ITERATIVE METHODS AND THEIR APPLICATIONS FOR SOLVING NON-LINEAR ILL-POSED EQUATIONS

Thesis

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by

MUHAMMED SAEED K

DEPARTMENT OF MATHEMATICAL AND COMPUTATIONAL SCIENCES

NATIONAL INSTITUTE OF TECHNOLOGY KARNATAKA, SURATHKAL

MANGALURU - 575025

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Dedicated to

All my maths teachers

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **ITERATIVE METHODS AND THEIR APPLICATIONS FOR SOLVING NON-LINEAR ILL-POSED EQUATIONS** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfilment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

(Muhammed Saeed K)
197031MA003
Department of Mathematical and Computational Sciences

Place: NITK, Surathkal
Date: December 27, 2023

CERTIFICATE

This is to *certify* that the Research Thesis entitled **ITERATIVE METHODS AND THEIR APPLICATIONS FOR SOLVING NON-LINEAR ILL-POSED EQUATIONS** submitted by **MUHAMMED SAEED K**, (Register Number: 197031MA003) as the record of the research work carried out by him is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Santhosh

Prof. Santhosh George

Research Guide

Jidesh P.

Dr. Jidesh P

Research Guide

29/12/23

Chairman - DRPC

(Signature with Date and Seal)

Chairman
DUGC / DPGC / DRPC
Dept. of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal
MANGALORE - 575 025

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Date: December 27, 2023

ABSTRACT

This thesis deals with iterative methods and their convergence for solving non-linear equations in Banach Spaces. As an application, it also deals with solving non-linear ill-posed equations in a Hilbert space setting. Under various assumptions, local and semi-local convergence analyses of some iterative schemes are studied. We have established the desired order of convergence using weaker assumptions than those available in the literature. We have also extended some of the methods efficiently. Computable radii of convergence and dynamics analysis using the basin of attractions are other highlights.

The first contribution of the thesis is the convergence analysis of a fifth-order iterative method using conditions only on the first Fréchet derivative. This increased the applicability of the method. In our second work, we used the iterative method for solving the regularized equation corresponding to a non-linear ill-posed equation. We introduced a new source condition and parameter choice strategy for the desired results. Thirdly, using Lipschitz-type assumptions on first and second derivatives instead of Taylor series expansion, we established third-order convergence of an iterative Homeier method. We further extended this method to the fifth and sixth order. Lastly, we studied another iterative method introduced by Traub. We established third-order convergence without using Taylor series expansion. We extended this method to the fifth and sixth order.

Keywords: *Iterative method, Order of convergence, Fréchet derivative, Banach space, Semi-local convergence, Recurrence relation, Ill-posed equation, Parameter choice strategy, Source condition, Fifth order convergence, Homeier method, Taylor expansion, Local convergence, Traub's method.*

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

In Natural Sciences and Engineering, it is important to solve operator equations of the form

$$F(x) = 0,$$

where F is an operator between Banach Spaces X and Y. Suppose F is a linear operator between finite-dimensional spaces. In that case, various direct methods exist (involving a finite number of arithmetic operations to reach an exact solution exempting round-off errors), like LU Decomposition, Gaussian elimination, etc. However, if F is a non-linear operator, except in rare cases, there are only a few direct methods. So, one has to consider iterative methods. Similar is the case with huge matrices or linear operators between infinite dimensional spaces. (A significant advantage of using iterative methods in the case of linear operators is that round-off errors are not given a chance to "accumulate," as they do in direct methods (see Burden and Faires (2012))).

Typically an iterative method constitutes three components, initial point, iteration function, and stopping criteria. A rigorous and generic definition is due to Ortega and Rheinboldt (1970) as follows.

Definition 1.1.1 (Ortega and Rheinboldt (1970))**.** *Let X be a Banach space, a family of* *operators* $\{G_k\}$,

$$G_k : D_k \subset X = \underbrace{X \times X \times X \ldots X}_{k+p \ times} \to X, \quad k = 0, 1, \ldots$$

defines an iterative process $\phi = (\{G_k\}, D^*, p)$, *with p initial points and with domain* $D^* \subset D_0$, *if* D^* *is not empty and if for any point* $(x_0, x_1, \ldots, x_{p+1}) \in D^*$, *the sequence* $\{x_k\}$ *generated by*

$$x_{k+1} = G_k(x_k, \ldots, x_{p+1}), \quad k = 0, 1 \ldots$$

exists, that is, if $(x_k, \ldots, x_{p+1}) \in D_k$ *for all* $k \geq 0$. *A point* x^* *such that* $\lim_{k\to\infty} x_k = x^*$ *is called a limit of the process and the set of all sequences* $\{x_k\}$ *which are generated by* ϕ *and converge to* x^* *are denoted by* $C(\phi, x^*)$.

The above definition is broad; we are only interested in particular subcases.

Definition 1.1.2 (Ortega and Rheinboldt (1970))**.** *An iterative process* $\Psi = (G_k, D^*, p)$ *is an m-step method if* $p = m$ *and if mappings of* G_k *are of the form*

$$G_k : D_k \subset X^m \longrightarrow X.$$

Definition 1.1.3 (Ortega and Rheinboldt (1970))**.** *An iterative process is sequential if the iterates are generated by*

$$x_{k+1} = G_k(x_k, \ldots, x_{k-m+1}), \quad k = 0, 1 \ldots,$$

and a sequential m-step method is stationary with iteration function G if $G_k = G, D_k = D, k = 1, 2, 3 \ldots, m = 1, 2, 3 \ldots$.

1.2 CONSTRUCTION AND CLASSIFICATION OF ITERATIVE METHODS

Iterative methods are classified based on various measures. Iterative methods are divided into stationary and non-stationary based on Definition 1.1.3. Further within sequential stationary m-step methods in $\mathbb{R}$, Traub (1964) classified as follows.

Definition 1.2.1. *Let $G : X \to X$ be the iteration function and if $x_{i+1} = G(x_i)$, x_i the previous iterate, we call this **one point iteration function**. If the iteration function is of the form, $x_{i+1} = G(x_i, x_{i-1}, \ldots, x_{i-n})$, then G is called **one-point iteration with memory** as it reuses information of past iterations. If x_{i+1} is determined by new information at $x_i, \omega_1(x_i), \ldots, \omega_k(x_i)$ and*

$$x_{i+1} = G[x_i, w_i(x_i), \ldots, w_k(x_i)].$$

*Then G is called a **multi-point iteration function**. If z_j represents $k+1$ quantities $x_j, w_1(x_j), \ldots, w_k(x_j)$ and*

$$x_{i+1} = G(z_i, z_{i-1}, \ldots, z_{i-n}).$$

*Then it is a **mult-point iteration function with memory**.*

Iterative methods are also classified based on the convergence speed into linear, quadratic, cubic, etc (see section 1.3). There are also nomenclatures based on construction techniques used to create iterative methods in the literature. These terminologies are based on techniques like interpolation, inverse interpolation, quadrature, adomain, etc.

1.3 ORDER OF CONVERGENCE

Order of convergence is a means to characterize the asymptotic 'speed' by which a sequence $\{x_n\}$ converges to its limit x^*. Early works on the order of convergence of Newton's method date back to Cauchy (see, Cauchy (1829)). There were also different attempts to characterize the speed of convergence. Some authors like Pták (1977); Potra and Ptak (1984) considered a function instead of a number to characterize the rate of convergence. There are several definitions of Order of Convergence. Analogous to ratio and root tests for sequences, there is Q-order or Quotient-Order defined as follows.

Definition 1.3.1 (cf. Ortega and Rheinboldt (1970)). *Let X be a Banach Space and* $\{x_k\} \subset X$ *be any convergent sequence with limit* x^**, then the quantities*

$$Q_p\{x_k\} = \begin{cases} 0, & \textit{if } x_k = x^* \textit{ for all but finitely many } k. \\ \lim_{k\to\infty} sup \frac{\|x_{k+1}-x^*\|}{\|x_k-x^*\|^p}, & \textit{if } x_k \neq x^* \textit{ for all but finitely many } k. \\ \infty, & \textit{otherwise}. \end{cases}$$

defined for all $p \in [1,\infty)$ *are called quotient convergence factors or Q-factors.*

The Root order or R-order is defined as

Definition 1.3.2 (cf. Ortega and Rheinboldt (1970)).

$$R_p\{x_k\} = \begin{cases} limsup\|x_k - x^*\|^{\frac{1}{k}}, & \textit{if } p = 1 \\ limsup\|x_k - x^*\|^{\frac{1}{p^k}}, & \textit{if } p > 1. \end{cases}$$

Schmidt (1963) introduced Q-order and it was also used by Traub (1964). R-order was used by Ortega and Rheinboldt (1970). R-order is more generic than Q-order, as in the case of the Root test and Ratio test. With the additional condition $0 < Q_p\{x_k\} < \infty$ there is equivalence between the above definition of $Q-$ and $R-$ orders (see Ortega and Rheinboldt (1970); Petković et al. (2014)).

