and v are part of a cycle in $G \setminus e$ which is a graph obtained by deleting the edge e from G. A graph is said to be chordless if it does not contain a chord.*

In other words, a graph is said to be chordless if every cycle in the graph is an induced cycle. Even if the graph contains one cycle that is not induced, then the graph surely has a chord in it. A chord being present in a cycle is the reason for the cycle to be not induced.

An example of a graph that is not chordless is shown in Figure 2.1. Observe that in the example, the edge vy is a chord because the vertices v and y are part of a cycle $vwxyzv$ in the graph $G \setminus vy$. Observe that the cycle $vwxyzv$ is not an induced cycle in the graph. Hence, according to both definitions, we can conclude that the graph in Figure 2.1 is not chordless.

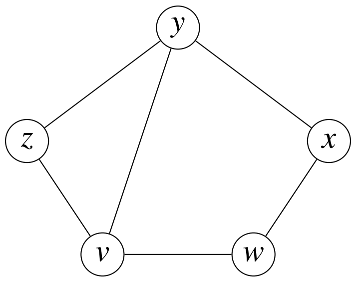

Figure 2.1: C_5 with a chord

An example of a chordless graph is shown in Figure 2.2. One can easily verify that there are no chords in this graph.

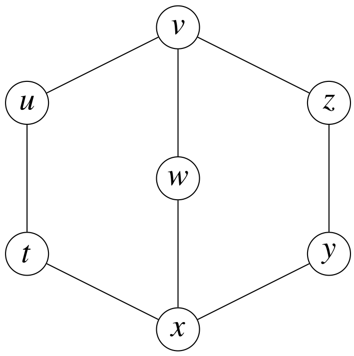

Figure 2.2: An example of a chordless graph

2.2 Overview

Recall that a 2-connected graph G is said to be minimally 2-connected if the subgraph $G \setminus e$ is no longer 2-connected for any edge e in G. The structural study of chordless graphs and minimally 2-connected graphs was initiated by Dirac (1967). He introduced the following terminology:

Definition 2.2 (Dirac, 1967)**.** *Let k be a positive integer. If $C_1, C_2, \dots C_{k+1}$ are distinct, connected graphs such that each graph C_i has at least two vertices, and the following condition is satisfied:*

$$C_i \cap C_j = \emptyset \quad \textit{if} \quad 1 \leq i \leq k+1, \quad 1 \leq j \leq k+1 \quad \textit{and} \quad |i-j| \geq 2$$

and $C_i \cap C_{i+1}$ is a singleton set (a vertex) for $i = 1, 2, \dots k$, then the graphs $C_1, C_2, \dots C_{k+1}$ are said to be in series in $C_1 \cup C_2 \cup \dots \cup C_{k+1}$ in the order of their indices. Further, C_1 and C_{k+1} are called the end-terms of the series.

In other words, in this definition, $C_1, C_2, \dots C_{k+1}$ is a $k+1$-ordered tuple of connected graphs with pairwise one vertex intersection for the consecutive pairs and no other intersections. After introducing this terminology, as a part of the study of minimally 2-connected graphs, Dirac (1967) obtained a necessary and sufficient condition for a graph G to be 2-connected and $G \setminus e$ not to be 2-connected for a particular edge e. This condition is presented in the form of the following theorem:

Theorem 2.3 (Dirac, 1967)**.** *Let G be a graph of order at least 3 and let $e = uv$ be an edge in G. Then G is a 2-connected graph, and $G \setminus e$ is not a 2-connected graph (In this case when $G \setminus e$ is not a 2-connected graph, the edge e is said to be a critical edge) if and only if the following conditions hold:*

- *$G \setminus e$ is connected.*
- *$G \setminus e$ has at least two but finite blocks (block is a maximal connected subgraph of a graph that does not have a cut vertex).*
- *The blocks of $G \setminus e$ are in series in the graph $G \setminus e$.*
- *The vertex u belongs to one end-term and does not belong to any other block of $G \setminus e$. The vertex v belongs to the other end-term and does not belong to any other block of $G \setminus e$.*

Dirac (1967) also proved the following lemma which is instrumental in obtaining a crucial characteristic of the minimally 2-connected graphs.

Lemma 2.4 (Dirac, 1967)**.** *Any edge e in a 2-connected graph G is a critical edge if and only if there is no cycle in the graph $G \setminus e$ that contains both the end vertices of e.*

Recall that for a given graph G, if there is no cycle in the graph $G \setminus e$ that contains both the end vertices of the edge e, then e is not a chord in G. Therefore, if every edge in a 2-connected graph G is a critical edge, then G is chordless. Thus we have the following lemma as a corollary of Lemma 2.4.

Lemma 2.5. *A chordless graph that is 2-connected is a minimally 2-connected graph.*

This lemma implies that the class of 2-connected chordless graphs is equivalent to the class of minimally 2-connected graphs signifying the importance of the study of chordless graphs and their structure.

Simultaneously, Plummer (1968) also studied these minimally 2-connected graphs but with different terminology.

Definition 2.6 (Plummer, 1968)**.** *A graph G is said to be a block if there is no vertex v in G such that the graph $G \setminus v$ is disconnected. A graph G is said to be a block-line-critical graph if $G \setminus e$ is not a block for any edge e in G.*

Observe that if $G \setminus v$ is disconnected, then v is a cut vertex in the connected graph G. Plummer (1968) related the block and a chord in a graph as follows:

Theorem 2.7 (Plummer, 1968)**.** *If G is a block and e is an edge in G, then $G \setminus e$ is also a block if and only if e is a diagonal in G (A chord in a graph was termed as a diagonal by Plummer).*

As an immediate corollary, we have that if $G \setminus e$ is not a block for some edge e in G, then e is not a chord in G. Therefore, the class of block-line-critical graphs is precisely chordless graphs. Plummer (1968) also gave a characterization of block-line-critical graphs as follows:

Theorem 2.8 (Plummer, 1968)**.** *Let e be an edge in a block-line-critical graph G and let x be a cut vertex in $G \setminus e$. Then*

- *x is on every cycle containing e.*
- *Every path containing e and x consists of an intermediate vertex (Any vertex in an (a,b)-path other than the end vertices a and b is said to be an intermediate vertex) of degree 2 in G.*

The subgraph of a block-line-critical graph G generated by the vertices of degree at least 3 is denoted by G'. Let $T_1, T_2, \ldots, T_k$ be the components of G'. By Theorem 2.8, we are sure that G' does not have any cycles which implies that each T_i is a tree. Let S be the set of vertices of degree 2 in G. A path is called an *S-path* if all of its intermediate vertices are of degree 2 in G. After defining these terms, Plummer (1968) proved the following theorem:

Theorem 2.9 (Plummer, 1968)**.** *If G is a block-line-critical graph and if T_i and S are as defined, then for any tree T_i, there is no S-path joining two vertices of T_i.*

By these results of Plummer (1968), one can conclude that a block-line-critical graph structurally consists of at least two mutually vertex-disjoint trees, i.e., $T_1, T_2, \ldots, T_k$ with $k \geq 2$, together with a set of paths such that each path P joins some pair T_i and T_j and the degree of all the intermediate (interior) vertices of P is 2.

Later, the structural study of chordless graphs was continued by Lévêque, Maffray and Trotignon (2012). The following definitions are necessary for further discussions.

Definition 2.10. *A k-cutset of a connected graph G is a set S of vertices of size k such that the vertex set of G can be partitioned into non-empty sets X, Y and S so that there is no edge between any vertex in X and any vertex in Y. In other words, a subset S of $V(G)$ with k vertices is said to be a k-cutset if the subgraph $G \setminus S$ is disconnected.*

If S is a k-cutset of G with X, Y and S being the partitions of $V(G)$ obtained as in the previous definition, then (X,Y,S) is called a *split* of the proper k-cutset S. Observe that a 1-cutset of a graph is the same as a cut vertex of the graph.

Definition 2.11. *A proper* 2*-cutset of a connected graph* G *is a* 2*-cutset* $\{a,b\}$ *of* G *with the vertex partitions* X*,* Y *and* $\{a,b\}$ *such that there is no edge between the vertices* a *and* b *in* G *and both the induced subgraphs* $G[X\cup\{a,b\}]$ *and* $G[Y\cup\{a,b\}]$ *contain an* ab*-path but neither* $G[X\cup\{a,b\}]$ *nor* $G[Y\cup\{a,b\}]$ *is an induced path.*

For simplicity, the split of the proper 2-cutset (a,b) is denoted as (X,Y,a,b) instead of $(X,Y,\{a,b\})$ throughout the literature. The pictorial representation of the proper 2-cutset of a graph is provided in Figure 2.3.

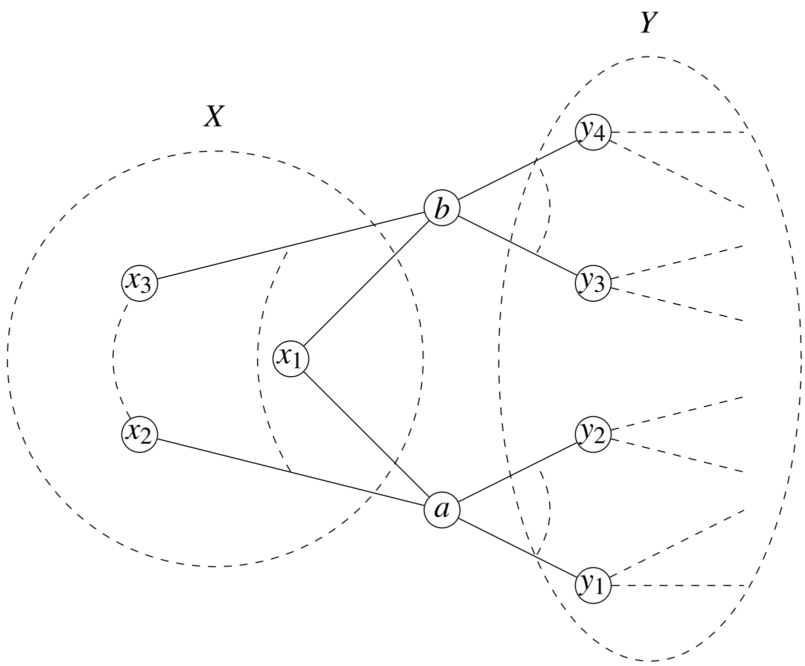

Figure 2.3: A proper 2-cutset

Definition 2.12. *For a given connected graph* G *with a proper* 2*-cutset* (a,b) *and a split* (X,Y,a,b)*, the block* $G_X(a,b)$ *is the graph obtained by taking the induced subgraph* $G[X\cup\{a,b\}]$ *and adding a new vertex* w *called the marker vertex, adjacent to both* a *and* b*. Similarly, the block* $G_Y(a,b)$ *is the graph obtained by taking the induced subgraph* $G[Y\cup\{a,b\}]$ *and adding a new vertex* w *called the marker vertex, adjacent to both* a *and* b*.*

The pictorial representations of block $G_X(a,b)$ and $G_Y(a,b)$ can be seen in Figure 2.4 and Figure 2.5, respectively.

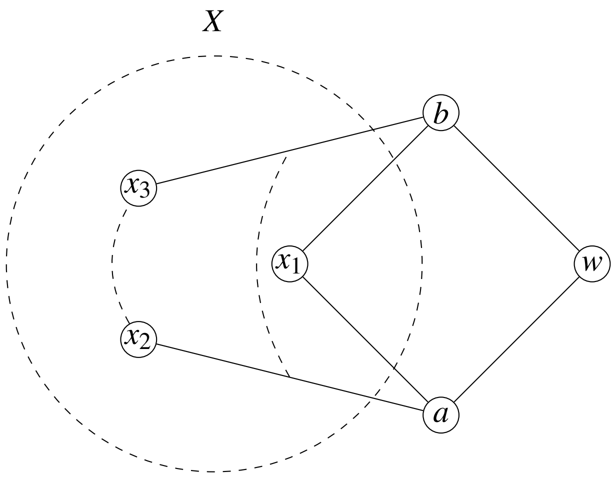

Figure 2.4: Block $G_X(a,b)$

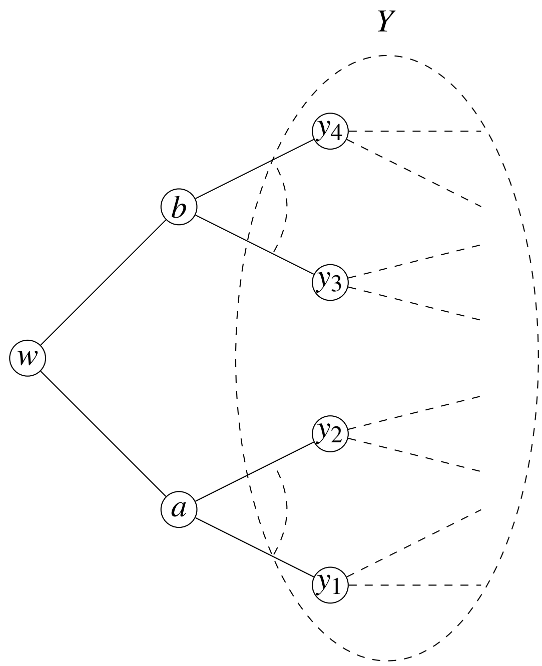

Figure 2.5: Block $G_Y(a,b)$

Now, we provide the definition of a strict subclass of chordless graphs.

Definition 2.13. *A graph G is called* 2*-sparse if every edge of G is incident on at least one vertex of degree at most* 2*. In other words, a graph G is said to be a* 2*-sparse graph if there is no edge in G with both end vertices having their degree at least* 3*.*

Lévêque et al. (2012) proved a structural property of chordless graphs which is found to be very useful. In particular, they proved the following lemma:

Lemma 2.14 (Lévêque et al., 2012)**.** *If G is a chordless graph, then either G is* 2*-sparse or G admits a* 1*-cutset or a proper* 2*-cutset.*

If we add the condition that the graph G is 2-connected, then G will not have any cut vertex (or a 1-cutset). Thus we can obtain the following lemma:

Lemma 2.15. *If G is a* 2*-connected chordless graph, then either G is* 2*-sparse or G admits a proper* 2*-cutset.*

Later, Machado et al. (2013) concentrated on the chromatic index of chordless graphs and proved that the edges of any connected chordless graph can be properly colored using exactly Δ colors unless G is an odd cycle. In fact, they gave the following theorem:

Theorem 2.16 (Machado et al., 2013)**.** *If G is a chordless graph with maximum degree* Δ *and either* $\Delta \neq 2$ *or G does not contain an odd cycle, then* $\chi'(G) = \Delta$.

To achieve this result, they refined the structure of chordless graphs given by Lévêque et al. (2012), which helped them in proving the result. Since any proper coloring requires at least Δ colors, this means that the chromatic index of any chordless graph can be exactly determined in linear time. Further, Machado et al. (2013) also gave a polynomial-time algorithm to color the edges of a given chordless graph with the optimum number of colors.

A detailed view of the structural refinement obtained by Machado et al. (2013) is provided in the next section which emphasizes the detailed study of the structural results and improvements on the class of chordless graphs.

2.3 The Structural Improvement

Recall that a graph G is called 2-sparse if every edge of G is incident on at least one vertex of degree at most 2. The class of 2-sparse graphs is very useful in obtaining a special structure in chordless graphs.

Observe that a chord of a cycle in a graph is an edge with both of its end vertices having their degree at least 3. Two edges corresponding to the cycle and one edge which is the chord itself are incident on both the end vertices of the chord. Hence, if there is no edge in the graph with both the end vertices having their degree at least 3, then there is no chord in the graph implying that the class of 2-sparse graphs is a subclass of the class of chordless graphs.

The structure provided in Lemma 2.15 is useful in proving many results on chordless graphs. But when Machado et al. (2013) tried to study the edge coloring of chordless graphs, they needed a refined structure with respect to the proper 2-cutsets than that is in Lemma 2.15. Hence, they came up with the following lemma on the structural improvement of chordless graphs.

Lemma 2.17 (Machado et al., 2013)**.** *Let G be a* 2*-connected, not* 2*-sparse chordless graph. Let* (X,Y,a,b) *be a split of a proper* 2*-cutset of G such that* $|X|$ *is the minimum among all such possible splits. Then both a and b have at least two neighbors in X and the block* $G_X(a,b)$ *is* 2*-sparse.*

This result helped them in proving that the edges of every chordless graph with a maximum degree of at least 3 can be properly colored using Δ colors. The extremal split helps them in the inductive proof. Their main idea is to extend the coloring of the block $G_Y(a,b)$ to the coloring of G by exploiting the structure of the block $G_X(a,b)$ which is 2-sparse.

The proof of acyclic edge coloring of chordless graphs provided in the next chapter also uses the method of induction. The proper 2-cutset and $G_X(a,b)$ play a major role in choosing the smaller subgraph. But since the coloring has to be acyclic, the cycles need to be limited to one block where they can be handled. Naturally, the intuition is to limit the possible cycles within $G_X(a,b) \setminus w$ since it is 2-sparse.

But the major hurdle is when all the edges in the 2-sparse side are incident to the vertices a and b. Note that when all the edges of $G_X(a,b)$ are incident to either a or b, then $G_X(a,b)$ is isomorphic to a complete bipartite graph $K_{2,t}$ for $t \geq 2$. When $t = 2$, the graph $G_X(a,b)$ is the cycle C_4.

In proper edge coloring, while extending the coloring of $G_Y(a,b)$ to G, the missing colors on the vertices a and b can be assigned to the edges in $G_X(a,b) \setminus w$ by appropriately permuting the colors wherever necessary. But the same technique does not work with respect to acyclic edge coloring since there is a possibility of cycles being created in the graph even when we permute the colors.

Therefore, when it comes to acyclic edge coloring of chordless graphs, a much more refined structure is needed with respect to the proper 2-cutset than that is required by Machado et al. The following lemma is the intended refinement of Lemma 2.17. This is the core result of the chapter.

Lemma 2.18. *Let G be a* 2*-connected, not* 2*-sparse chordless graph. Then there exists a split* (X,Y,a,b) *in G with the following properties.*

(i). $G_X(a,b)$ *is* 2*-sparse.*

(ii). $G_X(a,b)$ *is not isomorphic to* $K_{2,t}$ *for any* $t \geq 3$.

(iii). *In* $G_X(a,b)$*, we have* $deg(a) \geq 3$ *and either* $deg(b) \geq 3$ *or* $deg(b) = 2$ *with X being minimal.*

Proof. Since G is a 2-connected, not 2-sparse, chordless graph, the existence of splits with the property (i) follows from Lemma 2.15 and Lemma 2.17. Let $S(G)$ be the set of all splits (X,Y,a,b) of any proper 2-cutset of G such that $G_X(a,b)$ is 2-sparse. Further, the following terminologies are introduced in the thesis and they are essential in the proof.

A proper 2-cutset (a,b) of G is said to be an *isolating pair* if there exists a split (X,Y,a,b) such that $G_X(a,b)$ is isomorphic to $K_{2,t}$ for some $t \geq 3$. The set of all isolating pairs of G is represented as $I(G)$. For each $(a,b) \in I(G)$, we define the corresponding set of all degree 2 neighbors denoted by $N_G^2(a,b)$, as follows:

$$N_G^2(a,b) = \{x \in V(G) \mid N(x) = \{a,b\}\}$$

Note that $|N_G^2(a,b)| \geq 2$ for any $(a,b) \in I(G)$. For each $(a,b) \in I(G)$, we arbitrarily select a vertex from $N_G^2(a,b)$ as a representative vertex of the pair (a,b) and denote it as r_{ab}. An *ip-deleted subgraph* H of G is the graph obtained from G by deleting all the vertices in $N_G^2(u,v)$ other than r_{uv} for each isolating pair $(u,v) \in I(G)$. We make the following observations about the subgraph H. One can see that all these observations follow directly from the definition of an ip-deleted subgraph.

Observation 2.19. *If $(u,v) \in I(G)$, then r_{uv} is the unique degree 2 vertex in the set $N_H(u) \cap N_H(v)$.*

This is true because any other degree 2 vertex in the set $N_H(u) \cap N_H(v)$ is deleted by the choice of the graph H.

Observation 2.20. *The only vertices in H whose degrees decrease with respect to their degree in G are those which are part of an isolating pair in G.*

This is true because any vertex in H that is not a part of an isolating pair in G preserves its degree in H by choice of H.

Observation 2.21. *While obtaining H from G, all the vertices that are removed from G have degree 2 in G.*

This is true because if the degree of a vertex is not 2, then since G is 2-connected, the degree of the vertex is at least 3 by Lemma 1.3 and any vertex of degree at least 3 can not be in the set $N_G^2(a,b)$.

Now, to prove the property (ii) of the lemma, it is enough to prove that there exists a split $(X,Y,a,b) \in S(G)$ such that $G_X(a,b)$ is not isomorphic to $K_{2,t}$ for any $t \geq 3$. If such a split already exists, then we get a split satisfying the property (ii) of the lemma.

Otherwise, every split $(X,Y,a,b) \in S(G)$ is such that $G_X(a,b)$ is isomorphic to $K_{2,t}$ for some $t \geq 3$. Recall that H is the ip-deleted subgraph of G. By Lemma 1.3, we have $\delta(G) \geq 2$, which clearly implies that $\delta(H) \geq 2$. Now, the claim is that the subgraph H does not have any isolating pair.

Claim 2.1. *There exists no isolating pair in H, i.e., $I(H) = \emptyset$.*

Proof. By way of contradiction, assume that there exists an isolating pair (u,v) in H which implies the existence of a split (X,Y,u,v) in H such that $H_X(u,v)$ is isomorphic to $K_{2,t}$ for some $t \geq 3$. Since $(u,v) \in I(H)$, we have that $deg_H(u) \geq 3$ and $deg_H(v) \geq 3$. Since $|N_H^2(u,v)| \geq 2$, $N_H(u) \cap N_H(v)$ contains at least two degree 2 vertices.