Definition 1.3.3 (Potra (1989); Ezquerro and Hernández (2006); Jay (2001)). *A sequence* $\{x_n\}$ *converges to* x^* *with* $R-$*order at least p if there are constants* $C \in (0,\infty)$ *and* $\gamma \in (0,1)$ *such that*

$$\|z_{n+1} - z^*\| \leq C\gamma^{p^n},\ n = 0,1,2,\ldots. \tag{1.3.1}$$

Note that

$$\|z_{n+1} - z^*\| \leq c\|z_n - z^*\|^p,\ c > 0, n = 0,1,2,\ldots, \tag{1.3.2}$$

for $p \in \mathbb{N}$ implies (1.3.1) provided $\|z_0 - z^*\| < 1$.

Equivalently, we also have another definition.

Definition 1.3.4. *A sequence $\{x_n\}$ in X, with $\lim_{n\to\infty} x_n = x^*$ is said to be convergent of order $R > 1$, if there exist positive real numbers β_1, β_2 such that for all $n \in \mathbb{N}$,*

$$\|x_n - x^*\| \leq \beta_1 e^{-\beta_2 R^n}. \tag{1.3.3}$$

For the sake of simplicity of proofs in this thesis, we have used Definitions 1.3.1, Definition 1.3.2 and Definition 1.3.3 in different chapters. In practical computational cases, many authors have used an approximation of order of convergence introduced by Weerakoon and Fernando (2000) defined as follows.

Definition 1.3.5 (Weerakoon and Fernando (2000))**.** *The Computational Order of Convergence(COC) of a sequence $\{x_n\} \geq 0$ is defined by*

$$\overline{\rho} = \frac{\ln\left(\|\frac{x_{n+1}-x^*}{x_n-x^*}\|\right)}{\ln\left(\|\frac{x_n-x^*}{x_{n-1}-x^*}\|\right)},$$

where x_{n-1}, x_n, x_{n+1} are three consecutive iterates near root x^*.

One main drawback of the computational order of convergence is that it involves the exact root. To avoid that approximate computational order of convergence was defined as follows ;

Definition 1.3.6 (Cordero and Torregrosa (2007))**.** *The Approximated Computational Order of Convergence(ACOC) of a sequence $\{x_n\} \geq 0$ is defined by*

$$\overline{\rho} = \frac{\ln\left(\|\frac{x_{n+1}-x_n}{x_n-x_{n-1}}\|\right)}{\ln\left(\|\frac{x_n-x_{n-1}}{x_{n-1}-x_{n-2}}\|\right)},$$

where $x_{n-2}, x_{n-1}, x_n, x_{n+1}$ are four consecutive terms near the root x^*.

It is shown that the computational order of convergence and approximated computational order of convergence are the same as the order of convergence as $n \longrightarrow \infty$ in

Grau-Sánchez et al. (2010). COC and ACOC are not reliable measures as oscillating behaviour of approximations and slow convergence of iterates in the initial stage may result in inaccurate computation of COC and ACOC (Petković (2013); Petković et al. (2014)).

It is essential to have faster convergence in many applied areas. So the study of the higher-order convergence method is vital. Increasing the order of convergence is possible using higher-order derivatives and many function evaluations (Traub (1964)). However, it is computationally expensive. In most cases, function or derivative evaluations are far more expensive in computation than the cost of arithmetic operations on the available data (Petković (2013)). Traub (1964) introduced the informational efficiency, which can be expressed in terms of the order of convergence (R) of the method and the number of function (and derivative) evaluations Θ_f. The informational efficiency of an iterative method $I(\phi)$ is defined as

$$I(\phi) = \frac{R}{\Theta_f}.$$

Ostrowski (1966) introduced a different measure called efficiency index or computational efficiency, defined as

$$E(\phi) = R^{\frac{1}{\Theta_f}}.$$

So, many authors have used multiple-step methods to get higher-order convergence instead of higher-order derivatives. A well-known conjecture in the area is due to Kung and Traub.

Conjecture 1.3.7. *Multi-point iterative methods without memory, costing* $n+1$ *function evaluations per iteration, have the order of convergence at most* 2^n .

Woźniakowski proved the conjecture for some class of functions in Woźniakowski (1976). Iterative methods that satisfy Kung Traub's conjecture are called optimal methods. So the goal is to obtain optimal or near-optimal iterative methods.

Another area to consider is the stability behaviour of the iterative method. Some important definitions in the area for the sake of completion are given below.

Definition 1.3.8 (cf. Blanchard (1984))**.** *A rational function* $\hat{R} : \hat{C} \to \hat{C}$*, where* $\hat{C}$ *is the*

Riemann sphere, the orbit of a point $z_0 \in \hat{C}$ *is defined as:*

$$\{z_0, \hat{R}(z_0), \hat{R}^2(z_0), \ldots, \hat{R}^n(z_0), \ldots\}.$$

The phase plane of the map is classified from the starting points using the asymptotic behavior of their orbits.

Definition 1.3.9 (cf. Blanchard (1984))**.** *A point* $z_0 \in \hat{C}$ *is a fixed point if* $\hat{R}(z_0) = z_0$*. A periodic point* z_0 *of period* $p > 1$ *is a point such that* $\hat{R}^p(z_0) = z_0$ *and* $\hat{R}^k(z_0) \neq z_0$*, for* $k < p$*. A pre-periodic point is a point* z_0 *that is not periodic but there exists a* $k > 0$ *such that* $R^k(z_0)$ *is periodic. A critical point* z_0 *is a point where the derivative of the rational function vanishes,* $R'(z_0) = 0$*. Moreover, a fixed point* z_0 *is called an attractor if* $|R'(z_0)| < 1$*, super attractor if* $|R'(z_0)| = 0$*, repulsor if* $|R'(z_0)| > 1$ *and parabolic if* $|R'(z_0)| = 1$*. The fixed points different from the roots of the polynomial* $p(z)$ *are called strange fixed points.*

Definition 1.3.10 (cf. Blanchard (1984))**.** *The basin of attraction of an attractor* α *is defined as:* $A(\alpha) = \{z_0 \in \hat{C} : R^n(z_0) \to \alpha, n \to \infty\}$*. The Fatou set of the rational function* $\hat{R}, \mathscr{F}(R)$*, is the set of points* $z \in \hat{C}$ *whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in* $\hat{C}$ *is the Julia set,* $J(R)$*. That means that the basin of attraction of any fixed point belongs to the Fatou set, and the boundaries of these basins of attraction belong to the Julia set.*

In short, in a good iterative scheme, the roots of the functions become attracting fixed points, and orbits of 'close enough' points converge to this attractor. In this thesis, for the sake of simplicity, we use two-dimensional real cases instead of complex dynamics. Various authors have detailed works related to dynamics of iterative methods(see Amat et al. (2004); Chun et al. (2012); Varona (2002); Scott et al. (2011); Ardelean (2011); Chicharro et al. (2013)).

1.4 CONVERGENCE ANALYSIS OF ITERATIVE METHODS

Another critical question regarding an iterative method is its convergence. How to ensure the convergence of the method in a particular domain? There are three types of convergence analysis in literature. **Global convergence** analysis is the convergence study using conditions only on iteration function. The iterative method converges regardless of initial points in the domain. This is achievable in the case of Linear maps (for example, Picard's iteration), with very stringent conditions on the iteration function. However, in the case of non-linear functions, there are no feasible global convergence results (see Ortega and Rheinboldt (1970)).

The second type of convergence analysis is **local convergence** analysis. Here we assume conditions on iterative function and use the information around the solution. Typically we assume the existence of a solution in a very small neighborhood. Usually, this convergence analysis also involves the use of Taylor series expansion and assumptions on the existence of higher-order derivatives. Also, computable error estimates are rare here. In Chapter 4 and Chapter 5, we study the local convergence analysis of two iterative methods.

The third type, **semi-local convergence** analysis, is recent. The first significant work in this area was due to Kantorovič (1949). In this case, we find feasible regions of initial points using assumptions on iterative functions, solutions, and initial points. There are mainly two approaches in the area, majorization and recurrence relations. In the recurrence relations technique, the problem in Banach spaces is simplified to a more straightforward problem using real-valued functions and sequences. On the other side, in the majorizing sequences approach, the sequence of iterations produced by an iterative function is majorized by employing the same scheme on a scalar-valued function. In Chapter 2, we study semi-local convergence analysis of an iterative method using recurrence relations.

Typical assumptions on iterative functions are the existence of higher-order derivatives. Usually convergence analysis involves Taylor series and assumptions on the existence of higher derivatives even though the iterative methods themselves do not demand such (see Argyros (2007); Magreñán and Argyros (2018); Argyros (2022); Argyros et al. (2022, 2023)). Nevertheless, these assumptions are somewhat restrictive. For instance, if $X = Y = \mathbb{R}$, $\Delta = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Δ by

$$f(x) = \begin{cases} x^3 \log x^2 + x^5 - x^4 & if\, x \neq 0 \\ 0 & if\, x = 0. \end{cases}$$

Then,

$$f'''(x) = 6 \log x^2 + 60x^2 - 24x + 22.$$

Clearly, $f'''(x)$ is not bounded on Δ. So assumptions on higher order derivatives restrict the applicability of the iterative method. Also, higher-order derivatives are far more computationally expensive in the case of higher-order matrices. We use a novel approach in this thesis by considering Lipschitz type or Central Lipschitz type assumptions instead of assumptions using higher order derivatives. There are also attempts to ease the conditions further using Hölder-type conditions and $\omega-$ continuity conditions (see Argyros (2007); Magreñán and Argyros (2018); Argyros (2022); Argyros et al. (2022, 2023)).

Since an exact solution may not be reached after finite steps in the case of iterative methods, another area of importance in practical computation is the stopping criteria chosen. The stopping criteria standard in literature is the norm of error less than some pre-ascribed small value or norm of relative error less than a given value. Using relative error in some pathological cases is better, as pointed out in Burden and Faires (2012). However, our examples are primarily for illustrative purposes only, so we use conditions on the norm of absolute error. A deep dive into stopping criteria can be seen in Hamming (1986).

1.5 HISTORY AND APPLICATIONS OF ITERATIVE METHODS

Iterative methods have a long history dating back to ancient Babylon (see, Boyer and Merzbach (1991)). An excellent survey into the history of iterative methods is Young (1989), particularly linear iterative methods. Iterative methods are ubiquitous in engineering and natural sciences. It has a wide range of applications spanning different fields like fluid dynamics, structural analysis, dynamical modeling, kinetics theory of gases, queuing theory, etc. It is also used in novel areas like machine learning for tasks such as optimization of deep neural networks, training support vector machines, and clustering. Another area where iterative methods are applied is ill-posed equations. We discuss the topic in brief in the subsequent section.

1.6 ITERATIVE METHODS FOR SOLVING NON-LINEAR ILL-POSED EQUATIONS

We are also interested in approximately solving the ill-posed equation

$$F(u) = y, \tag{1.6.1}$$

when F is non-linear, Fréchet differentiable and monotone operator on a real Hilbert Space. Some terminologies and definitions are introduced.

An equation is ill-posed if it violates any of Hadamard's criteria (Hadamard (1953)), i.e., the existence of the solution, uniqueness of the solution, and continuous inverse. Ill-posed problems naturally originate from inverse problems. A simple, non-rigorous definition is given by Keller (1976) as "two problems are inverse to each other if the formulation of each involves all or part of the solution of the other". Ill-posed problems pose severe numerical difficulties (Engl et al. (1996)).There is a vast literature on inverse and ill-posed problems, including books, monographs (Ramm (2005); Engl et al. (1996); Anger (1990)), and journals.

For the sake of completeness, we first discuss linear ill-posed problems. A typical example of a linear ill-posed problem would be the Fredholm integral equation of the first kind, with $T : L^2[a,b] \longrightarrow L^2[a,b]$ defined by

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt, \ a \leq s \leq b.$$

where $k(.,.)$ is a non-degenerate square integrable function and $y(.)$ is a known function. Here, in the case of linear ill-posed equations, we can tackle the existence and uniqueness issue using the best approximate solutions defined as follows.

Definition 1.6.1. *(cf. Nair (2009)) Let $T : X \longrightarrow Y$ be a bounded linear operator.*

*(i) $x \in X$ is called **least-squares solution** of equation if*

$$||Tx - y|| = inf\{||Tz - y||, \ z \in X\}$$

*(ii) $x^\dagger \in X$ is called **best-approximate solution** of $Tx = y$,if x is a least-squares solution of $Tx = y$ and*

$$||x^\dagger|| = inf\{||z||, z \text{ is least-squares solution of } Tx = y\}$$

holds.

Definition 1.6.2. *(cf. Nair (2009)) Let $T \in \mathscr{L}(X,Y)$, set of all bounded linear functions from X to Y. The operator $T^\dagger : D(T^\dagger) \subseteq Y \longrightarrow X$, where $D(T^\dagger) = R(T) + R(T)^\perp$, defined by $T^\dagger y = x^\dagger$, where $x^\dagger$ is the best approximate solution of the equation $Tx = y$, is called the **generalized inverse** of T.*

Generalized inverse is not sufficient enough because even if $T^\dagger$ exists, $T^\dagger y^\delta$ is not a good approximation of $T^\dagger y$, because $T^\dagger$ may be unbounded. So we need some regularization method to obtain approximations of $x^\dagger$, say x_α^δ with the property $x_\alpha^\delta \longrightarrow x^\dagger$, where α is some parameter. Naively, regularization approximates an ill-posed problem by a family of well-posed problems.

Definition 1.6.3. *(cf. Engl et al. (1996)) Let* $T : X \longrightarrow Y$ *be a bounded linear operator between Banach spaces* X *and* Y *and* $\alpha_0 \in (0, +\infty)$*. For every* $\alpha \in (0, \alpha_0)$*, let* $R_\alpha : Y \longrightarrow X$ *be a continuous operator. The family* $\{R_\alpha\}$ *is called a* ***regularization*** *or a* ***regularization operator****, if there exists a parameter choice rule* $\alpha = \alpha(\delta, y^\delta)$ *such that*

$$\lim_{\delta \longrightarrow 0} \sup\{\|R_{\alpha(\delta,y^\delta)} y^\delta - x^\dagger\| \mid y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0$$

holds. Here, $\alpha : \mathbb{R}^+ \times Y \longrightarrow (0, \alpha_0)$ *is such that*

$$\lim_{\delta \longrightarrow 0} \sup\{\alpha(\delta, y^\delta) \mid y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0.$$

Thus, a regularization method consists of choosing; regularization operator R_α and regularization parameter α.

Regularization parameters can be chosen in two ways ; a parameter choice rule for choosing α is called an a-priori parameter choice rule if it does not depend on y^δ, but only on δ. Otherwise, we call the rule a-posteriori parameter choice rule. In a-priori parameter choice, knowledge of unknown element $x^\dagger$ is required. In a-posteriori parameter choice, the parameter α is chosen based on available data. More details about linear ill-posed problems can be found in Nair (2009).

Since we have studied the iterative method for non-linear ill-posed equations involving monotone operators, we focus on the discussion on non-linear ill-posed equations. In particular, consider the non-linear ill-posed equation

$$F(x) = y, \tag{1.6.2}$$

where $F : X \longrightarrow Y$ is a non-linear operator from Hilbert space X to Hilbert Space Y. In this case, instead of the best approximate or minimum-norm solution, we consider a more general x^***minimum-norm solution** $\hat{x}$ such that

$$\begin{aligned} F(\hat{x}) &= y \\ \|\hat{x} - x^*\| &= min\{\|x - x^*\| : F(x) = y\}. \end{aligned}$$

Observe that (Engl et al. (1996)) even the least square solution may not exist in the non-linear case. Available apriori information about solutions of $F(x) = y$ is considered for choosing x^*. In the case of multiple solutions, x^* plays the role of selection criterion.

Definition 1.6.4 (cf. Kantorovich and Akilov (1982))**.** *Let X and Y be normed linear spaces and suppose that E_1 is an open subset of X. A function $F : E_1 \to Y$ is said to be* ***Fréchet differentiable*** *at $x_0 \in X$ if there exist an operator $F'(x_0) \in \mathscr{L}(X,Y)$ such that*

$$\lim_{\|h\|\to 0} \frac{\|F(x_0+h) - F(x_0) - F'(x_0)h\|}{\|h\|} = 0.$$

Note that $F'(x_0)$ is the total derivative at x_0 if X and Y are finite dimensional. If F is non-linear and Fréchet differentiable, the operator is said to be ill-posed if the inverse of $F'(u)$ is not bounded for any point u in the domain of F. If $F'(u)^{-1}$ is bounded, then a local homeomorphism of F exist (Ramm (2005)). There are several examples of non-linear ill-posed problems (Tikhonov (1987)), and we provide one example below.