Suppose $(u,v) \in I(G)$. Then by Observation 2.19, r_{uv} is the unique degree 2 vertex in $N_H(u) \cap N_H(v)$, a contradiction to the fact that $|N_H^2(u,v)| \geq 2$. Therefore, we infer that $(u,v) \notin I(G)$.

Since $(u,v) \in I(H)$ and $(u,v) \notin I(G)$, there exists a vertex $z \in N_H^2(u,v)$ such that $deg_G(z) \geq 3$; but $deg_H(z) = 2$. Hence by Observation 2.20, z is part of an isolating pair, say (z,w) in G. Note that $r_{zw} \in V(H)$ and $zr_{zw} \in E(H)$. But we already know that $N_H(z) = \{u,v\}$. Hence, we can infer that $r_{zw} \in \{u,v\}$. But $deg_H(r_{zw}) = 2$, a contradiction since $deg_H(u) \geq 3$ and $deg_H(v) \geq 3$.

Therefore, our assumption that there exists an isolating pair (u,v) in H is wrong and we can infer that $I(H) = \emptyset$ implying the validity of the claim. ■

Further, the next claim is that there exists a vertex of degree 2 in H which has a higher degree in G.

Claim 2.2. *There exists an edge $ac \in E(G)$ satisfying $deg_H(a) = 2$, $deg_G(a) > 2$ and $deg_G(c) > 2$.*

Proof. Suppose H is 2-sparse. Since G is not 2-sparse, there exists an edge $ac \in E(G)$ such that $deg_G(a) > 2$ and $deg_G(c) > 2$. Thus by Observation 2.21, we can infer that $a,c \in V(H)$. Since H is 2-sparse, one of these vertices, either a or c should have degree 2 in H. Without loss of generality let it be a. Therefore, we have the desired edge ac satisfying the statement of the claim.

On the other hand, suppose H is not 2-sparse. Then since $I(H) = \emptyset$, by Lemma 2.17, we obtain a split $(\tilde{X},\tilde{Y},u,v)$ in $S(H)$ such that $G_{\tilde{X}}(u,v)$ is not isomorphic to $K_{2,t}$ for any $t \geq 3$. Since every split $(X,Y,a,b) \in S(G)$ is such that $G_X(a,b)$ is isomorphic to $K_{2,t}$ for some $t \geq 3$, we have $(\tilde{X},\tilde{Y},u,v) \notin S(G)$ which implies that $G_{\tilde{X}}(u,v)$ is not 2-sparse. Hence, there exists an edge $ac \in E(G_{\tilde{X}}(u,v))$ such that $deg_{G_{\tilde{X}}(u,v)}(a) > 2$ (therefore $deg_G(a) > 2$) and $deg_{G_{\tilde{X}}(u,v)}(c) > 2$ (therefore $deg_G(c) > 2$). Therefore, by Observation 2.21, we infer that $a,c \in V(H)$.

Further, $H_{\tilde{X}}(u,v)$ is 2-sparse since $(\tilde{X},\tilde{Y},u,v) \in S(H)$. Therefore, one of these vertices, either a or c should have degree 2 in $H_{\tilde{X}}(u,v)$ and also in H. Without loss of generality let the vertex be a. Hence, we have the desired edge ac satisfying the statement of the claim.

Therefore, in any case, there exists an edge $ac \in E(G)$ satisfying $deg_H(a) = 2$, $deg_G(a) > 2$ and $deg_G(c) > 2$ implying the validity of the claim. ■

By Claim 2.2, there exists a degree 2 vertex a in H which has a higher degree in G. By Observation 2.20, the vertex a was part of an isolating pair, say (a,b) in G with a corresponding split (X,Y,a,b). Since $(a,b) \in I(G)$, we have the following:

$$X = N_G^2(a,b), \qquad Y = V(G) \setminus (\{a,b\} \cup N_G^2(a,b))$$

We know that $r_{ab} \in H$. Since $deg_G(c) > 2$ and $ac \in E(G)$, we infer that $c \in Y$. Since $deg_H(a) = 2$ and $r_{ab} \in H$, c is the unique neighbor of a in Y. Hence $N_H(a) = \{c, r_{ab}\}$.

Now, consider the vertex pair (b,c). If $bc \in E(G)$, then either c is a cut vertex or bc is a chord, a contradiction since G is a 2-connected chordless graph. Thus $bc \notin E(G)$. Now, consider the following split:

$$(X',Y',b,c) \qquad \text{where} \qquad X' = X \cup \{a\}, \qquad Y' = Y \setminus c$$

Now, the claim is that $G_{X'}(b,c)$ is 2-sparse. Since $G_X(a,b)$ is 2-sparse, it is enough to prove that $deg_{G_{X'}(b,c)}(c) = 2$. Since c is the unique neighbor of a in Y and $bc \notin E(G)$, we have that $N_G(c) \cap X' = \{a\}$. Thus, $deg_{G_{X'}(b,c)}(c) = 2$ which implies that $G_{X'}(b,c)$ is 2-sparse. Therefore by definition, $(X',Y',b,c) \in S(G)$. Notice that $b \notin N_{G_{X'}(b,c)}(a)$, which implies that $G_{X'}(b,c)$ is not isomorphic to $K_{2,t}$ for any $t \geq 3$. But this is a contradiction to the fact that every split in $S(G)$, in particular (X',Y',b,c) is such that $G'_X(b,c)$ is isomorphic to $K_{2,t}$ for some $t \geq 3$.

Hence, we can conclude that there exists a split $(X,Y,a,b) \in S(G)$ such that $G_X(a,b)$ is not isomorphic to $K_{2,t}$ for any $t \geq 3$ implying that there exists a split in $S(G)$ which satisfies the condition (ii) of the lemma. Now, let $S'(G)$ be the set of all splits (X,Y,a,b) of any proper 2-cutset of G such that $G_X(a,b)$ is 2-sparse and is not isomorphic to $K_{2,t}$ for any $t \geq 3$.

Consider a split $s = (X,Y,a,b)$ in $S'(G)$ such that X is minimal. Since G is a chordless graph, every (a,b)-path in $G_X(a,b)$ is an induced path. Hence, an (a,b)-path P of maximum length in $G_X(a,b)$ is also an induced path. Since $(X,Y,a,b) \in S'(G)$, P has at least 3 edges; otherwise, the corresponding $G_X(a,b)$ is isomorphic to $K_{2,t}$ for some $t \geq 3$, a contradiction.

Hence, there exist $x, y \in X$ such that x is adjacent to a but not b and y is adjacent to b but not a. Note that both a and b have at least one neighbor each in X and at least one neighbor each in Y as per the definition of a proper 2-cutset. Now, we make the following claim about the neighbors of a and b.

Claim 2.3. *At least one among a or b has at least two neighbors in X.*

Proof. By way of contradiction, assume that both a and b have unique neighbors in X. Since the graph $G[X \cup \{a,b\}]$ is not an induced path in G which is a chordless graph, the unique neighbors of a and b in X are two distinct non-adjacent vertices in G. This implies that x and y are the unique neighbors of a and b in X respectively.

Now, consider the following split:

$$(X', Y', x, b) \quad \text{where} \quad X' = X \setminus x, \quad Y' = Y \cup \{a\}$$

Observe that the graph $G_{X'}(x,b)$ is 2-sparse since the graph $G_X(a,b)$ is 2-sparse. Further, since $deg_{G_X(a,b)}(b) = deg_{G_{X'}(x,b)}(b) = 2$, we have that $G_{X'}(x,b)$ is not isomorphic to $K_{2,t}$ for any $t \geq 3$. Thus we can infer that $(X', Y', x, b) \in S'(G)$, which is a contradiction to the minimality of s in $S'(G)$. Therefore, our assumption that both a and b have unique neighbors in X, is wrong and the claim holds. ■

Therefore, by Claim 2.3 we have that at least one among a or b has at least two neighbors in X, implying that either $deg(a) \geq 3$ or $deg(b) \geq 3$ in $G_X(a,b)$. Without loss of generality let $deg(a) \geq 3$ in $G_X(a,b)$.

If $deg(b) \geq 3$ in $G_X(a,b)$, then we are done. Otherwise, let $deg(b) = 2$ in $G_X(a,b)$. This implies that y is the unique neighbor of b in X. Now, consider the following split:

$$(X'', Y'', a, y) \quad \text{where} \quad X'' = X \setminus y, \quad Y'' = Y \cup \{b\}$$

If $G_{X''}(a,y)$ is not isomorphic to $K_{2,t}$ for any $t \geq 3$, then since $G_X(a,b)$ is 2-sparse, we have that $G_{X''}(a,y)$ is 2-sparse. This implies that $(X'', Y'', a, y) \in S'(G)$, a contradiction to the minimality of s in $S'(G)$.

Therefore, $G_{X''}(a,y)$ is isomorphic to $K_{2,t}$ for some $t \geq 3$, implying the minimality of X, as desired. Thus we can conclude that there exists a split in $S'(G)$ which satisfies property (iii) of the lemma.

Since we have obtained a split satisfying all the three desired properties by the collective arguments, this concludes the proof of Lemma 2.18. ■

The following lemma gives an upper bound on the number of edges present in a chordless graph.

Lemma 2.22. *For a chordless graph G with n vertices and m edges, we have $m \leq 2n-3$.*

Proof. Since the graph G is chordless, every subgraph of G is also chordless. Further, one can observe that the special structure present in a chordless graph as per Lemma 2.17 guarantees that every subgraph of G has a vertex of degree at most 2. Therefore, the class of chordless graphs is a proper subclass of the class of 2-degenerate graphs. Hence, G is also a 2-degenerate graph.

Consider a 2-degeneracy ordering of $V(G)$. Now, if we start from the last vertex in this ordering and keep on deleting the vertices in the reverse order of this ordering until exactly two vertices remain, we will be deleting at most two edges at every step with respect to a vertex and the number of steps is $n-2$. There can be at most one edge within the last two remaining vertices. Hence, $m \leq 2(n-2)+1 = 2n-3$ for a chordless graph. ■

Notice that C_3 is an example of a graph that attains the upper bound in Lemma 2.22. Further, Lemma 2.22 implies that the class of chordless graphs is one of the sparse graph classes where the number of edges is not too high. This property helps in proving some important results on chordless graphs.

2.4 Conclusion

The family of chordless graphs is introduced with an example illustration. The overview of the structural work done so far on the chordless graphs in the literature is presented. The extremal 2-sparse structure present in chordless graphs given by Machado et al. (2013) has been improved and presented in the form of a lemma in the chapter. This extremal structure is useful in studying the acyclic chromatic index of chordless graphs which is presented in the next chapter. This extremal 2-sparse structure might be a useful, independent and interesting result because one might study some other coloring problems on chordless graphs with the help of this structure which can constitute a good research problem.

Chapter 3

Acyclic Chromatic Index of Chordless Graphs

Recall that a chord in a graph is an edge between any two non-consecutive vertices in a cycle present in the graph. A graph is said to be chordless if it does not contain a chord. This chapter is a study of chordless graphs and aims at exactly determining the acyclic chromatic index of chordless graphs. Proving that except for a special case, the acyclic chromatic index of chordless graphs is exactly equal to Δ is the goal of the chapter. In particular, the essence of this chapter is stated in the form of a theorem provided in the next section.

3.1 The Theorem

Basavaraju and Chandran (2010) proved that the edges of any 2-degenerate graph can be acyclically colored using $\Delta+1$ colors. Recall that any chordless graph is also 2-degenerate. Therefore, $a'(G) \leq \Delta+1$ for a chordless graph G. But we already have a lower bound of Δ for the same. Thus we know that the acyclic chromatic index of a chordless graph is either Δ or $\Delta+1$. The outcome of Theorem 2.16 by Machado et al. (2013) is the exact determination of the chromatic index of chordless graphs. In a similar way, the following theorem helps in exactly determining the acyclic chromatic index of chordless graphs by pointing out the case for which it is not Δ.

Theorem 3.1. *Let G be a chordless graph with maximum degree Δ. Then $a'(G) = \Delta$, unless $\Delta = 2$ and G contains a cycle, in which case $a'(G) = \Delta+1 = 3$.*

Once the theorem is found to be valid with a proof, determining the acyclic chromatic index of a chordless graph boils down to determining the maximum degree of the given graph which can be done in linear time as in the case of chromatic index. Together with Theorem 2.16, Machado et al. (2013) also gave the sketch of a polynomial-time algorithm to color the given chordless graph with an optimum number of colors. In a similar but more involved way, the sketch of a polynomial-time algorithm to color the edges of a given chordless graph G with exactly $a'(G)$ colors, is also provided after the proof of the theorem. The running time of the proposed algorithm is proven to be $O(n^4)$.

3.2 Proof of Theorem 3.1

This section is completely about proving the core result of the chapter which is Theorem 3.1. The proof is a detailed case analysis covering all the necessary cases.

Proof. Let G be the given chordless graph with n vertices and m edges. Since the acyclic edge coloring of the components of a graph can be easily extended to the coloring of the whole graph, it is enough to prove the statement for connected graphs.

If $\Delta = 1$, then G has only one edge which requires only one color for any coloring, indicating that $a'(G) = 1 = \Delta$. If $\Delta = 2$ and G is acyclic, then since G is connected, it is a path. One can see that any path can be acyclically edge colored using 2 colors.

If $\Delta = 2$ and G is not acyclic, then since G is connected, it is a cycle. Note that for the acyclic edge coloring of a cycle, we need at least three colors by the definition of acyclic edge coloring. It is easy to see that three colors are also sufficient. Therefore, $a'(G) = 3 = \Delta + 1$. Hence, one can assume that $\Delta \geq 3$. Further, induction on the number of edges m is used in the proof, when necessary.

If G is not 2-connected, then there exists a vertex $x \in V$ which is a cut vertex. Let $C_1, C_2, ..., C_k$ be the components in $G \setminus x$. For each $i \in \{1, 2, ..., k\}$ let us define C'_i as $G[V(C_i) \cup x]$. Since each C'_i is chordless and has less than m edges, by Induction Hypothesis (I.H.), we can acyclically color all C'_i's with Δ colors since $\Delta(C'_i) \leq \Delta, \forall i$. Let this coloring be g.

Now, let us extend g to a coloring f of G. Observe that the coloring of each C'_i is independent of each other except for the edges incident on x. Hence, we can permute the colors in each C'_i appropriately so that the edges incident on x receive different colors, to get the coloring f of G. One can easily verify that the coloring f is proper and acyclic, and hence we are done. Thus, one can also assume that G is 2-connected. This also implies that $\delta(G) \geq 2$ by Lemma 1.3.

If G has an edge uv with both of its end vertices having degree 2, then we obtain a graph H from G by contracting the edge uv and thereby obtaining a new vertex k_{uv}. Let u' be the neighbor of u other than v and let v' be the neighbor of v other than u in G. Note that H is chordless and has less than m edges. Hence by I.H., the graph H can be colored using Δ colors, since $\Delta(H) \leq \Delta$. Let g be one such coloring.

Now, we try to extend the coloring g to a coloring f of G. Assign $f(uu') = g(k_{uv}u')$, $f(vv') = g(k_{uv}v')$ and assign a color other than these two colors, to the edge uv (we have at least three colors since $\Delta \geq 3$). For any other edge e in G, $f(e) = g(e)$. It is easy to see that the coloring f is acyclic and hence, we are done. Therefore, one can also assume that G does not have an edge with both of its end vertices having degree 2.

The following observations and the subsequent lemma are used multiple times further down the proof. First, let us prove these individually and then come back to the main proof later.

Observation 3.2. *Let G be a 2-connected, 2-sparse graph such that no edge is incident on two vertices of degree 2. Then G is bipartite.*

Proof. Since G is 2-sparse and 2-connected, by Lemma 1.3, we have $\delta(G) = 2$. Also since G is 2-sparse, the set of vertices $\{x \in V(G) \mid deg(x) \geq 3\}$ form an independent set. Now, since there is no edge between any two vertices of degree 2, the set of degree 2 vertices also form an independent set leading to the formation of the bipartition. Therefore, G is bipartite. ■

Observation 3.3. *Let G be a 2-connected, not 2-sparse chordless graph such that no edge is incident on two vertices of degree 2 and let (X,Y,a,b) be a split of G such that $G_X(a,b)$ is 2-sparse, $deg(a) \geq 3$ and $deg(b) \geq 3$. Then any (a,b)-path in $G_X(a,b)$ is of even length.*

Proof. Since G is a 2-connected, not 2-sparse graph, by Lemma 2.17 we have a split (X,Y,a,b) of G such that $G_X(a,b)$ is 2-sparse. Since there is no edge incident on two vertices of degree 2 in G and $deg(a) \geq 3$ and $deg(b) \geq 3$, there is no edge incident on two vertices of degree 2 in $G_X(a,b)$ as well. Hence, by Observation 3.2, $G_X(a,b)$ is bipartite with a and b on the same side of the bipartition. Therefore, any (a,b)-path should be of even length, as desired. ■

Lemma 3.4. *Let g be a partial acyclic edge coloring of a graph G using at most Δ colors. Let $P = v_1v_2\cdots v_k$ be a maximal bichromatic path with $g(v_{2i-1}v_{2i}) = \alpha$ and $g(v_{2i}v_{2i+1}) = \beta$ for $1 \leq i \leq \lfloor \frac{k-1}{2} \rfloor$. For all $v_{2i} \in V(P)$, let the neighbors of v_{2i} be all degree 2 vertices. Let $N(v_1) = \{v_2, s\}$, $deg(s) \geq 3$ and $s \notin V(P)$. Further, let the edge v_1s be not colored with respect to the coloring g. Then there exists a valid partial acyclic edge coloring f of G using at most Δ colors such that all the edges colored in g and the edge v_1s are colored in f.*

Proof. If there is a valid color for the edge v_1s, then we can assign the same to v_1s to get the required coloring f. Hence, from now on, we assume that there is no valid color for the edge v_1s. Therefore, either there is no candidate color for the edge v_1s which implies that $|F_s \cup F_{v_1}| = \Delta$ and $\alpha \notin F_{v_1s}$, or no candidate color, say η is valid for the edge v_1s which implies that $\alpha \in F_{v_1s}$ and there exists an (α, η, v_1s)-critical path in G with respect to the coloring g. Note that in the latter case, $\eta \neq \beta$ since the (α, β)-bichromatic path P does not reach the vertex s.

Now, let us obtain a coloring g' from g, by exchanging the colors α and β along the path P so that $F_{sv_1}(g') = \{\beta\}$ and $F_{v_1s}(g') = F_{v_1s}(g)$. If there is no bichromatic cycle created by this exchange, then we let coloring $g'' = g'$.

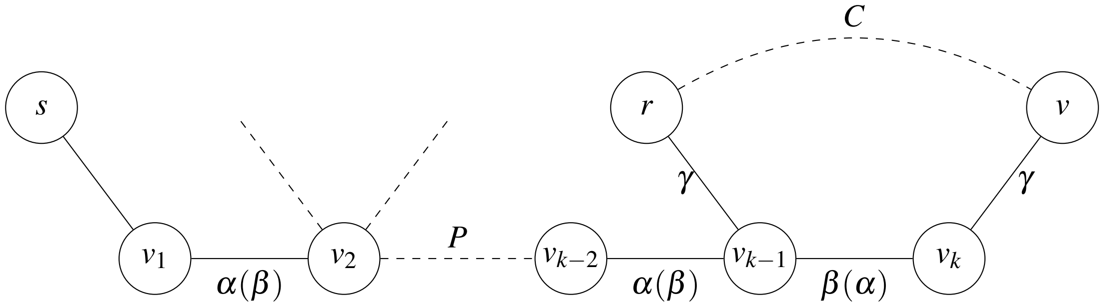

Figure 3.1: Path P and its neighborhood

Otherwise, let C be a bichromatic cycle formed because of the exchange (an instance is depicted in Figure 3.1). Since P is maximal, it is clear that C is not an (α, β)-bichromatic cycle. Note that one of the colors in C is either α or β because we did not change any color other than α and β. Therefore, since the alternate degree 2 vertices in P see only the colors α and β, we have the following:

$$V(P) \cap V(C) \subseteq \{v_1, v_2, v_{k-1}, v_k\}$$

Hence, cycle C may contain only the first or the last edge of the path P.

Further, cycle C can not contain the first edge of P since the edge v_1s is not assigned any color in the coloring g as well as in g'. Therefore, we can infer that the cycle contains only the last edge (which is $v_{k-1}v_k$) of P with $V(P) \cap V(C) = \{v_{k-1}, v_k\}$. This also implies that $deg(v_{k-1}) \geq 3$ and $deg(v_k) = 2$. We can also infer that C is the unique bichromatic cycle formed by the color exchange in P because any cycle formed has to contain the vertex v_k and $deg(v_k) = 2$.

Let r be the neighbor of v_{k-1} in the cycle C but not in the path P. Clearly, $deg(r) = 2$ and $deg(v_{k-2}) = 2$. It is easy to see that $g(v_{k-1}v_k) = \beta$. Hence, $g'(v_{k-1}v_k) = \alpha$. Note that $F_{v_{k-1}v_{k-2}} = \{\alpha\}$. Let $g'(v_{k-1}r) = \gamma$. Therefore, C is an (α, γ)-bichromatic cycle. Now, swap the colors with respect to the edges $v_{k-1}r$ and $v_{k-1}v_{k-2}$ to get a coloring g''

which clearly kills the bichromatic cycle C.

Any new bichromatic cycle C' formed should involve at least one of the edges $\{v_{k-1}r, v_{k-1}v_{k-2}\}$ and hence at least one of the vertices $\{r, v_{k-2}\}$. Note that since $F_r = \{\alpha, \beta\}$ and $F_{v_{k-2}} = \{\alpha, \gamma\}$, we can infer that the cycle C' contains α as one of the colors. Since $g''(v_{k-1}v_k) = \alpha$, the cycle C' contains the vertex v_k. But $F_{v_k} = \{\alpha, \gamma\}$, implying that C' is an (α, γ)-bichromatic cycle. This is a contradiction since the (α, γ)-bichromatic path from v_{k-1} going towards the vertex v_k ends at r. Hence, the coloring g'' is a valid partial coloring of G.

Now, we need to assign a color to the edge v_1s. Note that either $g''(v_1v_2) = \beta$ or $g''(v_1v_2) = \gamma$ with $k = 3$.

If $\alpha \notin F_{v_1s}$, then we assign the color α to the edge v_1s to get the coloring f. Notice that F_{sv_1} is either $\{\beta\}$ or $\{\gamma\}$.

If $F_{sv_1} = \{\beta\}$, the only possible bichromatic cycle with respect to the coloring c is an (α, β)-cycle. But we know that the path P ends at v_k. In the coloring g'', the (β, α)-bichromatic path from v_1 would end at v_k or v_{k-2}, both of which are not the same as s. Thus we conclude that there does not exist a (β, α, v_1s)-critical path with respect to g''.

Otherwise, we have $F_{sv_1} = \{\gamma\}$. Therefore, the only possible bichromatic cycle with respect to the coloring f is an (α, γ)-cycle. But we know that in the coloring g'', the (α, γ)-bichromatic path from v_1 would go through v_k and end at r. Further, since $deg(r) = 2$ and $deg(s) \geq 3$, $r \neq s$. Thus we can conclude that there does not exist a (γ, α, v_1s)-critical path with respect to g''. Hence, the coloring f is the required coloring for G.

On the other hand, suppose $\alpha \in F_{v_1s}$. Then we have the following:

$$|F_{v_1s}(g) \cup F_{sv_1}(g)| \leq \Delta - 1$$

Hence, there exists a candidate color η for the edge v_1s with respect to the coloring g as well as g''. Recall that $\eta \neq \beta$.

Assume that $F_{sv_1} = \{\beta\}$. Suppose the color η is not valid for the edge v_1s. Then there exists a (β, η, v_1s)-critical path Q with respect to g''. Let

$$Q = v_1v_2u_1u_2\ldots s$$

Note that $deg(u_1) = 2$ and hence $g''(u_1u_2) = \beta$. Recall that there existed an (α, η, v_1s)-critical path with respect to g which implies $g(u_1u_2) = \alpha$. The only edges that are recolored from α to β while obtaining g'' from g are the edges in P. If $k \leq 3$, then it is easy to see that P does not reach the vertex u_1.

If $k > 3$, then observe that any edge colored α in P goes from a degree 2 vertex to a higher degree vertex with respect to the coloring g. But $deg(u_1) = 2$ implying that $v_k \neq u_1$. Hence, irrespective of the value of k, the path P does not reach the vertex u_1. Therefore, $g''(u_1u_2) = \alpha$, a contradiction to the fact that $g''(u_1u_2) = \beta$. Hence, the color η is valid for the edge v_1s.

Now, we have $F_{sv_1} \neq \{\beta\}$. Then, $F_{sv_1} = \{\gamma\}$ with $k = 3$. Note that $g''(v_2r) = \beta$, $F_{v_2r} = \{\alpha\}$ and $g''(v_2v_3) = \alpha$. Recall that there existed an (α, γ)-bichromatic cycle C' involving the vertices $\{r, v_2, v_3\}$ with respect to the coloring g' which in turn implied that there was an (α, γ)-bichromatic path starting from v_1 going through r ending at v_3 with respect to the coloring g.

Suppose $\eta = \gamma$. Then, since there is no valid color for the edge v_1s with respect to g, there exists an (α, γ, v_1s)-critical path. Therefore, in the coloring g, the (α, γ)-bichromatic path starting from v_1 going through r ends at s, a contradiction since $v_3 \neq s$. Thus we can infer that $\eta \neq \gamma$.

Suppose the color η is not valid for the edge v_1s. Then there exists a (γ, η, v_1s)-critical path R with respect to g''. Let

$$R = v_1v_2w_1w_2 \dots s$$

Note that $deg(w_1) = 2$ and hence $g''(w_1w_2) = \gamma$. Recall that there existed an (α, η, v_1s)-critical path with respect to g which implies $g(w_1w_2) = \alpha$. Note that the only edge that was colored α in g which was recolored to γ in g'' is the edge v_1v_2. Therefore, $g''(w_1w_2) = \alpha$, a contradiction to the fact that $g''(w_1w_2) = \gamma$. Hence, the color η is valid for the edge v_1s in g''.

In any case, i.e., irrespective of whether $F_{sv_1} = \{\beta\}$ or not, the color η is valid for the edge v_1s in g''. Thus we can assign η to the edge v_1s to get the required valid coloring f of G which completes the proof of the lemma. ■

Now, let us get back to our main proof. Recall that we have a 2-connected graph G with $\Delta \geq 3$, $\delta \geq 2$, and no edge incident on both degree 2 vertices. Depending upon whether G is 2-sparse or not, we have the following cases:

Case 3.1. *G is 2-sparse.*

Consider any edge xy of G. Since G is 2-sparse and no edge in G incident on both degree 2 vertices, either $deg(x) = 2$ or $deg(y) = 2$. Without loss of generality assume that $deg(x) = 2$. Then we have $3 \leq deg(y) \leq \Delta$. Since $deg(x) = 2$, we have $|N(x) \setminus y| = 1$. Let x' be the neighbor of x other than y.

Let $G' = G \setminus xy$. Since G' is chordless and has less than m edges, by I.H., we can acyclically color G' with Δ colors because $\Delta(G') \leq \Delta$. Let g be an acyclic partial edge coloring of G corresponding to the subgraph G'. Let $g(xx') = \alpha$. Now, let us try to extend g to a coloring f of G.

If $F_{yx} \cap F_{xy} \neq \emptyset$, then $\alpha \in F_{yx} \cap F_{xy}$, which implies that $|F_{yx} \cup F_{xy}| \leq \Delta - 1$. Hence, there exists a candidate color γ for the edge xy. If there is no (α, γ, xy)-critical path, then γ is also valid for the edge xy by Lemma 1.16. Otherwise, there exists an (α, γ, xy)-critical path, say P. Let y' be the neighbor of y along P. Then $g(yy') = \alpha$ and $F_{yy'} = \{\gamma\}$ since $deg(y') = 2$. Let β be a color other than α and γ (β exists because $\Delta \geq 3$). Let Q be the (α, β)-maximal bichromatic path starting from the vertex x. Since $F_{yy'} = \{\gamma\}$, we are sure that Q does not reach y through an edge colored α.

If $F_{yx} \cap F_{xy} = \emptyset$, then there is no (α, β, xy)-critical path in G for any β. Therefore, if there exists at least one candidate color γ for the edge xy, then we are done since γ is also valid by Lemma 1.16. Otherwise, no color is a candidate color for the edge xy. Thus $|F_{yx} \cup F_{xy}| = \Delta$. Let β be a color other than α. Let Q be the (α, β)-maximal bichromatic path starting from the vertex x. Since there is no (α, β, xy)-critical path in G, we are sure that Q does not reach y through an edge colored α.

Hence, irrespective of whether $F_{yx} \cap F_{xy} = \emptyset$ or not, we have an (α, β)-maximal bichromatic path Q starting from x which does not reach y through an edge colored α. If Q reaches y through an edge colored β, we have an odd cycle in a 2-sparse graph G, a contradiction to Observation 3.2. Thus we can infer that Q does not reach y.

Since G is 2-sparse, the alternate vertices in Q are of degree 2 with the degree of x being equal to 2 and all the neighbors of any higher degree vertex in Q are of degree 2. Further, we have the following:

$$N(x) = \{x', y\}, \qquad deg(y) \geq 3 \qquad \text{and} \qquad y \notin V(Q).$$

Also since the edge xy is not colored in g, the path Q and the coloring g satisfy the conditions required by Lemma 3.4. Hence, we can obtain an acyclic partial edge coloring f of G with a valid color for the edge xy as per Lemma 3.4. Since all the edges of G have been colored in f, the coloring f is also an acyclic edge coloring of G.

Case 3.2. *G is not 2-sparse.*

By Lemma 2.15, G admits a proper 2-cutset with a corresponding split. Let S be the set of all splits (X, Y, a, b) of G such that $G_X(a, b)$ is 2-sparse, is not isomorphic to $K_{2,t}$ for any $t \geq 3$ and in $G_X(a, b)$, $deg(a) \geq 3$ and either $deg(b) \geq 3$ or $deg(b) = 2$ with X being minimal. By Lemma 2.18, we know that $S \neq \emptyset$.

If there exists a split (X,Y,a,b) in S such that $deg(a) \geq 3$ and $deg(b) \geq 3$, then consider such a split (X,Y,a,b). Otherwise consider a split (X,Y,a,b) in S such that $deg(a) \geq 3$ and $deg(b) = 2$ with X being minimal in S.

Let x be a vertex in $G_X(a,b)$ which is adjacent to the vertex a but not b, and let y be a vertex in $G_X(a,b)$ which is adjacent to the vertex b but not a. Note that the existence of the vertices x and y is guaranteed by Lemma 2.18. Since $deg(a) \geq 3$ in G as well as in $G_X(a,b)$, any neighbor of the vertex a in X should have degree 2 because $G_X(a,b)$ is 2-sparse. This implies that $deg(x) = 2$. By a similar argument, if b has at least two neighbors in X (i.e. if $deg(b) \geq 3$), then any neighbor of b in X (in particular, the vertex y) is of degree 2.

Now, consider the graph $G' = G \setminus xa$. Since G' is chordless and has less than m edges, by I.H., we can acyclically color the edges of G' with $\Delta(G)$ colors since $\Delta(G') \leq \Delta(G)$. Let g be an acyclic partial edge coloring of G corresponding to the subgraph G'. Since $deg(x) = 2$ and x is adjacent to the vertex a but not b, x should have a neighbor u other than the vertex a. Let $g(xu) = \alpha$. Now, let us try to extend g to a coloring f of G. The following claim is useful further down the proof.

Claim 3.1. *Let R be the (α,β)-maximal bichromatic path starting from the vertex x and let T be the (α,γ)-maximal bichromatic path starting from the vertex x. Then either R or T does not reach the vertex b.*

Proof. By way of contradiction, assume that both R and T reach the vertex b. Since G does not have an edge with both of its end vertices having degree 2 and $G_X(a,b)$ is 2-sparse, the alternate vertices in R and T are of degree 2. These paths start from a degree 2 vertex x such that:

The edge from a degree 2 vertex to a higher degree vertex is colored α and the edge from a higher degree vertex to a degree 2 vertex is colored β and γ in R and T respectively. (3.1)

Suppose R and T reach the vertex b through an edge colored α. Then by 3.1, we have that $deg(b) \geq 3$. We can infer that any neighbor of b in X is of degree 2. Let r be a neighbor of b in X such that $g(rb) = \alpha$. Since the vertex r is of degree 2, r can see at most one color in $\{\beta,\gamma\}$, a contradiction to our assumption that both R and T reach the vertex b.

Thus we can infer that R and T can not reach the vertex b through an edge colored α indicating that R and T reach the vertex b through edges colored β and γ respectively. Therefore, since b has at least one neighbor in G which is not in $G_X(a,b)$, we have that $deg(b) \geq 3$. Since we already have that $deg(a) \geq 3$, we are sure that R (or T) together with the edge (a,x) is an (a,b)-path of odd length in $G_X(a,b)$, a contradiction to Observation 3.3.

Hence, we arrive at a contradiction in any case. Thus we can infer that either R or T does not reach the vertex b, as claimed. ■

Suppose there does not exist a candidate color for the edge xa. Then, we have $|F_{ax} \cup F_{xa}| = \Delta$, which implies $F_{ax} \cap F_{xa} = \emptyset$. Let β and γ be two colors other than α. Clearly, there is no (α,β,xa)-critical path and no (α,γ,xa)-critical path in G since $F_{ax} \cap F_{xa} = \emptyset$.

Let R and T be the (α,β)-maximal bichromatic path and the (α,γ)-maximal bichromatic path starting from the vertex x. Then by Claim 3.1, either R or T does not reach the vertex b. Without loss of generality assume that R does not reach b. Since R reaches neither the vertex a nor the vertex b, R is completely in $G_X(a,b)$ which is 2-sparse.

Note that since $G_X(a,b)$ is 2-sparse, the alternate vertices in R are of degree 2 with the degree of x being equal to 2 and all the neighbors of any higher degree vertex in R are of degree 2. Further, we have the following:

$$N(x) = \{u,a\}, \qquad deg(a) \geq 3 \qquad \text{and} \qquad a \notin V(R).$$

Also since the edge xa is not colored in g, the path R and the coloring g satisfy the conditions required by Lemma 3.4. Hence, we can obtain an acyclic partial edge coloring f of G with a valid color for the edge xa as per Lemma 3.4. Since all the edges of G have been colored in f, the coloring f is also an acyclic edge coloring of G.

On the other hand, suppose there exists a candidate color γ for the edge xa. If there is no (α,γ,xa)-critical path in G, then γ is also valid and we are done. Hence, we can assume that there exists an (α,γ,xa)-critical path P in G. Let v be the neighbor of the vertex a along the path P. Observe that $g(av) = \alpha$ and $\gamma \in F_{av}$. Let β be a color other than α and γ. Let Q be the (α,β)-maximal bichromatic path starting from the vertex x.

Assume that Q does not reach the vertex b. If Q reaches a through an edge colored β, then we have an odd cycle in $G_X(a,b)$ which is 2-sparse, a contradiction to Observation 3.2. On the other hand, if Q reaches a through an edge colored α, then the edge $av \in Q$ which implies $v \in X$. Since $v \in N(a)$, $deg(v) = 2$. Therefore, $F_{av} = \{\beta\}$, a contradiction to the fact that $\gamma \in F_{av}$.

Thus we can infer that Q reaches neither the vertex a nor the vertex b. Therefore, Q is entirely in $G_X(a,b)$ which is 2-sparse. Hence, the alternate vertices in Q are of degree 2 with the degree of x being equal to 2 and all the neighbors of any higher degree vertex in Q are of degree 2. Further, we have the following:

$$N(x) = \{u, a\}, \qquad deg(a) \geq 3 \qquad \text{and} \qquad a \notin V(Q).$$

Also since the edge xa is not colored in g, the path Q and the coloring g satisfy the conditions required by Lemma 3.4. Hence, we obtain an acyclic partial edge coloring f of G with a valid color for the edge xa as per Lemma 3.4. Since all the edges of G have been colored in f, the coloring f is also an acyclic edge coloring of G.

Therefore, we can safely assume that Q reaches the vertex b. Note that since P is an (α, γ, xa)-critical path, it is also an (α, γ)-maximal bichromatic path starting from x. Hence by Claim 3.1, P does not reach the vertex b implying that P is entirely in $G_X(a,b)$ which is 2-sparse. Therefore, any edge from $G_Y(a,b)$ incident on the vertex a can not be colored α. Let z and w be the successors of the vertex u along P and Q respectively.

Suppose $w \neq b$. Now, we perform a color exchange with respect to the edges uz and uw in the coloring g to obtain a partial coloring g' of G corresponding to the subgraph $G \setminus xa$. Since $F_{uw} = F_{uz} = \{\alpha\}$, the color exchange is valid. Note that by this color exchange, we have removed the (α, γ, xa)-critical path, and since $F_{uw} = \{\alpha\}$, there is no new (α, γ, xa)-critical path formed. Therefore, we can infer that γ is a valid color for the edge xa in g'. Thus we can obtain an acyclic edge coloring f of G from g', by assigning the color γ to the edge xa.

On the other hand, suppose $w = b$. This implies the following:

$$y = u, \qquad deg(b) = 2 \qquad \text{and} \qquad g(yb) = g(uw) = \beta.$$

Thus y is the unique neighbor of b in X, implying that (X', Y', a, y) with $X' = X \setminus y$ and $Y' = Y \cup \{b\}$ is a split in G. Note that $deg(a) \geq 3$ and $deg(y) \geq 3$. Further, the graph $G_{X'}(a,y)$ is 2-sparse, since the graph $G_X(a,b)$ is 2-sparse. Therefore, if $G_{X'}(a,y)$ is not isomorphic to $K_{2,t}$ for any $t \geq 3$, then $(X', Y', a, y) \in S$, a contradiction to the minimality of X in S. Hence, $G_{X'}(a,y)$ is isomorphic to $K_{2,t}$ for some $t \geq 3$.

Thus we can infer that any neighbor of y in X' is also a neighbor of a. Hence, $uz = yz$ is colored γ and since $g(av) = \gamma$ and $v \in X$, we have $z = v$. (refer Figure 3.2). Based on the value of $\Delta(G)$, the proof is further divided into the following subcases:

Case 3.2.1. $\Delta(G) \geq 4$.

In this case, we have a color $\eta \notin \{\alpha, \beta, \gamma\}$. If no edge incident on the vertex y is colored η, then we change the color of the edge xy to η and assign the color γ to the edge xa to get a coloring f of G. Since $f(az) = \alpha$, there is no new (γ, η)-bichromatic cycle formed in the coloring f because of the recoloring. Thus f is the required acyclic edge coloring of G. Hence, we can assume that there exists a vertex $k \in N(y) \cap X$ such that $g(yk) = \eta$.

Since any neighbor of y in X is also a neighbor of the vertex a, the vertices k and z are adjacent to the vertex a. Since P is an (α, γ, xa)-critical path, we have $g(az) = \alpha$ which implies that $g(ak) \neq \alpha$. Also, since γ is a candidate color for the edge xa, we have $g(ak) \neq \gamma$ (refer to Figure 3.2).

Now, we perform a color exchange with respect to the edges yx and yk in the coloring g to obtain a partial coloring g' of G corresponding to the subgraph $G \setminus xa$. The coloring g' is proper because $g'(ak) = g(ak) \neq \alpha$. Further, since $g'(ak) \neq \gamma$, there is no (α, γ, xa)-critical path with respect to g'. Since $g'(az) = \alpha$, there is no new (γ, η, xa)-critical path formed by the color exchange.

Therefore, we can infer that γ is a valid color for the edge xa in g'. Hence, we can obtain an acyclic edge coloring f of G from the coloring g', by assigning the color γ to the edge xa.

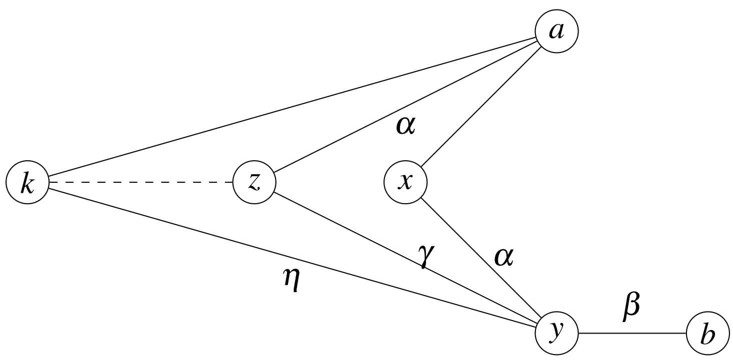

Figure 3.2: $G[X \cup \{a, b\}]$ when $\Delta \geq 4$

Case 3.2.2. $\Delta(G) = 3$.

In this case, $G_{X'}(a, y)$ is isomorphic to $K_{2,3}$. Therefore, clearly, $v = z$ is the common neighbor of a and y in X' other than x. Let p be the unique neighbor of a in Y'. Now for this subcase, let us color separately by redefining the split and the coloring.

Consider the following split:

$$(X'', Y'', p, b) \quad \text{where} \quad X'' = \{x, v, y, a\} \quad \text{and} \quad Y'' = Y \setminus p.$$

It is easy to see that the graph $G_{X''}(p, b)$ is 2-sparse since the graph $G_X(a, b)$ is 2-sparse.

Also, note that we have the following:

$$deg(x) = 2, \quad deg(v) = 2, \quad deg(a) = 3 \quad \text{and} \quad deg(y) = 3.$$

Now, consider the graph $G'' = G_{Y''}(p,b)$. Note that G'' is chordless and has less than m edges. Thus by I.H., we obtain an acyclic edge coloring g' of G'' with Δ colors, since $\Delta(G'') \leq \Delta$.

Let w be the marker vertex in $G_{Y''}(p,b)$. Let $g'(pw) = \alpha$ and $g'(bw) = \beta$. Let γ be a color other than α and β. Now, let us try to extend g' to a coloring f of G. As a first step, for any edge e in G which was a part of the graph $G'' \setminus w$, we assign $f(e) = g'(e)$. One can look at Figure 3.3 for the visualization. For the remaining edges, we perform the following color assignments:

$$\begin{aligned} f(pa) = g'(pw) &= \alpha \\ f(by) = g'(bw) &= \beta \\ f(av) &= \beta \\ f(yx) &= \alpha \\ f(ax) = f(yv) &= \gamma \end{aligned}$$

With some fundamental inspection, one can easily verify that there are no bichromatic cycles in G with respect to the coloring f, which implies that f is an acyclic edge coloring of G.