Example 1.6.5. Inverse Gravimetry problem:*(cf. Vasin and George (2014)).*

The problem of structural inverse gravimetry for a two-layer medium involves finding a function $x_3 = u(x_1, x_2)$ that describes the interface between two media with different densities (σ_1 and σ_2). The function $u(x_1, x_2)$ is the solution to an equation that involves integrals and the unknown gravitational field $\Delta g(x_1, x_2)$ caused by a deviation in the interface H from a horizontal asymptotic plane ($x_3 = S$). The equation involves variables x_1, x_2, and x_3 in a Cartesian coordinate system with the vertical axis x_3 pointing downwards. The equation has the form given below.

$$\Gamma\Delta\sigma \int\int_D \frac{1}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + H^2]^{1/2}} dx_1' dx_2'$$

$$- \int\int_D \frac{1}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + u^2(x_1', x_2')]^{1/2}} dx_1' dx_2' = \Delta g(x_1, x_2), \tag{1.6.3}$$

where Γ is the gravity constant and $\Delta\sigma$ is the difference in density between the two media. $\Delta g(x_1, x_2)$ is the unknown gravitational field caused by some disparity in the interface H from horizontal asymptotic plane $x_3 = S$, i.e., for the sought-for solution

$\hat{u}(x_1,x_2)$ *the following equality holds*

$$\lim_{|x_1|,|x_2|\to\infty} |\hat{u}(x_1,x_2) - H| = 0,$$

$g(x_1,x_2)$ *is given on the domain D. By considering (1.6.3) the first term does not depend on* $u(x_1,x_2)$ *so equation can be modified as*

$$F(u) \equiv -\int\int_D \frac{1}{[(x_1-x_1')^2+(x_2-x_2')^2+u^2(x_1',x_2')]^{1/2}} dx_1' dx_2' = f(x_1,x_2),$$

where $f(x_1,x_2) = \Delta g(x_1,x_2) + F(H)$.

In practice, one has to deal with noisy data y^δ instead of y with the condition $\|y - y^\delta\| \leq \delta$. Corresponding equation of (1.6.2) with y^δ is

$$F(x) = y^\delta. \tag{1.6.4}$$

Tikhonov regularization is a well known regularization method for solving (1.6.4). In Tikhonov regularization the minimizer x_α^δ of the functional

$$J_\alpha(x) = \underset{x\in D(F)}{min} \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2,$$

is taken as an approximation for $\hat{x}$.

If the operator in (1.6.4) is monotone and $X = Y$, then one may use Laurentiev regularization in which the solution of the equation

$$F(x) + \alpha(x - x_0) = y^\delta, \tag{1.6.5}$$

x_α^δ is taken as an approximate for $\hat{x}$ (see, Tautenhahn (2002)). The existence and uniqueness of the solution are due to the Minty-Browder theorem (see, Alber and Ryazantseva (2006)). Recall that an operator F on a Hilbert Space H is monotone if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \text{ for all } x, y \in D(F).$$

To obtain an error estimate for $\|\hat{x}-x_{\alpha}^{\delta}\|$, we need some assumptions on $\hat{x}-x_0$, called source conditions. Well-known source conditions for the Laurentiev regularization method are

$$x_0-\hat{x}\in R(F'(\hat{x})^{\nu}) \qquad 0<\nu\leq 1$$

and

$$x_0-\hat{x}\in R(F'(x_0)^{\nu}) \qquad 0<\nu\leq 1.$$

If $\alpha=\delta^{\frac{1}{\nu+1}}$, then the optimal order error estimate for $\|\hat{x}-x_{\alpha}^{\delta}\|$ is $\mathscr{O}\left(\delta^{\frac{\nu}{\nu+1}}\right)$. But such a choice is not possible since ν is unknown. We introduce a new source condition and parameter choice strategy in Chapter 3, which gives optimal order without knowing the unknown ν.

It is difficult to get closed form solution for (1.6.5), so iterative methods for solving the regularized equation (1.6.5) are studied in the literature (see Mahale and Nair (2009); Qi-nian (2000); Bakushinsky and Kokurin (2004)). One advantage of iterative method is the inherent regularization where stopping index act as a regularization parameter. We modified the iterative method developed in Chapter 2 suitably to approximate the solution x_{α}^{δ} of (1.6.5).

RESEARCH OBJECTIVES

The main objectives of this study are

(1) To obtain the order of convergence of iterative methods for operator equations in Banach Spaces using only the first Fréchet derivative of the operator involved.

(2) To apply the iterative methods for solving non-linear ill-posed equations in a Hilbert Space setting.

(3) To obtain the convergence order of iterative methods without using Taylor series expansion.

1.7 ORGANIZATION OF THE THESIS

The results obtained are presented in Chapter 2 to Chapter 5. The notation used in each Chapter is independent of other Chapters. So, some assumptions may repeat in the forthcoming Chapters. This thesis consists of six Chapters. Chapter 1 consists of a basic introduction to the topics discussed in this study and the necessary preliminaries. In Chapter 2, we study an iterative method introduced by Singh et al. (2016). The authors used conditions on the second Fréchet derivative and established fifth-order convergence. In this study, we established the fifth-order convergence of the method using assumptions only on the first Fréchet derivative of the involved operator. In Chapter 3, we introduced a new source condition and a new parameter choice strategy for the Lavrentiev regularization method for non-linear ill-posed operator equations involving a monotone operator in a Hilbert space setting. Also, we modified the fifth-order method in Chapter 2 for approximately solving the Lavrentiev regularized equation in Chapter 3. A numerical example is given to demonstrate the performance of the method. In Chapter 4, the local convergence analysis of a method studied in Homeier (2004) is revisited using only assumptions on the first and second derivatives in a Banach space setting. We obtained third-order convergence with a technique not involving Taylor series expansion. We further introduce two extensions of the method with orders five and six, and radii of convergence and basins of attraction are provided for illustrative examples. In Chapter 5, we study local convergence analysis of an extension of Newton's method introduced in Traub (1964) (also known as the Arithmetic-Mean Newton's Method and Weerakoon and Fernando method). We established the desired order of convergence in the Banach space setting. Our technique did not involve the use of higher-order Taylor series expansion. We also studied two of its extensions and provided estimates for the radii of convergence. Basins of attraction, comparison of iterations of similar iterative methods, approximate computational order of convergence (ACOC), and a representation of the number of iterations are also provided. The final chapter is the conclusion and the scope of future work.

CHAPTER 2

CONVERGENCE ANALYSIS OF A FIFTH-ORDER ITERATIVE METHOD USING RECURRENCE RELATIONS AND CONDITIONS ON THE FIRST DERIVATIVE

2.1 INTRODUCTION

In Singh et al. (2016), Singh et al. studied the iterative method:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1} F(x_k), \\ z_k &= y_k - F'(x_k)^{-1} F(y_k), \\ x_{k+1} &= z_k - F'(y_k)^{-1} F(z_k), \end{aligned} \tag{2.1.1}$$

for approximating a solution of the equation

$$F(x) = 0, \tag{2.1.2}$$

where $F : \Omega \subseteq X \longrightarrow Y$ is a Fréchet differentiable operator between the Banach spaces X and Y. Equation (2.1.2) is studied extensively in the literature (Amat et al. (2008); Argyros et al. (2018, 2015); Argyros and Magreñán (2017, 2016); Chen et al. (2011); Chun et al. (2011); Cordero et al. (2012, 2015b); Ezquerro and Hernández-Verón (2016);

Hueso and Martínez (2014); Jaiswal (2016); Ostrowski (1973); Proinov and Ivanov (2015); Parida and Gupta (2007); Traub (1964); Wang et al. (2011); Zheng and Gu (2012a,b)). The assumptions in Singh et al. (2016) limit the applicability of the method (2.1.1). Consider the function f defined on the interval $\left[-\frac{1}{2}, \frac{3}{2}\right]$ by:

$$f(x) = \begin{cases} x^2 \log x + x^4 - x^3, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then, f'' is unbounded on $\Omega = \left[-\frac{1}{2}, \frac{3}{2}\right]$. To extend the applicability of the method (2.1.1), Argyros and George (2018) studied the convergence analysis of method (2.1.1) using assumptions only on the first derivative of F. However, the order of convergence in Argyros and George (2018) is pessimistic, and hence, they proposed to use computational order of convergence (COC) defined as:

$$\xi = \ln\left(\frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|}\right) / \ln\left(\frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|}\right)$$

or the approximate computational order of convergence (ACOC) :

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right)$$

to obtain the order of convergence.

In this Chapter, we developed an iteration using assumption on the first derivative of F and obtained the R-order of convergence five. A variety of examples are solved in Singh et al. (2016) to demonstrate the applicability of method (2.1.1). The rest of the Chapter is organized as follows. In Section 2.2, we provide the preliminaries required for the main result. In Section 2.3, convergence analysis is studied.