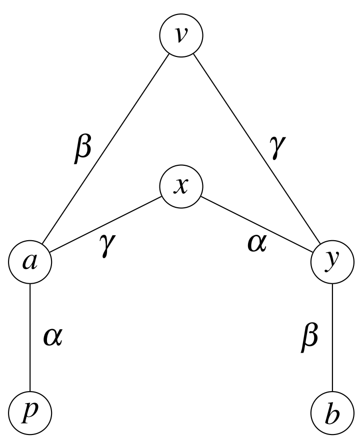

Figure 3.3: $G[X'' \cup \{p,b\}]$ when $\Delta = 3$

Therefore, in any case, we are able to color the given graph G with Δ colors, which marks the completion of the proof of Theorem 3.1. ■

The following corollary gives an equivalent condition for a chordless graph G to have an acyclic chromatic index exactly equal to Δ. One can easily see that the corollary is an immediate consequence of Theorem 3.1.

Corollary 3.5. *For a chordless graph G with maximum degree Δ, $a'(G) = \Delta$ if and only if either G is acyclic or $\Delta \neq 2$.*

3.3 Impact on Linear Arboricity

The theorem under consideration. i.e., Theorem 3.1 has a direct impact on a concept called the linear arboricity of a graph. Let us begin with the definition of the same.

Definition 3.6. *Linear arboricity of a graph G denoted by $la(G)$, is the minimum positive integer k such that the edges of G can be partitioned into k linear forests.*

It was conjectured by Akiyama et al. (1980) that for any graph G, $la(G) \leq \lceil \frac{\Delta+1}{2} \rceil$. By a result of Basavaraju and Chandran (2010), the conjecture is true for the class of 2-degenerate graphs. The following corollary that immediately follows from Theorem 3.1, highlights the effect of Theorem 3.1 on the linear arboricity of a chordless graph. The proof of the corollary is also included as a justification.

Corollary 3.7. *If G is a chordless graph with maximum degree Δ, then $la(G) = \lceil \frac{\Delta}{2} \rceil$, unless $\Delta = 2$ and G contains a cycle, in which case $la(G) = \lceil \frac{\Delta+1}{2} \rceil = 2$.*

Proof. Since the union of any two color classes in an acyclic edge coloring is a linear forest, we can infer that $la(G) \leq \left\lceil \frac{a'(G)}{2} \right\rceil$. Combining this with Theorem 3.1, we have the following:

$$la(G) \leq \left\lceil \frac{a'(G)}{2} \right\rceil = \left\lceil \frac{\Delta}{2} \right\rceil,$$

unless $\Delta = 2$ and G contains a cycle, in which case we have:

$$la(G) \leq \left\lceil \frac{a'(G)}{2} \right\rceil = \left\lceil \frac{\Delta+1}{2} \right\rceil = 2$$

Now, consider a vertex v of degree Δ in G. Any linear forest can contain at most two edges incident on the vertex v. Therefore, there exists at least $\lceil \frac{\Delta}{2} \rceil$ linear forests in any decomposition, which implies the following:

$$la(G) \geq \left\lceil \frac{\Delta}{2} \right\rceil$$

Either if $\Delta \neq 2$ or if G is acyclic, then since the upper and lower bounds match, we have $la(G) = \lceil \frac{\Delta}{2} \rceil$, as desired.

Otherwise, if $\Delta = 2$ and G has a cycle, then the edges of the cycle can not be in a single linear forest, by definition. Hence, there will be at least two linear forests in any decomposition which implies that $la(G) \geq 2$. Since the upper and lower bounds match again, we can conclude that the corollary is valid. ■

3.4 Algorithm and Complexity Analysis

In this section, we provide the sketch of a polynomial-time algorithm to acyclically color the edges of a chordless graph G with $a'(G)$ colors along with the complexity analysis of the same. As usual, n and m denote the number of vertices and edges of the input graph G respectively.

If $\Delta = 1$ or $\Delta = 2$ with G being acyclic, then G is either a matching or a set of paths. Note that since there is no cycle in both of these structures, they can be colored using $a'(G) = \Delta$ colors trivially because any proper edge coloring is sufficient.

If $\Delta = 2$ with G being non-acyclic, then each component in G is either a path or a cycle with at least one component being a cycle. Note that a cycle requires three colors and a path requires at most two colors for the acyclic edge coloring. Therefore, G can be colored using $a'(G) = \Delta + 1$ colors. Therefore, we can assume that $\Delta \geq 3$.

We can also assume that G is a 2-connected graph (and hence by Lemma 1.3, $\delta(G) \geq 2$), otherwise we can use the linear-time algorithm by Hopcroft and Tarjan (1973) to compute the 2-connected components of G, and the reconstruction of the coloring from the blocks to G is simple.

If G has an edge with both of its end vertices having degree 2, then we can contract the edge and obtain the coloring of the resultant graph. One can see that extending this coloring to the graph G is simple. Hence, we contract all such edges of G which are incident on both degree 2 end vertices.

Note that the colorings (when $\Delta \leq 2$) and all the extensions that are discussed so far, can be done in linear time. Hence, we have a 2-connected graph G with $\Delta \geq 3$ and no edge incident on both degree 2 vertices.

The algorithm for finding the chromatic index of a chordless graph in Machado et al. (2013) takes $O(n^3 m)$ time. But Lemma 2.22 implies that the time taken is actually $O(n^4)$. Now, we prove that the acyclic chromatic index of a chordless graph can also be found in $O(n^4)$ time.

First, check if graph G is 2-sparse which can be done in linear time. If G is 2-sparse, then let xy be an uncolored edge in G. Recall that we have an assumption that G does not have an edge with both of its end vertices having degree 2. Therefore, either $deg(x) = 2$ or $deg(y) = 2$. Without loss of generality assume that $deg(x) = 2$.

Now, we obtain a maximal bichromatic path Q starting from x which does not reach y (existence is proved in Case 3.1), and then obtain the partial acyclic edge coloring of the graph in which the edge xy is also colored, as per the strategy in Lemma 3.4. Note that the above-mentioned step can be done in linear time $O(m)$ since this is a mere combination of adjacent color checks and recoloring along a path. We repeat this step for each edge in the 2-sparse graph G. Hence, the coloring of a 2-sparse graph can be done in $O(m^2)$ time.

Otherwise, if G is not 2-sparse, then let (X,Y,a,b) be a split in G satisfying the properties in Lemma 2.18. Now, we choose a neighbor of the vertex a, say x and color the edge ax assuming a coloring of $G \setminus ax$. To do that, we obtain a maximal bichromatic path P starting from x which does not reach a and also does not reach b (existence follows from Claim 3.1), and then obtain the partial acyclic edge coloring of the graph in which the edge ax is also colored, as per the strategy in Lemma 3.4. This step of obtaining a color for the edge ax can be done in linear time $O(m)$ since this is a mere combination of adjacent color checks and recoloring along a path.

Now, we analyze the time required to get the desired split. The goal is to show that such a split can be obtained in $O(n^3)$ time. For each pair of non-adjacent vertices (a,b) in G, let S be the set of all components C in $G \setminus \{a,b\}$ which are 2-sparse in the graph $G[V(C) \cup \{a,b,w\}]$ (w is a marker vertex that is adjacent to both a and b).

Let C_1 be the set of all trivial components (a trivial component is an isolated vertex) in S. Let C_2 be the set of all non-trivial components in S in which every neighbor of a and every neighbor of b in $G[V(C) \cup \{a,b,w\}]$ is of degree 2. Let C_3 be the set of all non-trivial components in S in which there exists some neighbor of a or b in $G[V(C) \cup \{a,b,w\}]$ of degree at least 3.

If $|C_1 \cup C_2| \geq 2$ with $C_2 \neq \emptyset$, then let C' and C'' be any two components in $C_1 \cup C_2$ with at least one of them in C_2. Now, assign:

$$X = V(C') \cup V(C'') \qquad \text{and} \qquad Y = V(G) \setminus (X \cup \{a,b\})$$

One can see that (X,Y,a,b) is the required split with $deg(a) \geq 3$ and $deg(b) \geq 3$ in $G_X(a,b)$.

Otherwise, if $|C_1 \cup C_2| < 2$ and $C_2 \neq \emptyset$, then let $\tilde{C} \in C_2$. If a and b have at least two neighbors in $\tilde{C}$, then assign:

$$X = V(\tilde{C}) \qquad \text{and} \qquad Y = V(G) \setminus (X \cup \{a,b\})$$

Observe that (X,Y,a,b) is the required split with $deg(a) \geq 3$ and $deg(b) \geq 3$ in $G_X(a,b)$.

If both a and b have a respective unique neighbor in $\tilde{C}$, then discard the pair (a,b). Thus we can assume that exactly one among a or b has a unique neighbor in $\tilde{C}$. Without loss of generality let b has a unique neighbor b' in $\tilde{C}$. Note that since there is no edge incident on both degree 2 end vertices in G, $deg(b') \geq 3$. Now, consider the split:

$$(X',Y'a,b'), \qquad \text{where} \qquad X' = X \setminus b' \qquad \text{and} \qquad Y' = Y \cup \{b\}$$

If $G_{X'}(a,b')$ is not isomorphic to $K_{2,t}$ for any $t \geq 3$, then (X',Y',a,b') is the required split with $deg(a) \geq 3$ and $deg(b') \geq 3$ in $G_{X'}(a,b')$. Otherwise, if $G_{X'}(a,b')$ is isomorphic to $K_{2,t}$ for some $t \geq 3$, then (X,Y,a,b) is the required split with $deg(a) \geq 3$ and $deg(b) = 2$ with X being minimal.

Otherwise, if $C_2 = \emptyset$ and $C_3 \neq \emptyset$, then let C be a component in C_3. If both a and b have unique neighbors in X, then discard C and move to the next component in C_3, if it exists. Otherwise, exactly one of a or b (say a) has at least two neighbors in C. Let b'' be the unique neighbor of b in C. Now, assign:

$$X = V(C) \qquad \text{and} \qquad Y = V(G) \setminus (X \cup \{a,b\})$$

Note that since there is no edge incident on both degree 2 end vertices in G, $deg(b'') \geq 3$. Now, consider the split:

$$(X'',Y''a,b''), \qquad \text{where} \qquad X'' = X \setminus b'' \qquad \text{and} \qquad Y'' = Y \cup \{b\}$$

If $G_{X''}(a,b'')$ is not isomorphic to $K_{2,t}$ for any $t \geq 3$, then (X'',Y'',a,b'') is the required split with $deg(a) \geq 3$ and $deg(b'') \geq 3$ in $G_{X''}(a,b'')$. Otherwise, if $G_{X''}(a,b'')$ is isomorphic to $K_{2,t}$ for some $t \geq 3$, then (X,Y,a,b) is the required split with $deg(a) \geq 3$, $deg(b) = 2$ and X is minimal.

If none of the above-mentioned conditions hold, then discard the pair (a,b). Continue this for each non-adjacent pair (a,b) in G until we get the desired split satisfying the properties in Lemma 2.18.

Note that this step of verifying whether the given non-adjacent pair (a,b) has a corresponding split in G that is desired, can be done in linear time $O(n+m)$, which is the time required to characterize the components obtained by deleting the vertex pair (a,b) from G. By Lemma 2.22, we have $m = O(n)$ for a chordless graph. Therefore, since there are $O(n^2)$ such pairs, obtaining a split in G satisfying the properties in Lemma 2.18, can be done in $O(n^3)$ time.

Further, it is clear that we need to find at most n splits throughout the algorithm because in every step the size of the 2-sparse graph reduces by at least 1. Hence, the algorithm takes $O(n^4)$ time to find all the splits required throughout the algorithm.

Recall that to color each edge we require $O(m)$ time irrespective of whether the graph is 2-sparse or not. Therefore, to color all the edges we require $O(m^2)$ time. Notice that to extend the coloring of the blocks to the whole graph, we require $O(m)$ time. It is easy to see that we perform at most $O(n)$ such extensions. Therefore, the extensions take at most $O(mn)$ time.

Hence, the running time of the algorithm is $O(n^4 + m^2 + mn)$ in which the major contributing step is when we repeatedly find a split in the chordless graph G satisfying the properties in Lemma 2.18. Further, by Lemma 2.22, we have $m = O(n)$. Therefore, we can conclude that the global running time of the algorithm is $O(n^4)$.

3.5 Conclusion

The acyclic chromatic index of chordless graphs is studied extensively and arrived at a conclusion that for any chordless graph G, if $\Delta(G) = 2$ and G has a cycle, then $a'(G) = \Delta + 1$; otherwise, $a'(G) = \Delta$. This is an improvement of the result by Machado et al. (2013) who had studied the case of chromatic index of chordless graphs. After proving the proposed theorem, some immediate corollaries of the same are discussed and the effect on the linear arboricity of a graph is presented. Like Machado et al. (2013), a sketch of a polynomial-time algorithm to acyclically color the edges of a chordless graph with Δ colors (except for the simple excluded case of $\Delta(G) = 2$ and G having a cycle) is illustrated together with a justification for the proposed running time of $O(n^4)$ for the algorithm.

Chapter 4

Acyclic Chromatic Index of k-degenerate Graphs

Recall that a graph G is said to be a k-degenerate graph if every subgraph of G has a vertex of degree at most k. This chapter is a study of k-degenerate graphs and aims at improving the existing upper bound for the acyclic chromatic index of k-degenerate graphs. Let us begin with an overview of the literature and have a glance at the work that has been done so far and is relevant to the study of the acyclic chromatic index of k-degenerate graphs.

4.1 Overview

It is easy to see that the edges of a 1-degenerate graph can be acyclically colored using exactly Δ colors, using a simple greedy coloring. Hence, Conjecture 1.8 is obviously true for the family of 1-degenerate graphs. Further, Basavaraju and Chandran (2010) proved that the conjecture is true for the class of 2-degenerate graphs by giving a stronger upper bound of $\Delta+1$. Particularly, they proved that $a'(G) \leq \Delta+1$, for a 2-degenerate graph G.

Fiedorowicz (2011) proved the following theorem which gives an upper bound for the acyclic chromatic index of graphs which have a bounded number of edges relative to the number of vertices.

Theorem 4.1 (Fiedorowicz, 2011)**.** *If G is a graph that satisfies the following condition:*

$$|E(H)| \leq t|V(H)| - 1$$

for every subgraph $H \subseteq G$, where $t \geq 2$ is a given integer, and p is a constant given by $p = 2t^3 - 3t + 2$, then we have $a'(G) \leq (t-1)\Delta + p$.

The following lemma gives an upper bound on the number of edges present in a k-degenerate graph. Notice that in fact, this is a generalization of Lemma 2.22.

Lemma 4.2. *If G is a k-degenerate graph with n vertices and m edges, then $m \leq kn - 2k + 1$.*

Proof. Since the graph G is k-degenerate, we can consider a k-degeneracy ordering of $V(G)$. Now, if we start from the last vertex in this ordering and keep on deleting the vertices in the reverse order of this ordering until exactly two vertices remain, we will be deleting at most k edges at every step with respect to a vertex and the number of steps is $n-2$. There can be at most one edge within the last two remaining vertices. Therefore, $m \leq k(n-2)+1 = kn-2k+1$ for a k-degenerate graph, as desired. ■

Now, by Lemma 4.2, we infer that the class of k-degenerate graphs is a subclass of the class of graphs defined by Fiedorowicz (2011). Therefore, we can obtain an upper bound on the acyclic chromatic index of a k-degenerate graph G as

$$a'(G) \leq (k-1)\Delta + 2k^3 - 3k + 2$$

as per Fiedorowicz (2011). This is the existing upper bound on the acyclic chromatic index of a k-degenerate graph that is intended to be improved in this chapter. This improved upper bound is stated in the form of a theorem provided in the next section.

4.2 The Theorem

Theorem 4.3. *Let G be a k-degenerate graph with $k \geq 4$ and maximum degree Δ. Then $a'(G) \leq \lceil (\frac{k+1}{2})\Delta \rceil + 1$.*

Before beginning with the proof of this theorem, it is important to look at a lemma on the availability of a special edge in a k-degenerate graph. This special edge that we obtain by the following lemma is useful in the proof techniques in the next chapter.

Lemma 4.4. *If G is a k-degenerate graph, then there exists an edge xy in G such that $deg(x) \leq k$ and at most k neighbors of y have their degree strictly greater than k.*

Proof. Let G be the given k degenerate graph. By definition of G, there exists an edge xy in G such that $deg(x) \leq k$. By way of contradiction, assume that for every edge xy in G with $deg(x) \leq k$, at least $k+1$ neighbors of y have their degree strictly greater than k. Now, obtain a graph G' by deleting all the vertices of degree at most k from G. Clearly, G' has some edges in it because the edges between the vertex y and any of its higher degree neighbors will still be present in G'.

Since G' is a subgraph of G, we know that G' is also a k degenerate graph. Hence, there exists an edge uv in G' such that $deg(u) \leq k$. If the degree of u was at most k in the graph G, then by choice of G', the vertex u should have been deleted while obtaining G' from G. Since u is present in G', we are sure that the degree of u was at least $k+1$ in G. Hence, there exists a vertex w which is a neighbor of u in G but $w \notin V(G')$.

Since $w \notin V(G')$, w was deleted while obtaining G' from G. Thus $deg_G(w) \leq k$. In fact, any neighbor of u in G but not present in G' is of degree at most k in G. Therefore, the number of neighbors of u which have their degree at least $k+1$ is at most $deg_{G'}(u)$. Since $deg_{G'}(u) \leq k$, we have an edge wu in G with $deg_G(w) \leq k$ and at most k neighbors of u have their degree strictly greater than k in G, a contradiction to our initial assumption. Thus we can conclude that our assumption is wrong and the lemma is valid. ■

4.3 Proof of Theorem 4.3

This section is completely about proving our main theorem of the chapter which is Theorem 4.3. To achieve this, we make use of the minimum counterexample proof technique which is explained here. If a counterexample to some statement under consideration exists, then a minimum counterexample to the statement also exists. Conversely, if a minimum counterexample to a statement does not exist, then a counterexample to the statement also does not exist implying the validity of the statement. This method of minimum counterexample is widely used as a proof technique in the literature.

Proof. Let G be a minimum counterexample to Theorem 4.3 with respect to the number of edges. Then G is a k-degenerate graph. Let the order (number of vertices) and the size (number of edges) of G be n and m, and the maximum degree be Δ. We also have $k \geq 4$. Let us define the number p as follows:

$$p = \left\lceil \left(\frac{k+1}{2}\right)\Delta \right\rceil + 1$$

Notice that p is exactly the upper bound in Theorem 4.3 that we intend to prove. Let xy be an edge in G such that $deg(x) \leq k$. The existence of the edge xy is guaranteed because G is a k-degenerate graph.

Let $G' = G \setminus xy$, i.e., a graph formed by deleting the edge xy from G. Observe that G' is also a k-degenerate graph and has less than m edges. Since we did not add any edge or any vertex while obtaining G' from G, we have $\Delta(G') \leq \Delta(G)$. Therefore, since G is a minimum counterexample, we have an acyclic edge coloring g of G' with p colors.

Let C be the set of colors used in the coloring g, i.e., $C = \{1, 2, \ldots, p\}$.

Now, let us try to extend g to an acyclic edge coloring f of G by assigning a color to the edge xy from C, thereby arriving at a contradiction to the fact that G is a minimum counterexample. Let S be a set of vertices defined as follows:

$$S = \{u \in N(x) \setminus y \mid \exists v \in N(y) \text{ such that } g(xu) = g(yv)\}$$

Let E^* be the set of all edges in G which are incident on at least one vertex in the set $S \cup \{x, y\}$. Observe that all the edges in E^* except xy are colored in g. Let $g(E^*)$ be the set of all colors seen on the edges in E^* in the coloring g, excluding the repetitions. Now, we make the following claim about the validity of the colors which are not in $g(E^*)$ with respect to the edge xy.

Claim 4.1. *Any color that is not in $g(E^*)$, is a valid color for the edge xy in G.*

Proof. Let α be a color that is not in $g(E^*)$. Then clearly $\alpha \notin F_{xy}$ and $\alpha \notin F_{yx}$ by the choice of E^* and α. Therefore, α is a candidate color for the edge xy in G. By way of contradiction, assume that α is not a valid color for the edge xy in G.

This means that there exists a color β such that a (β, α, xy)-critical path exists in G'. Since the (β, α, xy)-critical path should be colored with the colors β and α only, there should exist three vertices x' and x'' and y' in G' distinct from x and y such that

$$g(xx') = \beta, \qquad g(x'x'') = \alpha \qquad \text{and} \qquad g(yy') = \beta$$

Observe that $x' \in N(x)$. But we also have $g(xx') = g(yy') = \beta$ which implies that x' is a vertex in S.

Therefore, $x'x'' \in E^*$. Since $g(x'x'') = \alpha$, this is a contradiction to our initial assumption that α was a color that is not in $g(E^*)$. Hence, we can conclude that our assumption was wrong and the claim holds, as desired. ■

Let C' be the set of candidate colors for the edge xy and let C'' be the set of colors in $F_{xy} \cup F_{yx}$. Observe that we have $C = C' \cup C''$. Since we have a total of p colors, it is easy to see that there exists some candidate color for the edge xy, i.e., $C' \neq \emptyset$. Note that any color in C' is not valid for the edge xy since G is a minimum counterexample. Hence, together with Claim 4.1, we have that every color in C is present in $g(E^*)$, i.e.,

$$|g(E^*)| = |C' \cup C''| = p$$

This also implies that every color in C' is present at some vertex in S. Let the set of

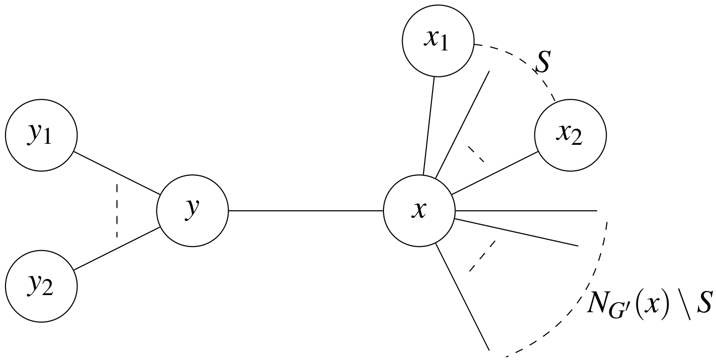

Figure 4.1: Neighborhood of the edge xy in G

colors in C' which appear only once on the edges which are incident on some vertex in $N_{G'}(x)$ be denoted by C^*. Notice that $|S| = |F_{xy} \cap F_{yx}|$. Now, the claim is that the size of the set S has a lower bound as follows:

Claim 4.2. $|S| > \frac{k-3}{2}$.