2.2 PRELIMINARIES

In this section, we define some functions and parameters, used for proving our results. For $K > 0$, let $h_1 : [0,\infty) \longrightarrow [0,\infty)$ be defined by:

$$h_1(t) = K\Pi(t)t^2 - 1,$$

where $\Pi(t) = K\left[\left(\frac{3}{2} + \frac{K}{4}t\right)\left(\frac{K}{2}t + 1\right) + 1\right]$. Then, $h_1(0) = -1 < 0$ and $h_1(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. Therefore, by the intermediate value theorem (Rudin (1976)), h_1 has a minimal zero r_1. Define functions $\xi, \delta, \Upsilon, \psi, \gamma \colon [0, r_1) \longrightarrow [0,\infty)$ by:

$$\begin{aligned}
\xi(t) &= \frac{K^2}{2}\left(1 + \frac{K}{4}t\right), \\
\delta(t) &= \psi(t)t + \frac{K}{2}, \\
\Upsilon(t) &= K\left[\left(\psi(t) + \frac{1}{2}\gamma(t)\right)\gamma(t)t + \delta(t)\xi(t)\right], \\
\psi(t) &= \left[1 + \frac{K}{1 - K\Pi(t)t^2}\left(\frac{K}{2}t^2 + \frac{1}{2}\xi(t)t^3\right)\right]\xi(t)t^3
\end{aligned}$$

and

$$\gamma(t) = K\left[\xi(t)t^3 + \frac{1}{2}\psi(t)t^3\right]\psi(t) + \frac{1}{1 - K\Pi(t)t^2}\xi(t).$$

Then, $\xi, \delta, \Upsilon, \psi, \gamma$ are monotonically increasing and continuous functions. Let $\Theta, h_2 : [0, r_1] \longrightarrow [0,\infty)$ be defined by:

$$\Theta(t) := \frac{K}{1 - K\Pi(t)t^2}(\Pi(t) + \xi(t)t)\xi(t)t^4$$

and

$$h_2(t) = \Theta(t) - 1.$$

Then, $h_2(0) = -1 < 0$ and $h_2(t) \longrightarrow \infty$ as $t \longrightarrow r_1^-$. By the intermediate value theorem (Rudin (1976)), h_2 has a minimal zero r in $(0, r_1)$. Then, for all $t \in [0, r]$, we have:

$$0 < K\Pi(t)t^2 < 1, \tag{2.2.1}$$

and

$$0 < \Theta(t) < 1. \tag{2.2.2}$$

2.3 CONVERGENCE AND ERROR ESTIMATE

Let $\eta_k := \|x_k - y_k\|$ and $\eta_0 < r$. First, we shall show using mathematical induction that the sequence $\{\eta_k\}$ converges with order of convergence five.

The following assumptions are used for proving our results.

Assumption 2.3.1. *There exists $k_0 > 0$, such that for every $x \in D(F)$, $\left\|F'(x)^{-1}\right\| \leq \beta$ and for every $u, v \in D(F)$:*

$$\left\|F'(u) - F'(v)\right\| \leq k_0\|u - v\|.$$

Assumption 2.3.2. *There exists a constant $K_0 > 0$, such that for every $u, v \in D(F)$ and $w \in X$, there exists an element $\phi(u, v, w) \in X$ satisfying*

$$\left[F'(u) - F'(v)\right] w = F'(v)\phi(u, v, w), \|\phi(u, v, w)\| \leq K_0\|w\|\|u - v\|$$

and $F'(v)^{-1}$ exists.

Let:

$$K = \begin{cases} \beta k_0 & \text{if Assumption 3.1 holds} \\ K_0 & \text{if Assumption 3.2 holds.} \end{cases} \tag{2.3.1}$$

The following auxiliary result is useful for the convergence analysis of method (2.1.1).

Lemma 2.3.3. *Let x_n, y_n, z_n be as in (2.1.1) and Assumption 2.3.1 or Assumption 2.3.2 hold. Let K be as in (2.3.1), and let ξ, Π, Θ, r be as defined in Section 2.2 and $K\Pi\left(\eta_0\right)\eta_0^2 <$ 1. Then, we have the following:*

(a) $\|z_k - y_k\| \leq \frac{K}{2}\|x_k - y_k\|$.

(b) $\|x_{k+1}-z_k\| \leq \xi(\eta_k)\eta_k^2, \forall k \geq 0.$

(c) $\|x_{k+1}-y_k\| \leq \Pi(\eta_k)\eta_k^2.$

(d) $\eta_k \leq \Theta(\eta_{k-1})\eta_{k-1}.$

(e) $\Theta(\eta_k) \leq \Theta(r)^{5^k}, \forall k \geq 0.$

(f) $\eta_k \leq \Theta(r)^{\frac{5^k-1}{4}}\eta_0.$

Proof. The proof is by induction. By (2.1.1), we have:

$$\begin{aligned}
z_0 - y_0 &= y_0 - x_0 - F'(x_0)^{-1}[F(y_0) - F(x_0)] \\
&= F'(x_0)^{-1}\left[F'(x_0)(y_0-x_0) - \int_0^1 F'(x_0+t(y_0-x_0))(y_0-x_0)\,dt\right] \\
&= -F'(x_0)^{-1}\int_0^1 \left[F'(x_0+t(y_0-x_0)) - F'(x_0)\right](y_0-x_0)\,dt \\
&= \begin{cases} -F'(x_0)^{-1}\int_0^1 [F'(x_0+t(y_0-x_0)) - F'(x_0)](y_0-x_0)\,dt \\ \qquad \text{if Assumption 3.1 holds} \\ -F'(x_0)^{-1}F'(x_0)\int_0^1 \varphi(x_0+t(y_0-x_0), x_0, y_0-x_0)\,dt, \\ \qquad \text{if Assumption 3.2 holds.} \end{cases}
\end{aligned}$$

Hence, we get:

$$\|z_0 - y_0\| \leq \frac{K}{2}\|x_0-y_0\|^2. \tag{2.3.2}$$

We have again by (2.1.1):

$$\begin{aligned}
x_1 - z_0 &= z_0 - y_0 - \left[F'(y_0)^{-1}F(z_0) - F'(x_0)^{-1}F(y_0)\right] \\
&= z_0 - y_0 - F'(y_0)^{-1}[F(z_0) - F(y_0)] + \left[F'(x_0)^{-1} - F'(y_0)^{-1}\right]F(y_0) \\
&= F'(y_0)^{-1}\left[F'(y_0)(z_0-y_0) - (F(z_0) - F(y_0))\right] \\
&+ F'(y_0)^{-1}\left[F'(y_0) - F'(x_0)\right]F'(x_0)^{-1}F(y_0) \\
&= F'(y_0)^{-1}\int_0^1 \left[F'(y_0) - F'(y_0+t(z_0-y_0))\right](z_0-y_0)\,dt \\
&- F'(y_0)^{-1}\left[F'(y_0) - F'(x_0)\right](z_0-y_0)
\end{aligned}$$

$$= \begin{cases} F'(y_0)^{-1}\int_0^1 [F'(y_0)-F'(y_0+t(z_0-y_0))](z_0-y_0)\,dt \\ -F'(y_0)^{-1}[F'(y_0)-F'(x_0)](z_0-y_0) \\ \text{if Assumption 2.3.1 holds} \\ -F'(y_0)^{-1}F'(y_0)\int_0^1 \varphi(y_0+t(z_0-y_0),y_0,z_0-y_0)\,dt \\ +F'(y_0)^{-1}F'(y_0)\,\varphi(x_0,y_0,z_0-y_0) \\ \text{if Assumption 2.3.2 holds.} \end{cases}$$

Therefore, by (2.3.2), we have

$$\begin{aligned} \|x_1-z_0\| &\le K\|z_0-y_0\|\left(\frac{\|z_0-y_0\|}{2}+\|x_0-y_0\|\right) \\ &\le \frac{K^2}{2}\|y_0-x_0\|^2\left[\frac{K}{4}\|x_0-y_0\|^2+\|x_0-y_0\|\right] \\ &= \xi(\eta_0)\,\eta_0^3. \end{aligned} \tag{2.3.3}$$

Furthermore, we have:

$$\begin{aligned} x_1-y_0 &= z_0-x_0-F'(y_0)^{-1}F(z_0)+F'(x_0)^{-1}F(x_0) \\ &= z_0-x_0-F'(y_0)^{-1}[F(z_0)-F(x_0)]+\left[F'(x_0)^{-1}-F'(y_0)^{-1}\right]F(x_0) \\ &= F'(y_0)^{-1}\left[F'(y_0)(z_0-x_0)-\int_0^1 F'(x_0+t(z_0-x_0))(z_0-x_0)\,dt\right] \\ &+F'(y_0)^{-1}\left[F'(y_0)-F'(x_0)\right]F'(x_0)^{-1}F(x_0) \end{aligned}$$

$$= \begin{cases} F'(y_0)^{-1}\int_0^1 [F'(y_0)-F'(x_0+t(z_0-x_0))](z_0-x_0)\,dt \\ -F'(y_0)^{-1}[F'(y_0)-F'(x_0)](z_0-y_0) \\ \text{if Assumption 2.3.1 holds} \\ -F'(y_0)^{-1}F'(y_0)\left[\int_0^1 \varphi(x_0+t(z_0-x_0),y_0,z_0-x_0)\,dt\right. \\ +\varphi(x_0,y_0,x_0-y_0)] \\ \text{if Assumption 2.3.2 holds.} \end{cases}$$

Hence, we get:

$$\|x_1-y_0\| \le K\left[\|x_0-y_0\|+\frac{1}{2}\|z_0-x_0\|\right]\|z_0-x_0\|+K\|x_0-y_0\|^2$$

$$\leq \Pi\left(\eta_0\right)\eta_0^2. \tag{2.3.4}$$

The last but one step follows from the inequality:

$$\left\|z_0-x_0\right\| \leq \left\|z_0-y_0\right\|+\left\|y_0-x_0\right\| \tag{2.3.5}$$

and (2.3.2).