Proof. By way of contradiction, assume that the inequality is not true. Then we have the following inequality:

$$|S| = |F_{xy} \cap F_{yx}| = q \leq \frac{k-3}{2}$$

Since $deg(x) \leq k$ and there are q colors common in the sets F_{xy} and F_{yx}, we have the following inequality:

$$|C''| = |F_{xy} \cup F_{yx}| \leq \Delta - 1 + k - q - 1 = \Delta + k - q - 2$$

The remaining colors in $g(E^*)$ are seen at the vertices in S. Since $|S| = q$, we have $|C'| \leq q(\Delta - 1)$. Thus we have the following inequality:

$$\begin{aligned}|C' \cup C''| &\leq (q(\Delta - 1)) + (\Delta + k - q - 2)\\ &\leq (q+1)\Delta + k - 2q - 2\end{aligned}$$

Since $k \leq \Delta$, we have $k - 2q - 2 \leq \Delta - 2$. Further, since $q \leq \frac{k-3}{2}$, we have $q + 1 \leq \frac{k-1}{2}$. Therefore, the inequality becomes as follows:

$$\begin{aligned}|C' \cup C''| &\leq (q+1)\Delta + k - 2q - 2\\ &\leq \left(\frac{k-1}{2}\right)\Delta + \Delta - 2\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{k+1}{2}\right)\Delta - 2 \\
&< p
\end{aligned}
$$

Observe that we have obtained an inequality $|g(E^*)| = |C' \cup C''| < p$, which is a contradiction to our assumption that $|g(E^*)| = p$. Therefore, we can conclude that our assumption is wrong and the claim holds. ■

Now, since $|S| > \frac{k-3}{2}$ by Claim 4.2, we have an upper bound on the size of the set $N_{G'}(x) \setminus S$ as follows:

$$|N_{G'}(x) \setminus S| \leq k - 1 - \left(\frac{k-3}{2}\right) = \frac{k+1}{2}$$

By combining this upper bound with the fact that $|F_{xy}| \leq \Delta - 1$, we can obtain the following inequality:

$$|C''| = |F_{xy} \cup F_{yx}| \leq \Delta - 1 + \frac{k+1}{2} = \Delta + \frac{k}{2} - \frac{1}{2}$$

Further, the claim is that the cardinality of the set C^* has a lower bound of two, i.e., there exist at least two colors in C^*.

Claim 4.3. $|C^*| \geq 2$.

Proof. By way of contradiction, assume that $|C^*| = q \leq 1$. Notice that the number of candidate colors that are not valid for the edge xy is exactly given by $|C'|$. Observe that except for the q colors in C^*, any of the remaining colors in C' appear at least twice at the edges incident on the vertices in $N_{G'}(x)$.

By this observation, one can obtain the following upper bound on the number of colors in the set C':

$$|C'| \leq \frac{(k-1)(\Delta-1) - q}{2} + q$$

Now, combining the two upper bounds obtained on the sets C' and C'', one can obtain the following inequality:

$$
\begin{aligned}
|C' \cup C''| &\leq \left(\frac{(k-1)(\Delta-1) - q}{2} + q\right) + \left(\Delta + \frac{k}{2} - \frac{1}{2}\right) \\
&\leq \left(\frac{k+1}{2}\right)\Delta + \frac{q}{2} \\
&\leq \left(\frac{k+1}{2}\right)\Delta + \frac{1}{2}
\end{aligned}
$$

$$< p$$

Since $|g(E^*)| = |C' \cup C''|$, we have $|g(E^*)| < p$, a contradiction to our initial assumption that $|g(E^*)| = p$. Therefore, our assumption that $|C^*| \leq 1$ is not valid and the claim holds, as desired. ■

The next claim is about the number of vertices in S whose edges see the colors in C^*. The claim is that the edges that are colored from the colors in C^* are not incident on a single vertex in S.

Claim 4.4. *There exist at least two vertices in S whose edges see the colors in C^*.*

Proof. Every color in C^* is present on some edge incident on a vertex in S, because $C^* \subseteq C'$ and every color in C' is present on some edge incident on a vertex in S. By way of contradiction, assume that every color in C^* is present on the edges incident on a single vertex in S.

Let x' be the vertex in S such that every color in C^* is in $F_{xx'}$ and let $g(xx') = \zeta$. Let α and β be any two colors in C^*. The existence of α and β is guaranteed by Claim 4.3. Since no color in C^* is valid for the edge xy in G, for each color $\gamma \in C^*$, there exists a (ζ, γ, xy)-critical path in G.

Now, the claim (subclaim in this context since we are already inside the proof of a claim) is that there exists some candidate color for the edge xy that is not present in the set $F_{xx'}$.

Subclaim 4.4.1. $C' \setminus F_{xx'} \neq \emptyset$.

Proof. By way of contradiction, assume that $C' \setminus F_{xx'} = \emptyset$. This means that every candidate color (there exist some candidate color; we already have $C' \neq \emptyset$) for the edge xy is in the set $F_{xx'}$, implying that:

$$|C'| \leq |F_{xx'}| \leq \Delta - 1$$

Further, recall that we also have an upper bound on the number of colors in C'' as follows:

$$|C''| \leq \Delta + \frac{k}{2} - \frac{1}{2}$$

Combining these two upper bounds, we have the following inequality obtained on the size of the set $C' \cup C''$:

$$|C' \cup C''| \leq (\Delta - 1) + \left(\Delta + \frac{k}{2} - \frac{1}{2}\right)$$

$$\leq 2\Delta + \frac{k}{2} - \frac{3}{2}$$

Recall that from the hypothesis of the theorem, we have $4 \leq k \leq \Delta$. This assumption on k gives us the following lower bound for p:

$$p = \left\lceil \left(\frac{k+1}{2}\right)\Delta \right\rceil + 1 \geq \lceil 2.5\Delta \rceil + 1$$

Now, depending on the value of k, we have the following cases. Since $k \geq 4$, the cases are exhaustive.

Case 4.1. $k = 4$

In this case, by substituting the value of $k = 4$ in the upper bound obtained on the size of the set $C' \cup C''$, one can infer the following:

$$|C' \cup C''| \leq 2\Delta + \frac{4}{2} - \frac{3}{2} = 2\Delta + 0.5 < p$$

Thus we have $|C' \cup C''| < p$.

Case 4.2. $k \geq 5$

In this case, by substituting the least possible value of k, i.e., $k = 5$ in the equation of p, one can obtain the following inequality:

$$p = \left\lceil \left(\frac{k+1}{2}\right)\Delta \right\rceil + 1 \geq 3\Delta + 1$$

Since the value of k will never exceed Δ in any k-degenerate graph, the following inequality is obviously true:

$$2\Delta + \frac{k}{2} - \frac{3}{2} < 2\Delta + \frac{\Delta}{2} < 3\Delta + 1$$

Combining this with the inequality on the size of the set $C' \cup C''$, the following set of inequalities is obtained:

$$|C' \cup C''| \leq 2\Delta + \frac{k}{2} - \frac{3}{2} < 3\Delta + 1 \leq p$$

Therefore, in any case, for $k \geq 4$, we have $|C' \cup C''| < p$, a contradiction to the fact that $|C| = |C' \cup C''| = p$. Hence, our assumption that $C' \setminus F_{xx'} = \emptyset$ was wrong and the subclaim holds, as desired. ■

Now, assume that there exists a color γ in $C' \setminus F_{xx'}$ that repeats at most twice on the edges incident on the vertices in $N_{G'}(x) \setminus x'$. Since every color in C^* is in $F_{xx'}$, we have $\gamma \notin C^*$, implying that γ repeats exactly twice on the edges incident on $N_{G'}(x) \setminus x'$. Let x_1 and x_2 be the vertices in $N_{G'}(x) \setminus x'$ such that $\gamma \in F_{xx_1}$ and $\gamma \in F_{xx_2}$.

Now, let us recolor the edges xx_1 and xx_2 with α and β respectively. Observe that for any color η in C^*, we have that $\eta \notin F_{xv}$ for every $v \in N_{G'}(x) \setminus x'$. Hence, this is particularly true for the colors α and β in C^*. Therefore, the recoloring is proper. Since for every color $\eta \in C^*$, the (ζ, η)-bichromatic path in G starting from x ends at y, there is no new bichromatic cycle formed by this recoloring by Lemma 1.13. Hence, the recoloring is valid.

Since $\gamma \in C' \setminus F_{xx'}$, we have that $\gamma \notin \{\alpha, \beta\}$, which implies that γ is a candidate color for the edge xy. Further, for any vertex $v \in N_{G'}(x) \setminus \{x_1, x_2\}$, we have $\gamma \notin F_{xv}$. Therefore, γ is also valid for the edge xy, a contradiction to the fact that G is a minimum counterexample.

Hence, we can safely assume that there does not exist a color in $C' \setminus F_{xx'}$ that repeats at most twice on the edges incident on the vertices in $N_{G'}(x) \setminus x'$. By Subclaim 4.4.1, there exists some color in $C' \setminus F_{xx'}$. This means that every color in $C' \setminus F_{xx'}$ repeats at least three times on the edges incident on the vertices in $N_{G'}(x) \setminus x'$. Therefore, we can infer the following:

$$
\begin{aligned}
|C' \setminus F_{xx'}| &\leq \frac{(k-2)(\Delta-1)}{3} \\
&\leq \left(\frac{k-2}{3}\right)\Delta - \frac{k}{3} + \frac{2}{3}
\end{aligned}
$$

Recall that we already have $|C''| \leq \Delta + \frac{k}{2} - \frac{1}{2}$. Therefore, collectively we have the following inequality:

$$
\begin{aligned}
|C' \cup C''| &\leq |C' \setminus F_{xx'}| + |F_{xx'}| + |C''| \\
&\leq \left(\left(\frac{k-2}{3}\right)\Delta - \frac{k}{3} + \frac{2}{3}\right) + (\Delta - 1) + \left(\Delta + \frac{k}{2} - \frac{1}{2}\right) \\
&\leq \left(\frac{k+4}{3}\right)\Delta + \frac{k}{6} - \frac{5}{6}
\end{aligned}
$$

Notice that if at all some color in $C' \setminus F_{xx'}$ has to repeat at least three times on the edges incident on the vertices in $N_{G'}(x) \setminus x'$, then it is necessary that $|N_{G'}(x) \setminus x'| \geq 3$. This implies that $k \geq 5$. Recall that we have $3\Delta + 1$ as a lower bound for p, whenever $k \geq 5$.

Now, depending on the value of k, we have the following cases. Since $k \geq 5$, the cases are exhaustive.

Case 4.3. $k = 5$

In this case, if we substitute the value of $k = 5$ in the upper bound obtained on the size of the set $C' \cup C''$, we get the following:

$$\left(\frac{k+4}{3}\right)\Delta + \frac{k}{6} - \frac{5}{6} = \left(\frac{5+4}{3}\right)\Delta + \frac{5}{6} - \frac{5}{6} = 3\Delta$$

Therefore, after the substitution, the modified upper bound on the size of the set $C' \cup C''$ for the case of $k = 5$, becomes $|C' \cup C''| \leq 3\Delta < p$. But this is a contradiction to the fact that $|C' \cup C''| = p$.

Case 4.4. $k \geq 6$

Since $k \leq \Delta$, we have $\frac{k}{6} \leq \frac{\Delta}{6}$. By substituting this value in the upper bound obtained on the size of the set $C' \cup C''$, we get the following:

$$\begin{aligned} |C' \cup C''| &\leq \left(\frac{k+4}{3}\right)\Delta + \frac{k}{6} - \frac{5}{6} \\ &\leq \left(\frac{k+4}{3}\right)\Delta + \frac{\Delta}{6} - \frac{5}{6} \\ &\leq \left(\frac{2k+9}{6}\right)\Delta - \frac{5}{6} \end{aligned}$$

With some basic calculations, one can see that whenever $k \geq 6$, we have $\frac{2k+9}{6} \leq \frac{k+1}{2}$. Since $k \geq 6$ for this case, we can use this inequality and infer the following:

$$|C' \cup C''| \leq \left(\frac{2k+9}{6}\right)\Delta - \frac{5}{6} \leq \left(\frac{k+1}{2}\right)\Delta - \frac{5}{6} < p$$

But this is a contradiction to the fact that $|C' \cup C''| = p$.

Therefore, in any case, we arrive at a contradiction. Hence, our assumption that every color in C^* is present on the edges incident on a single vertex in S, was wrong and the claim holds. ■

Recall that by Claim 4.3, we have that $|C^*| \geq 2$. Further, by Claim 4.4, we are sure that there exist at least two vertices in S whose edges see the colors in C^*. Let x_1 and x_2 be the vertices in S such that there exist two colors γ and η in C^* satisfying:

$$\gamma \in F_{xx_1}, \quad \eta \in F_{xx_2}, \quad g(xx_1) = \alpha \quad \text{and} \quad g(xx_2) = \beta$$

Since γ and η are not valid for the edge xy in G, there exists an (α, γ, xy)-critical path in G and also a (β, η, xy)-critical path in G.

Now, let us recolor the edge xx_2 to γ. This is still a proper coloring since $\gamma \in F_{xx_1}$ implies that $\gamma \notin F_{xx_2}$ by choice of γ. Now, since the (α, γ)-bichromatic path starting from x ends at y, by Lemma 1.13, there is no new bichromatic cycle created indicating that the recoloring is valid. Observe that η becomes a valid color for the edge xy because by this recoloring we have eliminated the unique xy-critical path in G that involves the color η, i.e., the (β, η, xy)-critical path in G has been eliminated by this recoloring.

Thus we can color the edge xy with color η and extend the coloring g to a coloring f of G with p colors. But this is a contradiction to the fact that G is a minimum counterexample. Therefore, we can conclude that a minimum counterexample to Theorem 4.3 does not exist which in turn implies the validity of Theorem 4.3. ■

4.4 Conclusion

The family of k-degenerate graphs is studied extensively in the chapter. The overview of the existing works in the literature relevant to the acyclic chromatic index of k-degenerate graphs is the starting point of the chapter. Moving on, the existing upper bound on the acyclic chromatic index of a k-degenerate graph by Fiedorowicz (2011) has been improvised and presented in the form of a theorem. The structural existence of a special edge in a k-degenerate graph is also a takeaway point of the chapter. Finally, a detailed proof of the main theorem is provided to justify the proposed bound.

Chapter 5

Acyclic Chromatic Index of 3-degenerate Graphs

Recall that a graph G is said to be a 3-degenerate graph if every subgraph of G has a vertex of degree at most 3. This chapter is a study of 3-degenerate graphs and aims at improving the existing upper bound for the acyclic chromatic index of 3-degenerate graphs.

Chapter 4 concluded with an upper bound on the acyclic chromatic index of a k-degenerate graph. However, the only exclusion was the case of $k = 3$, i.e., the class of 3-degenerate graphs. Any edge xy in a 3-degenerate graph has its neighborhood as shown in Figure 5.1.

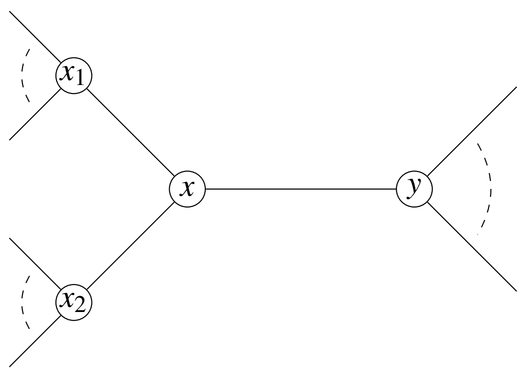

Figure 5.1: Neighborhood of any edge xy in a 3-degenerate graph

Let us consider the edges of a 3-degenerate graph in some arbitrary order and color each edge in the same order with the first available color. This type of coloring is called *greedy* coloring in the literature. Observe that a bichromatic cycle involving an edge xy requires the repetition of the color on the edge xy in F_{xx_1} or in F_{xx_2} (refer to Figure 5.1). Therefore, any color that is not available (a color that violates the condition of either proper coloring or acyclic coloring) for the edge xy is already seen on the edges incident on at least one of the vertices in $\{y, x_1, x_2\}$. Hence, the maximum number of colors that are not available for xy is $3\Delta - 1$ since xy is not yet colored and the maximum degree of any of these three vertices is Δ. Note that this argument is true for an arbitrary edge in a 3-degenerate graph.

Therefore, by using the greedy coloring algorithm, we can color any 3-degenerate graph with 3Δ colors, implying that 3Δ is an upper bound for the acyclic chromatic index of a 3-degenerate graph. However, this upper bound can also be improved by a rigorous work together with some case analysis which will be the main goal of this chapter.

5.1 The Theorem

The improved upper bound for the acyclic chromatic index of a 3-degenerate graph is presented in the form of a theorem in the section. This theorem is the core result of the chapter.

Theorem 5.1. *Let G be a* 3*-degenerate graph with maximum degree* Δ*. Then the acyclic chromatic index of G is at most* $\Delta + 5$*, i.e.,* $a'(G) \leq \Delta + 5$.

Recall that as per Conjecture 1.8, the upper bound is $\Delta + 2$ for any graph. Even though this theorem does not prove the conjecture, it brings the upper bound close to the conjectured value. Once the theorem is found to be valid with a proof, one can be sure that the acyclic chromatic index of a 3-degenerate graph is one among the six numerical values from Δ to $\Delta + 5$.

5.2 Proof of Theorem 5.1

This section is completely about proving the main theorem of the chapter which is Theorem 5.1. The principle of mathematical induction is used as a proof technique in the proof.

Proof. Let G be the given 3-degenerate graph with n vertices, m edges and maximum degree Δ. We use induction on the number of edges m of G to proceed with the proof. Let xy be an edge in G such that $deg(x) \leq 3$ and at most 3 neighbors of y have their degree strictly greater than 3. The existence of such an edge xy is guaranteed by Lemma 4.4. Further, we choose x as the neighbor of y that has the minimum degree among the vertices in $N(y)$.

Let $G' = G \setminus xy$, i.e., a graph obtained from G, by deleting the edge xy. Observe that G' is also a 3-degenerate graph and has less than m edges. Further, we have that $\Delta(G') \leq \Delta(G)$. Hence, by induction, we have an acyclic edge coloring g of G' with $\Delta + 5$ colors.

Let $N'(y)$ be the set of all neighbors of y in G' having their degree less than or equal to 3 and let $N''(y)$ be the set of all neighbors of y in G' having their degree strictly greater than 3. Notice that we have $|N''(y)| \leq 3$. Let S be the set of colors in F_y excluding those which belong to the set $\{g(yz) \mid z \in N''(y)\}$. Since $|N''(y)| \leq 3$, we are sure that $|F_y \setminus S| \leq 3$.

Now, let us try to extend g to an acyclic edge coloring f of G by assigning a color to the edge xy from the available $\Delta + 5$ colors. Further, depending on the degree of the vertex x in G, we have the following cases:

Case 5.1. $deg(x) = 1$.

In this case, let us assign the edge xy any color γ other than the colors in F_{xy}. Since $|F_{xy}| \leq \Delta - 1$ and we have $\Delta + 5$ available colors, such a color γ exists. Further, since $deg(x) = 1$, there is no bichromatic cycle created. Hence, by this assignment, we can extend g to the required coloring f of G.

Case 5.2. $deg(x) = 2$.

Let x' be the unique neighbor of x in G'. Let $g(xx') = \alpha$. If $\alpha \notin F_y$, then there is no possibility of the creation of a bichromatic cycle at all. This implies that we can assign any color satisfying the proper coloring to the edge xy and extend g to the required coloring f of G.

Therefore, we can assume that $\alpha \in F_y$. Let y' be the neighbor of y such that $g(yy') = \alpha$. Observe that the candidate colors which are not valid for the edge xy are precisely the colors in $F_{yy'}$. Further, since $\alpha \in F_{xy}$, the colors which are not candidate colors for the edge xy are the colors in F_{xy}.

Therefore, we are sure that any color that is not in the set $F_{xy} \cup F_{yy'}$ is valid for the edge xy. Depending on whether the color α is in the set S or not, we have the following cases:

Case 5.2.1. $\alpha \in S$.

Since $\alpha \in S$, the degree of the vertex y' is at most 3, i.e., $deg(y') \leq 3$. This implies that $|F_{yy'}| \leq 2$. Therefore, we have $|F_{xy} \cup F_{yy'}| \leq \Delta + 1$. We still have 4 colors available for the edge xy and by using any one of those 4 colors, we can extend g to the required coloring f of G.

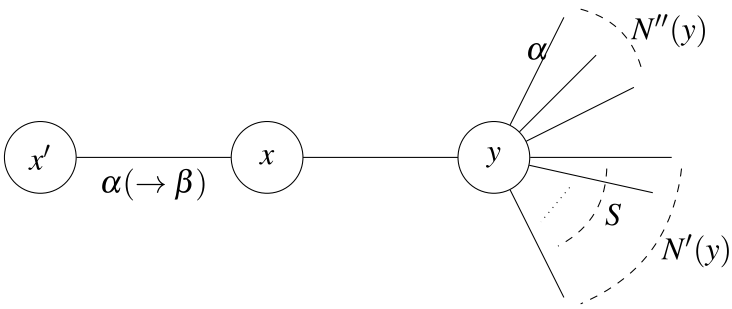

Figure 5.2: Neighborhood of the edge xy in G in Case 5.2.2

Case 5.2.2. $\alpha \notin S$.

Recall that we have $|F_{xx'}| \leq \Delta - 1$ and $|F_y \setminus S| \leq 3$. Combining both, we get the following inequality for the size of their union.

$$|F_{xx'} \cup (F_y \setminus S)| \leq \Delta + 2$$

Now, let us pick a color β that is not present in the set $F_{xx'} \cup (F_y \setminus S)$ (β exists since we have $\Delta + 5$ colors) and recolor the edge xx' from α to β. The recoloring is proper since $\beta \notin F_{xx'}$. The recoloring is valid because the edge xy is not yet colored, discarding the possibility of a bichromatic cycle.

Observe that if $\beta \notin F_y$, then since there is no possibility of a bichromatic cycle, we can assign any color satisfying the proper coloring to the edge xy and extend g to the required coloring f of G.

Thus we can assume that $\beta \in F_y$. Further, since β was picked satisfying:

$$\beta \notin F_{xx'} \cup (F_y \setminus S)$$

we can conclude that $\beta \in S$. But notice that this boils down to Case 5.2.1, and hence, we are done.

Case 5.3. $deg(x) = 3$.

Note that for this case, any neighbor of y which is not in $N''(y)$ has its degree exactly 3 by the choice of the vertex x. Let x_1 and x_2 be the neighbors of x in G other than y. Let $g(xx_1) = \alpha$ and let $g(xx_2) = \beta$. We define R to be the set of all colors from the total available $\Delta + 5$ colors that are not in $F_{xy} \cup F_{yx}$.

If any color in R is valid for the edge xy, then we can extend g to a coloring f of G by using that color. Thus we can assume that no color in R is valid for the edge xy. Depending on whether the colors $g(xx_1)$ and $g(xx_2)$ are in the set $F_{xy} \setminus S$ or not, we have the following cases:

Case 5.3.1. $\{g(xx_1), g(xx_2)\} \cap (F_{xy} \setminus S) = \emptyset$.

Recall that $g(xx_1) = \alpha$. Now, let $g(xx_2) = \beta$. For this case, no color in the set $\{\alpha, \beta\}$ is in $F_{xy} \setminus S$. This implies that every color in $\{\alpha, \beta\}$ is either present in S or not present in F_{xy}.

If no color in $\{\alpha, \beta\}$ is in S, then it also implies that no color in $\{\alpha, \beta\}$ is in F_{xy}. Since this nullifies the possibility of a bichromatic cycle being created, we can use any color satisfying the proper coloring for the edge xy and extend g to the required coloring f of G.

Otherwise, let exactly one color in $\{\alpha, \beta\}$ be in S. Without loss of generality, let $\alpha \in S$ which implies $\alpha \in F_{xy}$. Let y_m be the neighbor of y such that $g(yy_m) = \alpha$. Note that $y_m \in N'(y)$. Therefore, we have $|F_{yy_m}| \leq 2$. Further, one can see that the set of candidate colors that are not valid for the edge xy is given by F_{yy_m}, for this case. Since $\alpha \in F_{xy}$, we have $|F_{xy} \cup F_{yx}| \leq \Delta$. By combining the results, we can obtain the following upper bound for the size of the union:

$$|F_{xy} \cup F_{yx} \cup F_{yy_m}| \leq \Delta + 2$$

Since we have a total of $\Delta + 5$ colors, we have a valid color γ (any color that is not present in the set $F_{xy} \cup F_{yx} \cup F_{yy_m}$) for the edge xy since $|F_{xy} \cup F_{yx} \cup F_{yy_m}| \leq \Delta + 2$. By assigning γ to xy, we can extend g to the required coloring f of G.

Otherwise, let both the colors in $\{\alpha, \beta\}$ be in S. Hence, $\alpha \in S$ and $\beta \in S$ which implies that $\alpha \in F_{xy}$ and $\beta \in F_{xy}$. Let y_m and y_n be the neighbors of the vertex y such that $g(yy_m) = \alpha$ and $g(yy_n) = \beta$.

Since $\alpha, \beta \in S$, we have $y_m \in N'(y)$ and $y_n \in N'(y)$. Therefore, we have at most 4 colors in the set $F_{yy_m} \cup F_{yy_n}$, i.e., $|F_{yy_m} \cup F_{yy_n}| \leq 4$. Further, one can see that the set of candidate colors that are not valid for the edge xy is precisely given by $F_{yy_m} \cup F_{yy_n}$, for this case. Since both α and β are in F_{xy}, we have $|F_{xy} \cup F_{yx}| \leq \Delta - 1$. Now, by combining the results, we can obtain the following upper bound for the size of the union:

$$\begin{aligned} |F_{xy} \cup F_{yx} \cup F_{yy_m} \cup F_{yy_n}| &\leq \Delta - 1 + 2 + 2 \\ &= \Delta + 3 \end{aligned}$$

Since we have a total of $\Delta + 5$ colors, we have a valid color γ for the edge xy (any color that is not present in the set $F_{xy} \cup F_{yx} \cup F_{yy_m} \cup F_{yy_n}$). By assigning γ to xy, we can extend g to the required coloring f of G.

Since we are able to extend g to the required coloring f of G, irrespective of the number of common colors in $\{\alpha, \beta\}$ and S, we are done.

Case 5.3.2. $\{g(xx_1), g(xx_2)\} \cap (F_{xy} \setminus S) \neq \emptyset$.

In this case, at least one color in $\{\alpha, \beta\}$ is in $F_{xy} \setminus S$. Now, let us introduce the definition of a *freeable* color in S with respect to the edge xy. This concept of freeable color is very useful in the proof.

Definition 5.2. *For any vertex y' in $N'(y)$, the color $g(yy')$ in S is said to be freeable if we can recolor the edge yy' from $g(yy')$ to a color in R without forming any new bichromatic cycle.*

Observe that after this recoloring, $g(yy')$ becomes a candidate color for the edge xy in G. Now, let us make the following claim regarding the number of freeable colors in S. The result in this claim is crucial in completing the proof.

Claim 5.1. *There exists at most 2 colors in S which are not freeable.*

Proof. By way of contradiction, assume that there exist at least 3 colors in S that are not freeable. Let those colors be γ_1, γ_2 and γ_3 with the corresponding vertices $y_1, y_2, y_3 \in N'(y)$ such that:

$$g(yy_1) = \gamma_1, \quad g(yy_2) = \gamma_2 \quad \text{and} \quad g(yy_3) = \gamma_3$$

Throughout the proof of the claim, whenever we use the index i, we implicitly assume that for any i with $1 \leq i \leq 3$. Since $\gamma_i \in S$, we have $y_i \notin N''(y)$. This implies that $deg(y_i) = 3$. Let y_i' and y_i'' be the neighbors of y_i other than y and let $g(y_i y_i') = \nu_i$ and $g(y_i y_i'') = \eta_i$.

If exactly one color among α or β is in the set F_{xy}, then we have $|F_{xy} \cup F_{yx}| \leq \Delta$, which implies the following lower bound on the number of candidate colors for the edge xy:

$$|R| \geq (\Delta + 5) - \Delta = 5$$

Otherwise, if both the colors α and β are in the set F_{xy}, then we have $|F_{xy} \cup F_{yx}| \leq \Delta - 1$, which implies the following lower bound on the number of candidate colors for the edge xy:

$$|R| \geq (\Delta + 5) - (\Delta - 1) = 6$$

Recall that we have assumed that at least one among α or β belongs to F_{xy}. Therefore, in any case, there are at least five candidate colors for the edge xy, i.e., $|R| \geq 5$. Let $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ be any five colors in R.

Since $g(yy_i) = \gamma_i$ is not freeable, it means that if we recolor $g(yy_i)$ with any color in R, a new bichromatic cycle will be formed. Therefore, we have that for every color $\mu_j \in R$, either a (ν_i, μ_j, yy_i)-critical path exists or a (η_i, μ_j, yy_i)-critical path exists or both the above critical paths exist in G'. Therefore, at least three out of five yy_i-critical paths involve ν_i or at least three out of five yy_i-critical paths involve η_i. Recall that the statement is true for any i with $1 \leq i \leq 3$. Hence, without loss of generality, assume that at least three yy_1-critical paths involve ν_1, at least three yy_2-critical paths involve ν_2 and at least three yy_3-critical paths involve ν_3. Observe that at least three yy_i-critical paths that involve ν_i should reach y through a vertex $z_i \in N(y)$ with $deg(z_i) \geq 4$ which implies that $z_i \in N''(y)$ and $\nu_i \in F_{xy} \setminus S$.

Now, the claim (subclaim in the context, again) is that for any $1 \leq i, j \leq 3$, we have $\nu_i \neq \nu_j$.

Subclaim 5.1.1. *The colors ν_1, ν_2 and ν_3 are all distinct.*

Proof. By way of contradiction, without loss of generality assume that $\nu_1 = \nu_2 = \nu$ for some color ν. Let $y' \in N''(y)$ with $g(yy') = \nu$. Then there exist at least three (ν, μ_j, yy_1)-critical paths and at least three (ν, μ_k, yy_2)-critical paths for $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ in R.

Notice that we have five colors and at least six critical paths under consideration. Hence, we have a color μ_j in R such that there exists a (ν, μ_j, yy_1)-critical path and a (ν, μ_j, yy_2)-critical path. This implies that there exists a (ν, μ_j)-maximal bichromatic path starting from the vertex y, ending at the vertex y_1 and there exists a (ν, μ_j)-maximal bichromatic path starting from the vertex y, ending at the vertex y_2, a contradiction to Lemma 1.13.

Therefore, our assumption that $\nu_1 = \nu_2 = \nu$ is wrong and the subclaim holds. ■

Since any color μ_j in R with $1 \leq j \leq 5$, is not valid for the edge xy, there exists either an (α, μ_j, xy)-critical path or a (β, μ_j, xy)-critical path or both the critical paths exist in G'. Hence, there exist at least three (α, μ_j, xy)-critical paths or at least three (β, μ_j, xy)-critical paths. Without loss of generality, assume the existence of at least three (α, μ_j, xy)-critical paths.

Now, recall that we have $\nu_i \in F_{xy} \setminus S$. Since $|F_{xy} \setminus S| \leq 3$, Subclaim 5.1.1 implies that α is a color in $\{\nu_1, \nu_2, \nu_3\}$. Without loss of generality, let $\alpha = \nu_3$. Then there exists at least three (α, μ_j, yy_3)-critical paths together with the already assumed at least three (α, μ_j, xy)-critical paths for $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ in R.

Notice that we have five colors and at least six critical paths under consideration. Hence, we have a color μ_j in R such that there exists an (α, μ_j, yy_3)-critical path and an (α, μ_j, xy)-critical path. This implies that there exists an (α, μ_j)-maximal bichromatic path starting from the vertex y, ending at the vertex y_3 and there exists an (α, μ_j)-maximal bichromatic path starting from the vertex y, ending at the vertex x, a contradiction to Lemma 1.13.

Hence, our assumption that there exist at least 3 colors in S that are not freeable is wrong and the claim holds. ■

Let $S' \subset S$ be the set of all colors in S which are not freeable. Now, we define the set T to be $T = R \cup (S \setminus S')$. Further, the claim is that there are at least $\Delta - 1$ colors in the set T.

Claim 5.2. $|T| \geq \Delta - 1$.

Proof. Observe that the set T is precisely the set of all colors that are not present in $F_{yx} \cup (F_{xy} \setminus S) \cup S'$. By Claim 5.1, we have that $|S'| \leq 2$. We also have $|F_{xy} \setminus S| \leq 3$. Since $deg(x) = 3$, we have $|F_{yx}| = 2$. Precisely, $F_{yx} = \{\alpha, \beta\}$.

Recall that we have already assumed that at least one among α or β belongs to $F_{xy} \setminus S$. Therefore, there exists at most one color in $F_{yx} = \{\alpha, \beta\}$ which is not in $F_{xy} \setminus S$. With all these observations we can infer the following:

$$
\begin{aligned}
|T| &= \Delta + 5 - |F_{yx} \cup (F_{xy} \setminus S) \cup S'| \\
&\geq \Delta + 5 - (1 + 3 + 2) \\
&= \Delta - 1
\end{aligned}
$$

Thus we have the lower bound for the set T, as claimed. ■

Further, depending on how many colors among $\{g(xx_1), g(xx_2)\}$ belong to the set F_{xy}, we have the following cases:

Case 5.3.2.1. *Exactly one color in $\{g(xx_1), g(xx_2)\}$ belongs to F_{xy}.*

Recall that we have $g(xx_1) = \alpha$ and $g(xx_2) = \beta$. Without loss of generality, let $\alpha \in F_{xy}$ and $\beta \notin F_{xy}$. Let y_1 be the neighbor of y in G such that $g(yy_1) = \alpha$. Since we already have that at least one among α or β belongs to $F_{xy} \setminus S$, β not being present in F_{xy} will imply that $\alpha \notin S$. Collectively, we can infer that $\alpha \in F_{xy} \setminus S$.

Observe that since α is a color in F_{xy}, we have that $|F_{xy} \cup F_{yx}| \leq \Delta$ implying that $|R| \geq 5$. Let $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\} \in R$. If any color in R is valid for the edge xy, then we are done. Hence, we can assume that every candidate color for the edge xy in R is not valid. This implies that for every color μ_i in R, there exists an (α, μ_i, xy)-critical path in G' with respect to g.

Now, by Claim 5.2, we have $|T| \geq \Delta - 1$. If there exists a color $\zeta \in T$ such that there is no (α, ζ, xy)-critical path in G' with respect to g, then we can free the color ζ if necessary and assign ζ to the edge xy, and thereby extend g to the required coloring f of G. Therefore, we can assume that for every color ζ in T, there exists an (α, ζ, xy)-critical path in G' with respect to g.

Let us assume that $\beta \in F_{xx_1}$. Note that $|F_{xx_1}| \leq \Delta - 1$. Since $\beta \in F_{yx}$, we have that $\beta \notin T$. This together with the assumption that $\beta \in F_{xx_1}$ implies that there exists a color η such that $\eta \in T$ but $\eta \notin F_{xx_1}$. This implies that there can not be any (α, η, xy)-critical path in G' with respect to g. Since $\eta \in T$, this is a contradiction to our previous assumption that for every color ζ in T, there exists an (α, ζ, xy)-critical path in G' with respect to g.

Hence, our assumption that $\beta \in F_{xx_1}$ is not true, which implies that we are good to conclude that $\beta \notin F_{xx_1}$.

Since $|F_{xx_1} \cup \{\beta\}| \leq \Delta$ and $|F_{xy} \setminus S| \leq 3$, we are sure that there exists a color γ that satisfies the following:

$$\gamma \notin F_{xx_1} \cup \{\beta\} \cup (F_{xy} \setminus S)$$

Now, we recolor the edge xx_1 with γ. This recoloring is valid since $\beta \notin F_{xx_1}$.

If $\gamma \notin S$, then clearly $\gamma \notin F_{xy}$ which nullifies the possibility of a bichromatic cycle being created. Hence, by assigning any color to the edge xy which satisfies proper coloring, we can extend g to the required coloring f of G.

Otherwise, if $\gamma \in S$, then since $\beta \notin F_{xy}$, this boils down to Case 5.3.1, and hence, we are done.

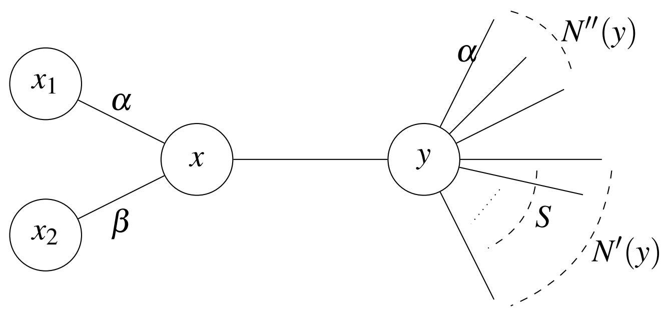

Figure 5.3: Neighborhood of the edge xy in G in Case 5.3.2.1

Case 5.3.2.2. *Both the colors in* $\{g(xx_1), g(xx_2)\}$ *belong to* F_{xy}.

Recall that we have $g(xx_1) = \alpha$ and $g(xx_2) = \beta$. Let y_1 and y_2 be the neighbors of y such that $g(yy_1) = \alpha$ and $g(yy_2) = \beta$. Since the colors α and β are seen at both the vertices x and y in G', we have $|F_{xy} \cup F_{yx}| \leq \Delta - 1$, implying that there are at least six candidate colors for the edge xy, i.e., $|R| \geq 6$. Let $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\} \in R$.

If any color in R is valid for the edge xy, then we are done. Hence, we can assume that every candidate color for the edge xy in R is not valid. This implies that for every color μ_i in R, there exists either an (α, μ_i, xy)-critical path or a (β, μ_i, xy)-critical with respect to g or both the critical paths exist in G'.

Further, if there exists a color $\zeta \in T$ such that there is no (α, ζ, xy)-critical path and there is no (β, ζ, xy)-critical path in G' with respect to g, then we can free the color ζ if necessary, and assign ζ to the edge xy, and thereby, extend g to the required coloring f of G. Hence, we can also assume that for every color ζ in T, there exists either an (α, ζ, xy)-critical path or a (β, ζ, xy)-critical path with respect to g or both the critical paths exist in G'.

Since we already have that at least one among α or β belongs to $F_{xy} \setminus S$, we are sure that at most one color in the set $\{\alpha, \beta\}$ is in S. This also implies that at least one color in the set $\{\alpha, \beta\}$ is not in S. Without loss of generality, assume that $\beta \notin S$. Then since $\beta \in F_{xy}$, we can infer that $\beta \in F_{xy} \setminus S$.

Let us assume that $\beta \in F_{xx_1}$. Note that $|F_{xx_1}| \leq \Delta - 1$. Since $\beta \in F_x$, we have $\beta \notin T$. This together with Claim 5.2 and the assumption that $\beta \in F_{xx_1}$ implies that there exists a color η such that $\eta \in T$ but $\eta \notin F_{xx_1}$. Therefore, there can not be an (α, η, xy)-critical path in G' with respect to g implying that there exists a (β, η, xy)-critical path, since $\eta \in T$. Hence, we can free the color η and recolor the edge xx_1 with η without forming any new bichromatic cycles, since the (β, η)-bichromatic path starting from the vertex x can not reach x_1 because it ends at y. Now, since $\eta \notin F_{xy}$ and $\beta \in F_{xy}$, this boils down to Case 5.3.2.1, and hence, we are done.

Now, assume that $\beta \notin F_{xx_1}$. Recall that we also have $|F_{xx_1} \cup \{\beta\}| \leq \Delta$ and $|F_{xy} \setminus S| \leq 3$. But since $\beta \in F_{xy} \setminus S$, we have the following upper bound:

$$|F_{xx_1} \cup \{\beta\} \cup (F_{xy} \setminus S)| \leq \Delta + 2$$

Further, we also have $|S'| \leq 2$ by Claim 5.1. Combining this with the previous inequality, we can obtain the following upper bound:

$$|F_{xx_1} \cup \{\beta\} \cup (F_{xy} \setminus S) \cup S'| \leq \Delta + 4$$

Finally, since we have a total of $\Delta + 5$ colors and the previous upper bound was $\Delta + 4$, we are sure that there exists a color γ satisfying the following:

$$\gamma \notin F_{xx_1} \cup \{\beta\} \cup (F_{xy} \setminus S) \cup S'$$

Now, let us free the color γ and recolor the edge xx_1 with γ. This recoloring is valid since $\beta \notin F_{xx_1}$. Moreover, since $\gamma \notin F_{xy}$, and $\beta \in F_{xy}$, this boils down to Case 5.3.2.1, and hence, we are done.

Therefore, in any case, we can extend the coloring g of G' to a coloring f of G with the same number of colors, which in turn confirms the validity of Theorem 5.1. ■

5.3 Conclusion

After a study of k-degenerate graphs with $k \neq 3$, this chapter addresses the excluded case. The family of 3-degenerate graphs is studied extensively in the chapter. The existing upper bound of 3Δ on the acyclic chromatic index of a 3-degenerate graph has been improvised and presented in the form of a theorem. To justify the proposed bound in the theorem, a detailed proof is provided that begins with the special edge in a k-degenerate graph. The existence of this special edge was proved in the previous chapter. However, the proof in the chapter involves an extensive case analysis.

Chapter 6

The Domination Number of a Graph

This chapter focuses on the study of the domination number of a graph. A classical upper bound for the domination number of a graph G having no isolated vertices is $\lfloor \frac{n}{2} \rfloor$. However, for several families of graphs, we have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$ which gives a substantially improved upper bound.

This chapter gives some conditions necessary for a graph G to have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$, and some conditions sufficient for a graph G to have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$. The chapter also includes a characterization of all connected graphs G with $\gamma(G) = \lfloor \sqrt{n} \rfloor$. One can see that the same characterization works for the graphs with $\gamma_t(G) = \lfloor \sqrt{n} \rfloor$ and for the graphs with $\gamma_c(G) = \lfloor \sqrt{n} \rfloor$ with some minor modifications. These variants of $\gamma(G)$ are defined in the next section.

6.1 Fundamentals

A subset D of V is said to be a dominating set of a graph $G = (V, E)$ if every vertex $v \in V \setminus D$ is adjacent to a vertex in D. The domination number of G denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. We omit G from the notations whenever the graph G is understood from the context. A dominating set D of G with $|D| = \gamma$ is called a γ-set of G.

Depending on the applications, there are several variants of the domination number of a graph. A couple of them are defined and discussed here.