Next, we shall prove that $\eta_1 < \eta_0$. Note that:

$$\begin{aligned}
x_1-y_1 &= z_0-x_1-F'\left(y_0\right)^{-1}F\left(z_0\right)+F'\left(x_1\right)^{-1}F\left(x_1\right)\\
&= z_0-x_1-F'\left(y_0\right)^{-1}\left[F\left(z_0\right)-F\left(x_1\right)\right]+\left[F'\left(x_1\right)^{-1}-F'\left(y_0\right)^{-1}\right]F\left(x_1\right)\\
&= F'\left(y_0\right)^{-1}\left[F'\left(y_0\right)\left(z_0-x_1\right)-\int_0^1 F'\left(x_1+t\left(z_0-x_1\right)\right)d\right]t\left(z_0-x_1\right)\\
&+F'\left(y_0\right)^{-1}\left[F'\left(y_0\right)-F'\left(x_1\right)\right]F'\left(x_1\right)^{-1}F\left(x_1\right)\\
&= \begin{cases}
F'\left(y_0\right)^{-1}\int_0^1\left[F'\left(y_0\right)-F'\left(x_1+t\left(z_0-x_1\right)\right)\right]\left(z_0-x_1\right)dt\\
-F'\left(y_0\right)^{-1}\left[F'\left(y_0\right)-F'\left(x_1\right)\right]\left(x_1-y_1\right)\\
\text{if Assumption 2.3.1 holds}\\
-F'\left(y_0\right)^{-1}F'\left(y_0\right)\int_0^1\varphi\left(x_1+t\left(z_0-x_1\right),y_0,z_0-x_1\right)dt\\
+\varphi\left(x_1,y_0,x_1-y_1\right)\\
\text{if Assumption 2.3.2 holds.}
\end{cases}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|x_1-y_1\right\| &\leq K\left(\left\|x_1-y_0\right\|+\frac{1}{2}\left\|x_1-z_0\right\|\right)\left\|x_1-z_0\right\|+K\left\|x_1-y_0\right\|\left\|x_1-y_1\right\|\\
&\leq K\left(\left\|x_1-y_0\right\|+\frac{1}{2}\left\|x_1-z_0\right\|\right)\left\|x_1-z_0\right\|+K\Pi\left(\eta_0\right)\eta_0^2\left\|x_1-y_1\right\|
\end{aligned}$$

$$\left(1-K\Pi\left(\eta_0\right)\eta_0^2\right)\left\|x_1-y_1\right\| \leq K\left(\Pi\left(\eta_0\right)+\xi\left(\eta_0\right)\eta_0\right)\xi\left(\eta_0\right)\eta_0^5. \tag{2.3.6}$$

That is, by (2.3.6) and (2.2.2), we have:

$$\eta_1 = \|x_1 - y_1\| \leq \Theta(\eta_0)\,\eta_0 < \eta_0. \tag{2.3.7}$$

The induction for (a), (b), (c), and (d) will be completed if we simply replace x_0, y_0, z_0, x_1, y_1 by $x_k, y_k, z_k, x_{k+1}, y_{k+1}$ in the preceding arguments.

To prove (e), we first observe that $\Theta(\mu t) \leq \mu^4 \Theta(t)$ for $0 < \mu < 1$, so we have:

$$\Theta(\eta_k) \leq \Theta(\eta_{k-1})^5 \leq \cdots \Theta(\eta_0)^{5^k} \leq \Theta(r)^{5^k}.$$

The proof of (f) is as follows:

$$\begin{aligned}
\eta_k &\leq \Theta(\eta_{k-1})\,\eta_{k-1} \\
&\leq \Theta(\eta_0)^{5^{k-1}}\,\Theta(\eta_{k-2})\,\eta_{k-2} \\
&\leq \Theta(\eta_0)^{5^{k-1}+5^{k-2}+\cdots+1}\,\eta_0 \\
&\leq \Theta(r)^{\frac{5^k-1}{4}}\,\eta_0.
\end{aligned}$$

□

Theorem 2.3.4. *Let $R = r + r^2 \max\left\{\frac{K}{2}, \Pi(r)\right\}$. Assumptions of Lemma 2.3.3 hold true. Then, $y_k, z_k, x_{k+1} \in B(x_0, R)$ for all $k \geq 0$ and:*

$$\|x_{k+2} - x_*\| \leq C e^{-\gamma 5^{k+2}}, \quad k \geq 2,$$

where $C = \frac{\eta_0}{\left(1-\Theta(r)^{5^k}\right)\Theta(r)^{5^2}} \Upsilon(r)$ and $\gamma_0 = -\log(\Theta(r))$. Furthermore, if $KR < 1$, then the solution of (2.1.2) is unique in $B(x_0, R)$.

Proof. Clearly, $x_0, y_0 \in B(x_0, r) \subset B(x_0, R)$. We have:

$$\|z_0 - x_0\| \leq \|z_0 - y_0\| + \|y_0 - x_0\| \leq \frac{K}{2}\|x_0 - y_0\|^2 + \|x_0 - y_0\| \leq R,$$

so $z_0 \in B(x_0, R)$. Now, since $\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq \Pi(\eta_0)\,\eta_0^2 + r < R$, we have $x_1 \in B(x_0, R)$. Simply replace y_0, z_0, x_1 by y_k, z_k, x_{k+1} in the above argument, to obtain $y_k, z_k, x_{k+1} \in B(x_0, R)$ for all $k \geq 0$.

To prove (2.3.4), first, we shall prove that $\left\| F'(x_1)^{-1} F'(y_0) \right\|$ is bounded.

Note that for $v \in X$:

$$\begin{aligned}
\left\| F'(y_0)^{-1} \left[F'(x_1) - F'(y_0) \right] v \right\| &= \left\| F'(y_0)^{-1} F'(y_0) \varphi(x_1, y_0, v) \right\| \\
&\leq K \|x_1 - y_0\| \|v\| \\
&\leq K\Pi(\eta_0) \eta_0^2 \|v\|,
\end{aligned}$$

so since $K\Pi(\eta_0) \eta_0^2 < 1$, by Banach Lemma on invertible operators (Kantorovich and Akilov (1982)), we have:

$$\left\| F'(x_1)^{-1} F'(y_0) \right\| \leq \frac{1}{1 - K\Pi(\eta_0) \eta_0^2}. \tag{2.3.8}$$

Observe that:

$$\begin{aligned}
y_1 - z_0 &= x_1 - z_0 - F'(x_1)^{-1} F(x_1) \\
&= x_1 - z_0 + F'(x_1)^{-1} \left[F(z_0) + \int_0^1 F'(z_0 + t(x_1 - z_0))(x_1 - z_0)\, dt \right] \\
&= x_1 - z_0 + F'(x_1)^{-1} \left[-F'(y_0)(x_1 - z_0) + \int_0^1 F'(z_0 + t(x_1 - z_0))(x_1 - \right. \\
&= \begin{cases} x_1 - z_0 + F'(x_1)^{-1} \int_0^1 \left[F'(z_0 + t(x_1 - z_0)) - F'(y_0) \right](x_1 - z_0)\, dt \\ \text{ if Assumption 2.3.1holds} \\ x_1 - z_0 + F'(x_1)^{-1} F'(y_0) \int_0^1 \varphi(z_0 + t(x_1 - z_0), y_0, x_1 - z_0)\, dt \\ \text{ if Assumption 2.3.2 holds} \end{cases}
\end{aligned}$$

and hence, we have:

$$\begin{aligned}
\|y_1 - z_0\| &\leq \|x_1 - z_0\| + \left\| F'(x_1)^{-1} F'(y_0) \right\| K \left(\|z_0 - y_0\| + \frac{\|x_1 - z_0\|}{2} \right) \|x_1 - z_0\| \\
&\leq \left[1 + \frac{K}{1 - K\Pi(\eta_0) \eta_0^2} \left(\frac{K}{2} \eta_0^2 + \frac{1}{2} \xi(\eta_0) \eta_0^3 \right) \right] \xi(\eta_0) \eta_0^3.
\end{aligned}$$