A dominating set D of G is said to be a total dominating set if the induced subgraph $G[D]$ has no isolated vertices. Observe that $G[D]$ having no isolated vertices also mean that every vertex in G is adjacent to a vertex in D. Further, a dominating set D of G is said to be a connected dominating set if the induced subgraph $G[D]$ is a connected subgraph of G.

The total domination number and the connected domination number of a graph G

denoted by $\gamma_t(G)$ and $\gamma_c(G)$ respectively, are defined as follows:

$$\gamma_t(G) = \min\ \{|D| : D \text{ is a total dominating set of } G\}$$
$$\gamma_c(G) = \min\ \{|D| : D \text{ is a connected dominating set of } G\}$$

A total dominating set D of G with $|D| = \gamma_t$ is called a γ_t-set of G. Further, a connected dominating set D of G with $|D| = \gamma_c$ is called a γ_c-set of G. Observe that γ_t is defined only for graphs without isolated vertices and γ_c is defined only for connected graphs. Further, since any connected dominating set of a graph is also a total dominating set and any total dominating set of a graph is also a dominating set of the graph, we have the following inequality for any graph G:

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$$

Observe that the example of a dominating set of C_6 given in Figure 1.13, is not a total dominating set of C_6 since the subgraph of C_6 induced by the vertices $\{u,x\}$ has isolated vertices in it (refer to Figure 6.1).

Figure 6.1: The subgraph of C_6 induced by the vertices $\{u,x\}$

Further, the dominating set of C_6 given in Figure 1.13 is also not a connected dominating set of C_6 since the subgraph of C_6 induced by the vertices $\{u,x\}$ is not a connected graph (refer to Figure 6.1).

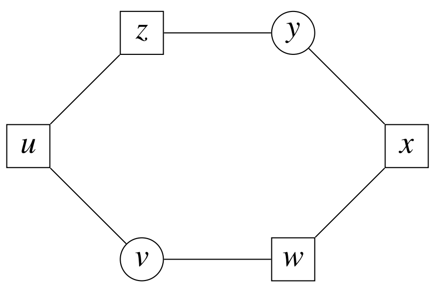

Figure 6.2: A minimum total dominating set of C_6

A minimum total dominating set of C_6 is shown in Figure 6.2. Vertices marked with a box are in the total dominating set. Observe that the subgraph of C_6 induced by the vertices $\{u, w, x, z\}$ does not have any isolated vertices in it (refer to Figure 6.3). Further, one can easily verify that by selecting at most three vertices, it is not possible to obtain a total dominating set of C_6, implying that the total domination number of C_6 is 4.

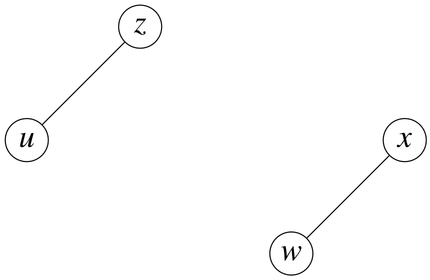

Figure 6.3: The subgraph of C_6 induced by the vertices $\{u, w, x, z\}$

Note that the total dominating set of C_6 given in Figure 6.2 is not a connected dominating set of C_6 since the subgraph of C_6 induced by the vertices $\{u, w, x, z\}$ is not a connected graph (refer to Figure 6.3).

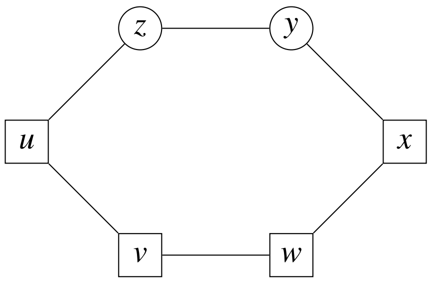

Figure 6.4: A minimum connected dominating set of C_6

A minimum connected dominating set of C_6 is shown in Figure 6.4. Vertices marked with a box are in the connected dominating set. Observe that the subgraph of C_6 induced by the vertices $\{u, v, w, x\}$ is a connected graph (refer to Figure 6.5). Further, one can easily verify that by selecting at most three vertices, it is not possible to obtain a connected dominating set of C_6, and hence the connected domination number of C_6 is 4.

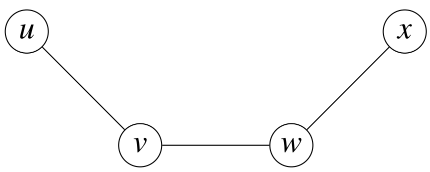

Figure 6.5: The subgraph of C_6 induced by the vertices $\{u, v, w, x\}$

6.2 Overview

Trivially, we have the range of the domination number of a graph G as $1 \leq \gamma(G) \leq n$. Further, $\gamma(G) = n$ if and only if $G = \overline{K_n}$ (complement of the complete graph K_n). A classical result of Ore (1962) gives an improved upper bound for graphs without isolated vertices.

Theorem 6.1 (Ore, 1962)**.** *If a graph G has no isolated vertices, then $\gamma(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

The concept of domination has significant applications in Network Science. Brewster, MacGillivray and Yang (2019) have investigated the concept of broadcast domination in the context of communication systems. For specific applications of dominating sets in social networks and biological networks, we refer to Kelleher and Cozzens (1988) and Nacher and Akutsu (2016). Cheng, Huang, Li, Wu and Du (2003) demonstrated that the problem of constructing a connected dominating set in a unit disc graph is the same as constructing a virtual backbone in Ad Hoc wireless networks. In all such applications, the number of vertices in the network is large and the crucial problem is to find a dominating set of small order. In this chapter, we prove that for most of the graphs, there exists a dominating set of order at most $\lfloor \sqrt{n} \rfloor$, which is quite significant in the context of the above mentioned applications.

In graph theory, *Nordhaus–Gaddum type* results are the study of the upper bound (maximum value) or the lower bound (minimum value); in general, the extremal values of the sum or the product of any parameter of a graph and its complement. An excellent survey of Nordhaus-Gaddum type results for various parameters is given by Aouchiche and Hansen (2013). In the classical paper Nordhaus and Gaddum (1956), Nordhaus and Gaddum established the first such result for the chromatic number. Nordhaus-Gaddum type results for the domination in graphs (graph parameter considered is the domination number), was published by Jaeger and Payan (1972) and is also given in the book by Haynes et al. (1998). We state only the multiplicative version (i.e., the product is considered) of the theorem.

Theorem 6.2 (Jaeger and Payan, 1972)**.** *Let G be a graph of order n with $\overline{G}$ being the complement of G. Then $\gamma(G)\gamma(\overline{G}) \leq n$.*

One can easily see that Theorem 6.2 leads to the following corollary, thereby giving a much improved upper bound for the domination number of a graph in comparison with the bound given in Theorem 6.1.

Corollary 6.3. *If G is any graph of order n with $\overline{G}$ being the complement of G, then either $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$ or $\gamma(\overline{G}) \leq \lfloor \sqrt{n} \rfloor$ or both.*

One of the important goals of the chapter is to investigate the structure of graphs satisfying $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$ and $\gamma(\overline{G}) \leq \lfloor \sqrt{n} \rfloor$. The following definitions and theorems are necessary for the discussion. Unless otherwise specified, these definitions and theorems throughout the section are taken from the book by Haynes et al. (1998).

The upper bound in the following theorem is proved in the book by Berge (1958) and the lower bound in the book by Walikar, Acharya and Sampathkumar (1979).

Theorem 6.4. *For any graph G with maximum degree Δ, we have*

$$\left\lceil \frac{n}{1+\Delta} \right\rceil \leq \gamma(G) \leq n - \Delta$$

A graph on n vertices is said to be a *star* if it is a tree with the maximum vertex degree equal to $n-1$. In other words, any edge in a star has one end vertex with its degree 1 and the other end vertex with its degree $n-1$. A graph is said to be a *galaxy* if each component in the graph is a star.

Theorem 6.5. *For any graph G with n vertices and m edges, we have*

$$n - m \leq \gamma(G) \leq n + 1 - \sqrt{1+2m}$$

Furthermore, $\gamma(G) = n - m$ if and only if G is a galaxy.

For any vertex $v \in V$, the set $N(v) = \{u \in V : uv \in E\}$ is called the open neighborhood of v. Also, $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. The following definition is crucial for the study of domination in graphs.

Definition 6.6. *Let $G = (V, E)$ be a graph of order n. Let S be a subset of V, i.e., $S \subseteq V$, and let $v \in S$. A vertex w is called a private neighbor of v if $N[w] \cap S = \{v\}$. If w is a vertex in $V \setminus S$, then w is called an external private neighbor of v.*

Let $pn[v, S]$ and $epn[v, S]$ denote the set of all private neighbors of v and the set of all external private neighbors of v, respectively. One can see that v is a private neighbor of itself if and only if v is an isolated vertex in the induced subgraph $G[S]$. The following theorem mentioned in the book by Haynes et al. (1998), is a consequence of a result proved by Ore (1962).

Theorem 6.7 (Ore, 1962)**.** *A dominating set D of the graph G is a minimal dominating set if and only if $pn[v, D] \neq \emptyset$ for all $v \in D$; If v is not an isolated vertex in the induced subgraph $G[D]$, then $epn[v, D] \neq \emptyset$.*

The following theorem is proved by Brigham, Chinn and Dutton (1988).

Theorem 6.8 (Brigham et al., 1988). *If $\gamma(\overline{G}) \geq 3$, then $diam(G) \leq 2$.*

Since the contrapositive of a statement is true if and only if the actual statement is true, one can come up with a corollary as follows:

Corollary 6.9. *If $diam(G) \geq 3$, then $\gamma(\overline{G}) \leq 2$.*

6.3 Main Results

In this section, the main structural results on the domination number of a graph are provided. To begin with, let us come up with some conditions necessary for a graph G to have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$.

Theorem 6.10. *Let G be a graph of order $n \geq 2$ and let $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$. Then*
(i). $\Delta \geq \lceil \sqrt{n} \rceil - 1$ and the bound is sharp.
(ii). $m \geq n - \lfloor \sqrt{n} \rfloor$.

Proof. (i). From Theorem 6.4, we have

$$\left\lceil \frac{n}{1+\Delta} \right\rceil \leq \gamma(G) \leq n - \Delta$$

Combining this with the fact that $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$, we can obtain the following:

$$\begin{aligned} &\left\lceil \frac{n}{1+\Delta} \right\rceil \leq \lfloor \sqrt{n} \rfloor \\ \Longrightarrow\ &\frac{n}{1+\Delta} \leq \lfloor \sqrt{n} \rfloor \\ \Longrightarrow\ &1+\Delta \geq \frac{n}{\lfloor \sqrt{n} \rfloor} \\ \Longrightarrow\ &1+\Delta \geq \lceil \sqrt{n} \rceil \\ \Longrightarrow\ &\Delta \geq \lceil \sqrt{n} \rceil - 1 \end{aligned}$$

Thus we have $\Delta \geq \lceil \sqrt{n} \rceil - 1$, as desired. Observe that for the cycle C_5, we have $\gamma = 2 = \lfloor \sqrt{n} \rfloor$ and $\Delta = 2 = \lceil \sqrt{n} \rceil - 1$. Thus we can conclude that the bound is sharp.

(ii). From Theorem 6.5, we have

$$n - m \leq \gamma(G) \leq n + 1 - \sqrt{1+2m}$$

Combining this with the fact that $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$, we get $n - m \leq \gamma(G) \leq \lfloor \sqrt{n} \rfloor$. Thus we have $n - m \leq \lfloor \sqrt{n} \rfloor$ which in turn implies that $m \geq n - \lfloor \sqrt{n} \rfloor$, implying the validity of the theorem. ■

For a given graph G, if $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$, then we are sure that these conditions in the theorem are satisfied. Now, let us proceed to characterize the connected graphs of order n with $\gamma(G) = \lfloor \sqrt{n} \rfloor$. For this purpose, let us introduce a family of graphs.

Definition 6.11. *Let H_1 be a graph of order k with $\{v_1, v_2, \ldots, v_k\}$ being the vertex set of H_1, i.e., $V(H_1) = \{v_1, v_2, \ldots, v_k\}$. Let H_2 be another graph satisfying the following:*

$$k(k-1) \leq V(H_2) \leq k^2 + k$$

Let $\{V_1, V_2, \ldots, V_k, V_{k+1}\}$ be a partition of $V(H_2)$ such that whenever v_i is not an isolated vertex in H_1, $V_i \neq \emptyset$ for any i with $1 \leq i \leq k$ (The set V_{k+1} may be empty and V_i may be empty for $1 \leq i \leq k$ if v_i is an isolated vertex in H_1).

For any subset $I \subseteq \{1, 2, \ldots, k\}$, we define two induced subgraphs as follows:

$$H_1^I = H_1[\{v_i : i \in I\}]$$
$$H_2^I = H_2[(\bigcup_{i \in I} V_i) \cup \{v \in V_{k+1} : N(v) \cap \{v_i : i \notin I\} = \emptyset\}]$$

For any two subsets $S \subseteq V(H_2)$ and $I \subseteq \{1, 2, \ldots, k\}$, we define two sets as follows:

$$N_{H_1^I}(\bar{I}) = \bigcup_{i \notin I} N_{H_1^I}(v_i) \quad \textit{and} \quad N_{H_1}(S) = \bigcup_{v \in S} N_{H_1}(v)$$

Let G be the graph obtained from the union of H_1 and H_2 satisfying the following conditions:

- (C_1). *For $1 \leq i, j \leq k$ with $i \neq j$, the vertex v_i is adjacent to all the vertices in V_i and not adjacent to any vertex in V_j.*
- (C_2). *Each vertex of V_{k+1} is adjacent to at least two vertices of H_1.*
- (C_3). *For any i, j with $1 \leq i < j \leq k$, the induced subgraph $G[V_i \cup V_j \cup V_{k+1} \cup \{v_i, v_j\}]$ is denoted by G^{ij}. The graph H_2 is such that $\gamma(G^{ij}) \geq 2$ for all i, j with $1 \leq i < j \leq k$.*
- (C_4). *For any subset $I \subseteq \{1, 2, \ldots, k\}$ with $|I| = s$, if S is a dominating set of H_2^I with $|S| \leq s - 1$, then $|N_{H_1}(S) \cup N_{H_1^I}(\bar{I})| \leq s - 1$.*

Let $\mathcal{F}$ denote the family of graphs G constructed as above. An example of a graph that belongs to the family $\mathcal{F}$ is shown in Figure 6.6.

Now that we have a new family of graphs, we proceed to prove that $\mathcal{F}$ is precisely the family of all graphs of order n with $\gamma(G) = \lfloor \sqrt{n} \rfloor$.

Theorem 6.12. *Let G be a connected graph of order n with $n \geq 4$. Then $\gamma(G) = \lfloor \sqrt{n} \rfloor$ if and only if $G \in \mathcal{F}$.*

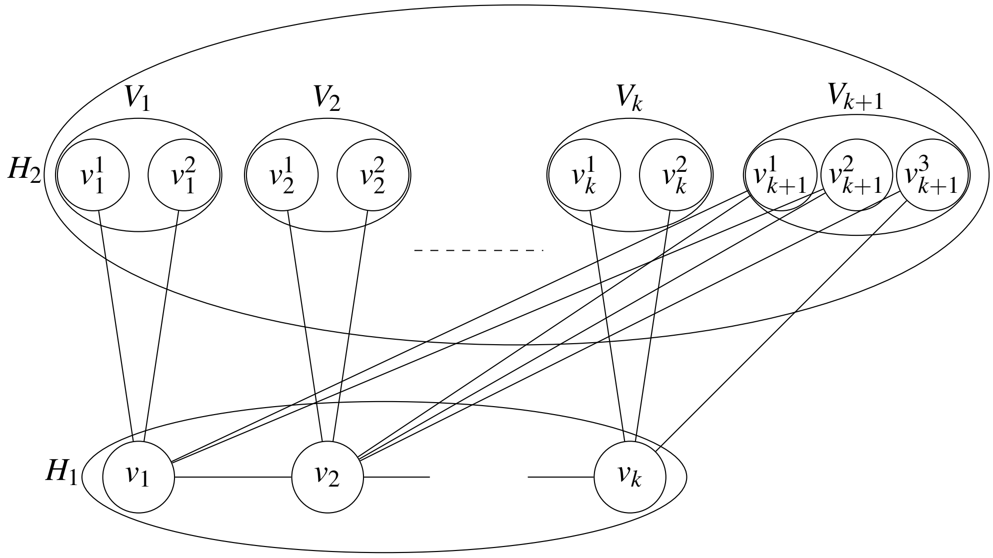

Figure 6.6: An example of a graph in the family $\mathcal{F}$

Proof. Let $k = \lfloor \sqrt{n} \rfloor$ and G be a connected graph of order n with $\gamma(G) = k$. Since $n \geq 4$, we have $k \geq 2$. Let $D = \{v_1, v_2, \ldots, v_k\}$ be a γ-set of G. Then $H_1 = G[D]$ is a graph of order k. Let $H_2 = G[V \setminus D]$.

Since D is a γ-set of G, if any $v_i \in D$ is not an isolated vertex in $G[D]$, then by Theorem 6.7, we have $epn(v_i) \neq \emptyset$. Let $V_i = epn(v_i)$, for $1 \leq i \leq k$. Let V_{k+1} be the set of all vertices in H_2 which are adjacent to at least two vertices in D. Observe that by choice of V_{k+1}, any vertex in V_{k+1} is not a private neighbor of any vertex in D. Note that the set V_{k+1} may be empty. Clearly, $\{V_1, V_2, \ldots, V_k, V_{k+1}\}$ is a partition of $V(H_2)$ and for any i with $1 \leq i \leq k$, whenever v_i is not an isolated vertex in H_1, we have $V_i \neq \emptyset$ $V_i \neq \emptyset$.

Now, if $|V(H_2)| < k(k-1)$, then we have the following:

$$n = |V(G)| = |V(H_1)| + |V(H_2)| < k + k(k-1) = k^2$$

Since $n < k^2$, we have $\lfloor \sqrt{n} \rfloor < k$, a contradiction. Thus we are sure that $|V(H_2)| \geq k(k-1)$.

Now, if $|V(H_2)| \geq k^2 + k + 1$, then we have the following:

$$n = |V(G)| = |V(H_1)| + |V(H_2)| \geq k + k^2 + k + 1 = k^2 + 2k + 1 = (k+1)^2$$

Since $n \geq (k+1)^2$, we have $\lfloor \sqrt{n} \rfloor \geq k+1$, a contradiction. Thus we are sure that $|V(H_2)| \leq k^2 + k$.

Since $epn(v_i) = V_i$, it follows that v_i is adjacent to each vertex in V_i and is not adjacent to any vertex in V_j for all $j \neq i$, and $1 \leq i, j \leq k$. Also, each vertex in V_{k+1} is adjacent to at least two vertices in the set $V(H_1)$. Hence, the conditions C_1 and C_2 of Definition 6.11 holds good for G.

Now, suppose the condition C_3 of Definition 6.11 fails for G. Then there exist i, j with $1 \leq i < j \leq k$ and $\gamma(G^{ij}) = 1$. Let $\{v\}$ be a dominating set of G^{ij}. Observe that the vertex v is adjacent to every vertex in $V_i \cup V_j \cup V_{k+1} \cup \{v_i, v_j\}$ other than itself. Now, the claim is that v is a vertex in V_{k+1}.

Claim 6.1. $v \in V_{k+1}$.

Proof. By way of contradiction, assume that $v \notin V_{k+1}$, which implies that v is a vertex in $V_i \cup V_j \cup \{v_i, v_j\}$.

If $v = v_i$ or $v = v_j$, then without loss of generality, let $v = v_i$. Notice that v can not be adjacent to any vertex in V_j because V_j is a set of private neighbors of v_j. This means that since $\{v\}$ is a dominating set of G^{ij}, we have that $V_j = \emptyset$, a contradiction since $j \neq k+1$.

If $v \in V_i$ or $v \in V_j$, then without loss of generality, let $v \in V_j$. Notice that v can not be adjacent to v_i because V_j is a set of private neighbors of v_j. This is a contradiction since $\{v\}$ being a dominating set of G^{ij}, is adjacent to every vertex in $V_i \cup V_j \cup V_{k+1} \cup \{v_i, v_j\}$ other than v itself.

In any case, we have a contradiction. Thus we can conclude that $v \in V_{k+1}$, as claimed. ■

Now, consider the set $D_1 = (D \setminus \{v_i, v_j\}) \cup \{v\}$. The next claim is that D_1 is a possible candidate for a dominating set of G.

Claim 6.2. *D_1 is a dominating set of G.*

Proof. We need to prove that every vertex that is not in $V(G) \setminus D_1$ is adjacent to a vertex in D_1. Let u be a vertex in $V(G) \setminus D_1$ chosen arbitrarily.

If $u \in V_i \cup V_j \cup V_{k+1} \cup \{v_i, v_j\}$, then by Claim 6.1, u is adjacent to the vertex v in D_1. Otherwise, we have $u \in V_r$ for some r satisfying $r \neq i$, $r \neq j$, $r \neq k+1$ and $1 \leq r \leq k$, But in that case, u is adjacent to the vertex v_r in D_1.

Therefore, in any case, u is adjacent to some vertex in D_1. Since the vertex u was chosen arbitrarily, we can conclude that D_1 is a dominating set of G, as claimed. ■

Thus we have a dominating set D_1 of G with $|D_1| = |D| - 1 = k - 1$, a contradiction to the fact that $\gamma(G) = k$. Therefore, for every i, j with $1 \leq i < j \leq k$, we have $\gamma(G^{ij}) \geq 2$, implying that the condition C_3 of Definition 6.11 holds good for G.