Hence,

$$\|y_1 - z_0\| \leq \psi(\eta_0) \eta_0^3. \tag{2.3.9}$$

Again, by (2.1.1), we have:

$$\begin{aligned}
z_1 - z_0 &= y_1 - z_0 - F'(x_1)^{-1} F(y_1) \\
&= F'(x_1)^{-1}\left[F'(x_1)(y_1 - z_0) - F(y_1)\right] \\
&= F'(x_1)^{-1}\left[F'(x_1)(y_1 - z_0) - (F(y_1) - F(z_0))\right] - F'(x_1)^{-1} F(z_0) \\
&= -F'(x_1)^{-1}\int_0^1 \left[F'(z_0 + t(y_1 - z_0)) - F'(x_1)\right](y_1 - z_0)\,dt - F'(x_1)^{-1} F(z_0) \\
&= \begin{cases} -F'(x_1)^{-1}\int_0^1 [F'(z_0 + t(y_1 - z_0)) - F'(x_1)](y_1 - z_0)\,dt \\ +F'(x_1)^{-1} F'(y_0)(x_1 - z_0) \\ \text{if Assumption 2.3.1 holds} \\ -\int_0^1 \varphi(z_0 + t(y_1 - z_0), x_1, y_1 - z_0)\,dt + F'(x_1)^{-1} F'(y_0)(x_1 - z_0) \\ \text{if Assumption 2.3.2 holds} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\|z_1 - z_0\| &\le K\left[\|x_1 - z_0\| + \frac{1}{2}\|y_1 - z_0\|\right]\|y_1 - z_0\| + \frac{1}{1 - K\Pi(\eta_0)\eta_0^2}\|x_1 - z_0\| \\
&\le \gamma(\eta_0)\eta_0^3, \qquad (2.3.10) \\
\|y_1 - y_0\| &\le \|y_1 - z_0\| + \|z_0 - y_0\| \\
&\le \psi(\eta_0)\eta_0^3 + \frac{K}{2}\eta_0^2 = \delta(\eta_0)\eta_0^2. \qquad (2.3.11)
\end{aligned}$$

Furthermore, by (2.1.1), we have:

$$\begin{aligned}
x_2 - x_1 &= z_1 - z_0 - F'(y_1)^{-1}[F(z_1) - F(z_0)] - \left[F'(y_1)^{-1} - F'(y_0)^{-1}\right] F(z_0) \\
&= F'(y_1)^{-1}\left[F'(y_1)(z_1 - z_0) - \int_0^1 F'(z_0 + t(z_1 - z_0))(z_1 - z_0)\,dt\right] \\
&\quad - F'(y_1)^{-1}\left[F'(y_0) - F'(y_1)\right] F'(y_0)^{-1} F(z_0)
\end{aligned}$$

$$
= \begin{cases}
-F'(y_1)^{-1} \int_0^1 \left[F'(z_0 + t(z_1 - z_0)) - F'(y_1)\right](z_1 - z_0)\,dt \\
-F'(y_1)^{-1}\left[F'(y_0) - F'(y_1)\right](x_1 - z_0) \\
\text{if Assumption 2.3.1 holds} \\
-F'(y_1)^{-1} F'(y_1) \int_0^1 \varphi(z_0 + t(z_1 - z_0), y_1, z_1 - z_0)\,dt \\
-F'(y_1)^{-1} F'(y_1)\,\varphi(y_0, y_1, x_1 - z_0) \\
\text{if Assumption 2.3.2 holds.}
\end{cases}
$$

So,

$$
\begin{aligned}
\|x_2 - x_1\| &\le K\left[\|z_0 - y_1\| + \frac{1}{2}\|z_1 - z_0\|\right]\|z_1 - z_0\| + K\|y_1 - y_0\|\,\|x_1 - z_0\| \\
&\le \Upsilon(\eta_0)\,\eta_0^5.
\end{aligned}
$$

Now, by simply replacing $y_0, y_1, z_0, z_1, x_1, x_2$ in the above argument by $y_k, y_{k+1}, z_k, z_{k+1}, x_{k+1}, x_{k+2}$, we obtain:

$$
\|x_{k+2} - x_{k+1}\| \le \Upsilon(\eta_k)\,\eta_k^5 \le \Upsilon(r)\eta_k^5.
$$

Hence:

$$
\begin{aligned}
\|x_{k+m} - x_{k+1}\| &\le \sum_{i=0}^{m-2} \|x_{k+i+2} - x_{k+i+1}\| \\
&\le \sum_{i=0}^{m-2} \Upsilon(r)\eta_{k+i}^5 \\
&\le \Upsilon(r)\left[\eta_k^5 + \eta_{k+1}^5 + \cdots + \eta_{k+m-2}^5\right] \\
&\le \Upsilon(r)\left[\Theta(r)^{\frac{5^k-1}{4}}\eta_0 + \Theta(r)^{\frac{5^{k+1}-1}{4}}\eta_0 + \cdots + \Theta(r)^{\frac{5^{k+m-2}-1}{4}}\eta_0\right] \\
&\le \Upsilon(r)\Theta(r)^{\frac{5^k-1}{4}}\eta_0 \frac{1 - \Theta(r)^{5^k}\frac{5^{m-2}-1}{4} + 1}{1 - \Theta(r)^{5^k}} \\
&\le \frac{1}{1 - \Theta(r)^{5^k}}\Upsilon(r)\Theta(r)^{\frac{5^k-1}{4}}\eta_0 \\
&\le \frac{\eta_0}{1 - \Theta(r)^{5^k}}\Upsilon(r)\Theta(r)^{5^k} \\
&\le \frac{\eta_0}{\left(1 - \Theta(r)^{5^k}\right)\Theta(r)^{5^2}}\Upsilon(r)\Theta(r)^{5^{k+2}} \\
&\le Ce^{-\gamma_0 5^{k+2}}. \qquad (2.3.12)
\end{aligned}
$$

Thus, sequence $\{x_n\}$ is a Cauchy's sequence in $B(x_0,R)$, and hence, it converges to a solution x_* of (2.1.2). By letting $m \longrightarrow \infty$ in (2.3.12), we obtain the result.
To prove the uniqueness of the solution, let $y_* \in B(x_0,R)$ be such that $F(y_*) = 0$. Set $T = \int_0^1 F'(y_* + t(x_* - y_*))\,dt$. Then, we have:

$$\begin{aligned}
\left\| F'(x_0)^{-1}\left(T - F'(x_0)\right)\right\| &= \left\| F'(x_0)^{-1} \int_0^1 \left[F'(y_* + t(x_* - y_*)) - F'(x_0)\right] dt \right\| \\
&\le K \int_0^1 \|y_* + t(x_* - y_*) - x_0\|\, dt \\
&\le K \int_0^1 \int_0^1 \left[(1-t)\|y_* - x_0\| + t\|x_* - x_0\|\right] dt \\
&\le \frac{K}{2}(R+R) < 1, \qquad (2.3.13)
\end{aligned}$$

so T^{-1} exists. Next, the result follows from the fact that $F(x_*) - F(y_*) = T(x_* - y_*)$.
□

Remark 2.3.5. *The semi-local convergence analysis of method (2.1.1) depends on the sufficient criterion:*

$$K\Pi(\eta_0)\,\eta_0^2 < 1$$

This inequality provides an estimate on how close x_0 should be to y_0 for convergence to be achieved. In any case, the convergence domain is small in general. Notice also that the smaller K is the greater the chance is for inequality in Remark (2.3.5) to be satisfied. Another problem with the preceding results is that $F'(x)^{-1}$ must exist for each $x \in D(F)$ which is a rather strong assumption. Below, we show how to address these problems.

Assumption 2.3.6. *There exist $\bar{k}_0 > 0, x_0 \in D(F), \beta_0 > 0$, such that $F'(x_0)^{-1}$ exists, $\left\|F'(x_0)^{-1}\right\| \le \beta_0$ and for each $x \in D(F)$:*

$$\left\|F'(x) - F'(x_0)\right\| \le \bar{k}_0 \|x - x_0\|.$$

Set $D_0(F) = D(F) \cap U\left(x_0, \frac{1}{\beta_0 \bar{k}_0}\right)$. Notice that $D_0(F) = D_0(F)\left(D(F), \bar{k}_0\right)$.

That is, it is a function of $\bar{k}_0$ and $D(F)$.