Now, suppose the condition C_4 of Definition 6.11 fails for G. Then there exists a subset $I \subseteq \{1, 2, \ldots, k\}$ with $|I| = s$, and a dominating set S of H_2^I with $|S| \leq s - 1$ and

$|N_{H_1}(S) \cup N_{H_1^I}(\bar{I})| \geq s$. Consider the set $S_1 = S \cup \{v_i : i \notin I\}$. The claim is that S_1 is a dominating set of G.

Claim 6.3. *S_1 is a dominating set of G.*

Proof. We need to prove that every vertex that is not in $V(G) \setminus S_1$ is adjacent to a vertex in S_1. Let v be a vertex in $V(G) \setminus S_1$ chosen arbitrarily.

If $v \in V(H_2^I)$, then v is adjacent to some vertex in S (and therefore in S_1) since S dominates all the vertices in $V(H_2^I)$. Otherwise, if $v \in V_j$ for some j with $j \notin I$, then we are sure that v is adjacent to the vertex v_j in S_1.

Otherwise, if $v \in V_{k+1}$, then since $v \notin V(H_2^I)$, we are sure that v is adjacent to a vertex v_l for some l with $l \notin I$, by choice of the graph H_2^I.

Otherwise, if $v \in V(H_1)$, then clearly v is a vertex satisfying $v \in v_i$ for some i with $i \in I$. By way of contradiction, assume that some vertex v_p for some p with $p \in I$, is not adjacent to any vertex in S_1. This implies that the vertex v_p neither belongs to the set $N_{H_1}(S)$ nor belongs to the set $N_{H_1^I}(I)$. But this will imply that

$$|N_{H_1}(S) \cup N_{H_1^I}(I)| < |I| = s$$

which is a contradiction. Therefore, every vertex v_p with $p \in I$ is adjacent to some vertex in S_1. In particular, since v is also adjacent to some vertex in S_1.

Therefore, in any case, v is adjacent to some vertex in S_1. Since the vertex v was chosen arbitrarily, we can conclude that S_1 is a dominating set of G, as claimed. ■

Now, let us try to obtain a bound on the size of S_1. Recall that we have $|S| \leq s-1$ and $|\{v_i : i \notin I\}| = k - |I|$. Thus we have the following:

$$|S_1| = |S| + |\{v_i : i \notin I\}| \leq (s-1) + (k - |I|) = (s-1) + (k-s) = k-1$$

Thus we have $|S_1| \leq k-1$ for the dominating set S_1 of G, a contradiction to the fact that $\gamma(G) = k$. Therefore, for any subset $I \subseteq \{1, 2, \ldots, k\}$ with $|I| = s$, if S is a dominating set of H_2^I with $|S| \leq s-1$, then $|N_{H_1}(S) \cup N_{H_1^I}(\bar{I})| \leq s-1$, implying that the condition C_4 of Definition 6.11 holds good for G. Hence, collectively we have concluded that $G \in \mathcal{F}$, which completes one direction of the proof.

Conversely, let $G \in \mathcal{F}$. Then, G is a graph obtained by the union on some graphs H_1 and H_2 as in Definition 6.11. Therefore, we have $n = |V(G)| = k + |V(H_2)|$ and $k(k-1) \leq |V(H_2)| \leq k^2 + k$. Hence, $k^2 \leq n < (k+1)^2$. Thus $k = \lfloor \sqrt{n} \rfloor$. One can see that $V(H_1)$ is a dominating set of G by choice of graphs H_1 and H_2. Therefore, we have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$.

Now, let D be a γ-set of G. Hence, we have $|D| \leq k$. Further, we claim that $|D| \geq k$ which in turn implies $|D| = k$. By way of contradiction, assume that $|D| \leq k-1$. If $k = 2$, then we have $|D| = 1$ which is a clear violation of condition C_3 of Definition 6.11, a contradiction since $G \in \mathcal{F}$. Thus we can assume that $k \geq 3$. Depending on the size of the intersection of the set D with $V(H_2)$, we have the following cases:

Case 6.1. $D \cap V(H_2) = \emptyset$.

In this case, $D \subseteq V(H_1)$. Since $|D| \leq k-1$, we have $D \subset V(H_1)$. But every vertex v_i in $V(H_1)$ is adjacent to a unique set of vertices V_i (set of private neighbors of v_i) for $1 \leq i \leq k$. Since $D \subset V(H_1)$, there exists a vertex $v_j \in V(H_1)$ for some j with $1 \leq j \leq k$ such that $v_j \notin D$.

Recall that $V_j \neq \emptyset$. Since $G \in \mathcal{F}$, the only neighbor of any vertex in V_j in $V(H_1)$ is v_j. But since $v_j \notin D$, we have that the vertex set V_j has no neighbors in D, implying that D is not a dominating set of G, a contradiction to the fact that D is a γ-set of G. Hence, our assumption that $|D| \leq k-1$ is wrong and we can conclude that $|D| \geq k$ (which trivially implies $|D| = |V(H_1)| = k$), as desired.

Case 6.2. $D \cap V(H_2) \neq \emptyset$.

In this case, D surely has some vertices of H_2. Let $I = \{i : i \in \{1,2,\dots,k\}, v_i \notin D\}$. Clearly, $I \subseteq \{1,2,\dots,k\}$. Let $|I| = s$. Since $D \cap V(H_2) \neq \emptyset$ and $|D| \leq k-1$, we have $|I| \neq \emptyset$. Let $S = D \setminus (\bigcup_{i \notin I} v_i)$. Now, the claim is that S dominates all the vertices in $V(H_2^I)$.

Claim 6.4. *S is a dominating set of H_2^I.*

Proof. We need to prove that every vertex that is not in $V(H_2^I) \setminus S$ is adjacent to a vertex in S. Let v be a vertex in $V(H_2^I) \setminus S$ chosen arbitrarily. Since D is a dominating set of G, there exists some vertex u in D such that v is adjacent to u in G.

Now, let us try to prove that u is also a vertex in the set S. By way of contradiction, assume that $u \notin S$. Then by choice of S, $u = v_j$ with $j \notin I$.

Suppose $v \notin V_{k+1}$, then since v is a vertex in $V(H_2^I)$, we have that $v \in V_i$ with $i \in I$. Since $j \notin I$ and $i \in I$, we have $i \neq j$. This implies that v is a private neighbor of v_i and hence, v is not adjacent to $u = v_j$, a contradiction since u was chosen to be a neighbor of v. Thus we have that $v \in V_{k+1}$.

Now, since v is a vertex in $V(H_2^I)$ and V_{k+1}, v is not in $N_{V_{k+1}}(I)$. This implies that v is not adjacent to any vertex v_l with $l \notin I$. In particular, v is not adjacent to $u = v_j$ since $j \notin I$, a contradiction since u was chosen to be a neighbor of v.

Hence our assumption that $u \notin S$ is wrong implying that u is also a vertex in S with the vertex v being adjacent to u. Since the vertex v was chosen arbitrarily, we can conclude that the set S dominates all the vertices in $V(H_2^I)$, as claimed. ■

Since $S = D \setminus (\bigcup_{i \notin I} v_i)$, we have deleted exactly $k-s$ vertices from D to obtain S. Hence, we have $|D| - |S| = k - s$. This together with the fact that $|D| \leq k-1$ implies the following inequality:

$$|S| = |D| - (k-s) \leq k-1-(k-s) = s-1$$

Thus we have $|S| \leq s-1$. Now, let v_i be a vertex with $i \in I$, chosen arbitrarily. Since D is a dominating set of G, the vertex v_i is adjacent to some vertex u in D. If $u = v_j$ with $j \notin I$, then v_i is a vertex in the set $N_{H_1^I}(I)$. Otherwise, if $u \neq v_j$ for any $j \notin I$, then u should be present in S by choice of the set S, implying that v_i is a vertex in the set $N_{H_1}(S)$.

In any case, the vertex v_i belongs to the set $N_{H_1}(S) \cup N_{H_1^I}(I)$. Notice that since the vertex v_i is chosen arbitrarily with $i \in I$, we are sure that $v_l \in N_{H_1}(S) \cup N_{H_1^I}(I)$ for any l with $l \in I$. Thus we have $|N_{H_1}(S) \cup N_{H_1^I}(I)| \geq |I| = s$, a contradiction to condition C_4 of the Definition 6.11 since $G \in \mathcal{F}$. Hence, our assumption that $|D| \leq k-1$ is wrong and we can conclude that $|D| \geq k$, as desired.

In any case, we have $|D| \geq k$. Hence, collectively we have concluded that $|D| = k = \lfloor \sqrt{n} \rfloor$, which completes the proof. ■

Now, let us come up with multiple conditions that are individually sufficient for a graph G to have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$. For a given graph G, if at least one of the following conditions holds, then we are sure that $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$.

Theorem 6.13. *Let G be a graph of order $n \geq 2$. If any one of the following conditions holds, then $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$.*

(i). *$\overline{G}$ is a disconnected graph.*

(ii). *diam$(\overline{G}) \geq 3$.*

Proof. (i). Let G be the graph given in the theorem and assume that $\overline{G}$ is a disconnected graph.

If $n = 2$, then let x and y be the two vertices in G. Since $\overline{G}$ is a disconnected graph, $xy \notin E(\overline{G})$, implying that $xy \in E(G)$. Hence, either $\{x\}$ or $\{y\}$ can be a dominating set in G. Hence, $\gamma(G) = 1 \leq \lfloor \sqrt{n} \rfloor = \lfloor \sqrt{2} \rfloor = 1$.

Otherwise, if $n = 3$, then let x, y and z be the three vertices in G. Since $\overline{G}$ is a disconnected graph on three vertices, the maximum number of edges in it is at most one. Therefore, there should exist an isolated vertex in $\overline{G}$. Without loss of generality, let this isolated vertex be x. One can see that $xy \in E(G)$ and $xz \in E(G)$, implying that $\{x\}$ is a dominating set in G. Hence, $\gamma(G) = 1 \leq \lfloor\sqrt{n}\rfloor = \lfloor\sqrt{3}\rfloor = 1$.

Now, we can safely assume that $n \geq 4$. Since $\overline{G}$ is disconnected, it has at least two components. Let u and v be two vertices in $\overline{G}$ which are in different components, say C_1 and C_2 respectively. Let w be an arbitrary vertex in G. If $w \in V(C_1)$, then since $v \notin V(C_1)$, we have $vw \in E(G)$. Otherwise, if $w \notin V(C_1)$, then since $u \in V(C_1)$, we have that $uw \in E(G)$. Therefore, since the vertex w was chosen arbitrarily, $\{u, v\}$ is a dominating set in G. Hence, $\gamma(G) \leq 2$. Since $n \geq 4$, we have $\gamma(G) \leq \lfloor\sqrt{n}\rfloor$, as desired.

(ii). Now, assume that $diam(\overline{G}) \geq 3$. Then by Corollary 6.9, we have $\gamma(G) \leq 2$. One can see that if $n \leq 3$, then $diam(\overline{G}) \geq 3$ is not achievable because we need at least 4 vertices for that to happen. Since we already have $diam(\overline{G}) \geq 3$, we can assume that $n \geq 4$. Thus we have the following inequality.

$$\gamma(G) \leq 2 \leq \lfloor\sqrt{n}\rfloor$$

We have $\gamma(G) \leq \lfloor\sqrt{n}\rfloor$, implying the validity of the theorem. ■

Now, let us proceed towards the problem of deciding for a given graph G whether $\gamma(G) \leq \lfloor\sqrt{n}\rfloor$ or $\gamma(\overline{G}) \leq \lfloor\sqrt{n}\rfloor$. To achieve this, let us define a class of graphs that is instrumental in the subsequent theorem.

Definition 6.14. *A graph G is said to be a t-complementary self-centered graph, denoted by $SCC(t)$, if both G and $\overline{G}$ are self-centered graphs satisfying the conditions $rad(G) = diam(G) = rad(\overline{G}) = diam(\overline{G}) = t$.*

Further, let us give a classification of a graph based on its domination number. This classification is useful in the subsequent theorem.

Definition 6.15. *A graph G is said to be in:*

(i). *domination type-I, if $\gamma(G) \leq \lfloor\sqrt{n}\rfloor$.*

(ii). *domination type-II, if $\gamma(\overline{G}) \leq \lfloor\sqrt{n}\rfloor$.*

Note that by Observation 6.3, if a graph G is not in domination type-I, then it has to be in domination type-II. Moreover, some graphs may be both in domination type-I and domination type-II. The following theorem helps us in deciding the domination type of a given graph.

Theorem 6.16. *If $G \notin SCC(2)$, then deciding whether G is in domination type-I or in domination type-II can be done in polynomial time.*

Proof. First, note that the determination of whether the given graph is connected or not can be done in polynomial time. Moreover, the determination of the diameter of the given graph can also be done in polynomial time since it requires only the comparison of lengths of a finite number of paths.

If G is a disconnected graph, then by Theorem 6.13, we have $\gamma(\overline{G}) \leq \lfloor\sqrt{n}\rfloor$. This implies that graph G is in domination type-II, and our statement is valid. Thus we can assume that G is connected.

If $\overline{G}$ is a disconnected graph, then by Theorem 6.13, we have $\gamma(G) \leq \lfloor\sqrt{n}\rfloor$. This implies that graph G is in domination type-I, and our statement is valid. Thus we can assume that $\overline{G}$ is connected.

Now, we have that the graphs G and $\overline{G}$ are connected. Hence, there exists a path between any pair of vertices in G as well as $\overline{G}$. Therefore, $diam(G)$ and $diam(\overline{G})$ are well-defined non-zero integers.

If $diam(G) \geq 3$, then by Theorem 6.13, we have $\gamma(\overline{G}) \leq \lfloor\sqrt{n}\rfloor$. This implies that graph G is in domination type-II, and our statement is valid. Thus we can assume that $diam(G) \leq 2$.

If $diam(\overline{G}) \geq 3$, then by Theorem 6.13, we have $\gamma(G) \leq \lfloor\sqrt{n}\rfloor$. This implies that graph G is in domination type-I, and our statement is valid. Thus we can assume that $diam(\overline{G}) \leq 2$.

If $diam(G) = 1$ or $rad(G) = 1$, then it means that distance from some vertex (say z) to all other vertices in G is exactly 1, implying that z is adjacent to all other vertices in G. Hence, $\{z\}$ is dominating set in G. Thus $\gamma(G) = 1 \leq \lfloor\sqrt{n}\rfloor$. This implies that graph G is in domination type-I, and our statement is valid. Thus we can assume that $diam(G) \neq 1$ and $rad(G) \neq 1$.

If $diam(\overline{G}) = 1$ or $rad(\overline{G}) = 1$, then it means that distance from some vertex (say w) to all other vertices in $\overline{G}$ is exactly 1, implying that w is adjacent to all other vertices in $\overline{G}$. Hence, $\{w\}$ is dominating set in $\overline{G}$. Thus $\gamma(\overline{G}) = 1 \leq \lfloor\sqrt{n}\rfloor$. This implies that graph G is in domination type-II, and our statement is valid. Thus we can assume that $diam(\overline{G}) \neq 1$ and $rad(\overline{G}) \neq 1$.

Observe that $diam(G)$ and $diam(\overline{G})$ are non-zero integers. Further, since we have $rad(G) \leq diam(G)$ by definition, we have collectively arrived at the following inference on the radius and the diameter G and $\overline{G}$ because of our assumptions.

$$rad(G) = diam(G) = diam(\overline{G}) = rad(\overline{G}) = 2$$

This implies that $G \in SCC(2)$, as desired.

Further, all the above steps yield an algorithm for deciding whether G is in domination type-I or domination type-II. For executing this algorithm, we need to calculate $diam(G)$, $diam(\overline{G})$, $rad(G)$, $rad(\overline{G})$ and the number of components in G and $\overline{G}$. One can see that these calculations can be done in polynomial time. Therefore, the whole process of deciding whether $G \notin SCC(2)$ is in domination type-I or domination type-II can be done in polynomial time, implying the validity of the theorem. ■

Moreover, there is no example of a graph in the literature that belongs to $SCC(2)$ but $\gamma(G) > \lceil \sqrt{n} \rceil$. Also, there is no example of a graph in the literature that belongs to $SCC(2)$ but $\gamma(\overline{G}) > \lceil \sqrt{n} \rceil$. Therefore, it can be conjectured as follows:

Conjecture 6.17. *If G is graph in $SCC(2)$, then $\gamma(G) \leq \lceil \sqrt{n} \rceil$ and $\gamma(\overline{G}) \leq \lceil \sqrt{n} \rceil$.*

Note that C_5 is an example of a graph in $SCC(2)$ which satisfies the conjecture.

6.4 Conclusion

The domination number of a graph is introduced with a couple of variants together with an example illustration wherever necessary. Some conditions necessary for a graph G to have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$ are presented followed by some conditions sufficient for a graph G to have $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$. A characterization of all connected graphs G with $\gamma(G) = \lfloor \sqrt{n} \rfloor$ is given with the introduction of a new class of graphs. Further, the chapter concludes with a result that for a graph G not satisfying the condition:

$$rad(G) = diam(G) = rad(\overline{G}) = diam(\overline{G}) = 2,$$

the task of deciding whether it is $\gamma(G) \leq \lfloor \sqrt{n} \rfloor$ or $\gamma(\overline{G}) \leq \lfloor \sqrt{n} \rfloor$ can be done in polynomial time, and conjectures that if graph G satisfies the above condition, then $\gamma(G) \leq \lceil \sqrt{n} \rceil$ and $\gamma(\overline{G}) \leq \lceil \sqrt{n} \rceil$.

Chapter 7

Conclusion and Future Scope

The thesis majorly deals with the study of two attributes of a graph namely the acyclic chromatic index and the domination number. The acyclic edge coloring is studied with an emphasis on the classes of chordless graphs, k-degenerate graphs and 3-degenerate graphs. With respect to the domination number of a graph, a partial attempt is made towards improving the upper bound for the same.

A study of the family of chordless graphs is presented in the thesis with an emphasis on the structure of the same. The existence of an extremal 2-sparse structure in a chordless graph is justified which marks an improvement of the structural result given by Machado et al. (2013). The benefit of this extremal structure is justified by utilizing it for the study of the acyclic chromatic index of chordless graphs. In the future, one can make use of this extremal 2-sparse structure in a chordless graph and try to study some other coloring problems on chordless graphs like strong edge coloring and star edge coloring. Particularly, one can try to come up with a better upper bound for the strong chromatic index and the star chromatic index of a chordless graph using this extremal 2-sparse structure.

The thesis presents an exact determination of the acyclic chromatic index of a chordless graph. To achieve this, the extremal 2-sparse structure present in a chordless graph is utilized. One can arrive at a conclusion that for any chordless graph G with the maximum degree $\Delta(G)$, if $\Delta(G) = 2$ and G has a cycle, then the acyclic chromatic index is $\Delta + 1$; otherwise, it is Δ, which marks an improvement of the result by Machado et al. (2013) who had given a similar result for the chromatic index of chordless graphs. A sketch of a polynomial-time algorithm to acyclically color the edges of a chordless graph with Δ colors (except for the simple excluded case of $\Delta(G) = 2$ and G having a cycle) is illustrated together with a justification for the proposed running time of $O(n^4)$ for the algorithm. In the thesis, the emphasis is mainly on obtaining only the sketch of a polynomial-time algorithm and not on the minimality of the running time or on coming up with an exact algorithm. In the future, one can make an attempt to come up with an exact algorithm to acyclically color the edges of a chordless graph with Δ colors, possibly having a better running time.

The family of k-degenerate graphs is examined and the existing upper bound on the acyclic chromatic index of a k-degenerate graph by Fiedorowicz (2011) has been improvised in the thesis. Further, the special case of the family of 3-degenerate graphs is considered and the existing trivial upper bound of 3Δ on the acyclic chromatic index of a 3-degenerate graph has been improved by proving an upper bound of $\Delta+5$. To achieve this, a concept of freeable colors is introduced which might be a useful tool for some other research scenarios. Even though the upper bound provided is not the conjectured value, it is close to that. In the future, one can have a go at either improving the existing upper bound for the acyclic chromatic index of a 3-degenerate graph or in general, a k-degenerate graph or proving Conjecture 1.8 for the same. Ideally, one can try to prove Conjecture 1.8 for an arbitrary graph G; but the existing tools and techniques in the literature do not seem sufficient in this direction.

Concerning the domination number of a graph, some necessary conditions and some sufficient conditions for a graph G to have $\gamma(G) \leq \lfloor\sqrt{n}\rfloor$ are presented followed by a characterization of all connected graphs G with $\gamma(G) = \lfloor\sqrt{n}\rfloor$, with the introduction of a new family of graphs. As a concluding remark towards the domination number of a graph, the thesis presents a conjecture that if a graph G belongs to the graph class $SCC(2)$, then G is in domination type-I as well as in domination type-II up to an addition of 1. The conjecture can be formally written in a detailed manner as follows:

Conjecture 7.1. *If G and $\overline{G}$ are self-centered graphs satisfying the condition:*

$$rad(G) = diam(G) = rad(\overline{G}) = diam(\overline{G}) = 2,$$

then $\gamma(G) \leq \lceil\sqrt{n}\rceil$ *and* $\gamma(\overline{G}) \leq \lceil\sqrt{n}\rceil$.

If the conjecture is true, then by Theorem 6.16, one can completely decide for a given graph G, whether $\gamma(G) \leq \lfloor\sqrt{n}\rfloor$ or $\gamma(\overline{G}) \leq \lfloor\sqrt{n}\rfloor$ in polynomial time up to an addition of 1. This makes the conjecture an interesting open problem that can be attempted by researchers in the future.

Having presented the desired results on the acyclic edge coloring and the domination number of a graph followed by the future scope for the research work, we have settled on a plan of action for the thesis.

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