Assumption 2.3.7. *There exist* $\bar{\bar{k}}_0 > 0, x_0 \in D(F), \beta_0 > 0$, *such that* $F'(x_0)^{-1}$ *exists,* $\left\|F'(x_0)^{-1}\right\| \leq \beta_0$ *and for each* $x, y \in D_0(F)$ *:*

$$\|F'(y) - F'(x)\| \leq \bar{\bar{k}}_0 \|y - x\|.$$

Remark 2.3.8. *Notice that*

$$\bar{k}_0 \leq k_0 \tag{2.3.14}$$

and

$$\bar{\bar{k}}_0 \leq k_0, \tag{2.3.15}$$

Since,

$$D_0(F) \subseteq D(F). \tag{2.3.16}$$

Moreover, by the Banach Lemma on invertible operators (Kantorovich and Akilov (1982)), $F'(x)^{-1}$ exists on $D_0(F)$ and:

$$\|F'(x)^{-1}\| \leq \frac{\beta_0}{1 - \beta_0 \bar{k}_0 \|x - x_0\|}. \tag{2.3.17}$$

Define $D_1(F) = D(F) \cap U\left(x_0, \frac{\alpha}{\beta_0 k_0}\right)$ for some $\alpha \in (0, 1)$. Then, we can define:

$$\tilde{\beta} := \frac{\beta_0}{1 - \alpha} \tag{2.3.18}$$

so we have

$$\|F'(x)^{-1}\| \leq \tilde{\beta}. \tag{2.3.19}$$

Set $\tilde{k}_0 = \max\left\{\bar{k}_0, \bar{k}_0\right\}$. Then, still, we have:

$$\tilde{k}_0 \leq k_0. \tag{2.3.20}$$

Therefore, clearly $\tilde{\beta}, \tilde{k}_0$, Assumption 2.3.6 and Assumption 2.3.7 can replace β, k_0 Assumption 2.3.1, respectively. In view of (2.3.20) (and if we set $\beta = \tilde{\beta}$), the results in this setting will be finer. Similar benefits are obtained if we use Assumption 2.3.6' obtained from Assumption 2.3.6 but holding on $D_0(F)$ instead of $D(F)$. Then, we obtain again by (2.3.16):

$$\tilde{K}_0 \leq K_0. \tag{2.3.21}$$

Hence, $\tilde{K}_0$ can replace K_0 in the proof of Lemma 2.3.3. Concerning the uniqueness, we can do better than in Theorem(2.3.4) using Assumption 2.3.6. Indeed, as in 2.3.13, we arrive at:

$$\left\| F'(x_0)^{-1}\left(T - F'(x_0)\right) \right\| \leq \frac{\beta_0}{2}\left[\|x_* - x_0\| + \|y_* - x_0\|\right] < 1,$$

if $\frac{\beta_0}{2}(R + R_1) < 1$ for some $R_1 \geq R$. Then, the solution x_* is unique in $D(F) \cap \bar{U}(x_0, R_1)$. Notice that even if $R_1 = R$, still the uniqueness is extended, since:

$$\beta_0 \leq \beta. \tag{2.3.22}$$

It is worth noticing that these advantages are obtained under the same computational effort, since, in practice, the computations of the constants in Assumption 2.3.1 and Assumption 2.3.2 require the computation of the new constants as special cases.

2.4 CONCLUSION

In this work,we studied a fifth order iterative method and established fifth order of convergence of the method using assumptions only on the first Fréchet derivative of the involved operator.

CHAPTER 3

AN APRIORI PARAMETER CHOICE STRATEGY AND A FIFTH ORDER ITERATIVE SCHEME FOR LAVRENTIEV REGULARIZATION METHOD

3.1 INTRODUCTION

In this Chapter, we consider a new source condition and a new apriori parameter choice strategy and use the iterative method in Singh et al. (2016) for Lavrentiev regularization in the nonlinear ill-posed equation.

$$F(u) = y, \tag{3.1.1}$$

where $F : D(F) \subseteq E \to E$ is a monotone nonlinear operator with positive self adjoint Fréchet derivative $F'(x), x \in D(F)$ and E is a Hilbert space. The norm on E, is denoted by $\|.\|$ and the inner product by $\langle .,. \rangle$. Throughout the chapter $R(F'(x))$ denote the range of the linear operator $F'(x)$, $x \in D(F)$. Recall that an operator F is said to be monotone if

$$\langle F(x) - F(y), x - y \rangle \geq 0, \text{ for all } x, y \in D(F).$$

For $\alpha > 0$ and for fixed $y \in E$,

$$F(u) + \alpha(u - u_0) = y \tag{3.1.2}$$

has a unique solution (George (2010); George and Elmahdy (2012); Mahale and Nair (2009); Semenova (2010); Tautenhahn (2002); Vasin and George (2014)) denoted by u_α where u_0 is the initial guess of the exact solution $\hat{u}$ of the equation (3.1.1). It is known (George (2010); George and Elmahdy (2012); Mahale and Nair (2009); Semenova (2010); Tautenhahn (2002); Vasin and George (2014)) that u_α is an approximation for $\hat{u}$ (i.e., $u_\alpha \longrightarrow \hat{u}$ as $\alpha \longrightarrow 0$).

In practice, the available data is y^δ with

$$\|y - y^\delta\| \leq \delta. \tag{3.1.3}$$

So, one has to deal with the equation

$$F(u) + \alpha(u - u_0) = y^\delta \tag{3.1.4}$$

instead of (3.1.2). The unique solution of (3.1.4) denoted by u_α^δ, is a good approximation for $\hat{u}$ provided α is chosen appropriately (George (2010); George and Elmahdy (2012); Mahale and Nair (2009); Semenova (2010); Tautenhahn (2002); Vasin and George (2014)).

In fact, we have the following estimates (see Tautenhahn (2002))

$$\|u_\alpha^\delta - u_\alpha\| \leq \frac{\delta}{\alpha} \tag{3.1.5}$$

and

$$\|u_\alpha - \hat{u}\| \leq \|\hat{u} - u_0\|. \tag{3.1.6}$$

In earlier studies such as George (2010); George and Elmahdy (2012); Mahale and Nair (2009); Semenova (2010); Tautenhahn (2002); Vasin and George (2014), the convergence rate for $\|u_\alpha - \hat{u}\|$ is obtained under the following source condition:

$$u_0 - \hat{u} \in R(F'(\hat{u})^\nu), \quad 0 < \nu \leq 1. \tag{3.1.7}$$

or

$$u_0 - \hat{u} \in R(F'(u_0)^\nu), \quad 0 < \nu \leq 1. \tag{3.1.8}$$

In this study, we introduce a new source condition (see Section 2), i.e., we assume that, there exists $\rho > 0$ such that

$$u_0 - \hat{u} = A^\nu z, \ \|z\| \leq \rho, \quad 0 < \nu \leq 1, \tag{3.1.9}$$

where $A = \int_0^1 F'(\hat{u} + t(u_0 - \hat{u}))dt$. This source condition enables us to obtain error estimate under the apriori parameter choice strategy introduced in Section 3 for choosing regularization parameter α in (3.1.4). Most of the known apriori parameter choice strategies are depending on the unknown parameter ν. The advantage of using the above source condition is that, our apriori parameter choice strategy is independent of ν (see Section 3).

One can prove that (see George et al. (2022a)) (3.1.9) implies

$$x_0 - \hat{x} = \begin{cases} F'(x_0)^{\nu_1}\xi_z, \ \|\xi_z\| \leq \rho_0 \text{ for} & 0 < \nu_1 < \nu < 1 \\ F'(x_0)\xi_{z_1}, \ \|\xi_{z_1}\| \leq \rho_1 \text{ for} & \nu = 1, \end{cases}$$

and

$$x_0 - \hat{x} = \begin{cases} F'(\hat{u})^{\nu_1}\xi_z, \ \|\xi_z\| \leq \rho_0 \text{ for} & 0 < \nu_1 < \nu < 1 \\ F'(\hat{u})\xi_{z_1}, \ \|\xi_{z_1}\| \leq \rho_1 \text{ for} & \nu = 1, \end{cases}$$

for some constants ρ_0 and ρ_1.

Remark 3.1.1.

(a) In aposteriori parameter choice strategy, the regularisation parameter α (depending on δ and y^δ) is choosen at the time of solving u_α^δ (see, Anderssen and de Hoog (1990)). In our approach, we choose the parameter α (depending on δ and y^δ) before solving u_α^δ. So, we consider our proposed method as an apriori parameter choice strategy.

(b) The new source condition enables us to choose the regularization parameter α before computing u_α^δ (see Section 3.2).

(c) Notice that, the operator A and A^ν are used to obtain an estimate for $\|\hat{u} - u_\alpha\|$. In actual computation of the approximation u_{n+1} (see (3.1.10) below) and α (see Sect. 3) we do not require the operator A or A^ν. However, one can use Dunford integral (see e.g. Plato (1995)) for fractional powers of $F'(.)$ i.e.,

$$F'(.)^\nu = \frac{\pi}{\sin(\nu\pi)} \int_0^\infty s^{\nu-1}(F'(.) + sI)^{-1} F'(.) ds, \quad 0 < \nu < 1$$

and using the above formula, we have

$$A^\nu = \frac{\pi}{\sin(\nu\pi)} \int_0^1 \int_0^\infty s^{\nu-1}(F'(\hat{u} + t(u_0 - \hat{u})) + sI)^{-1} F'(\hat{u} + t(u_0 - \hat{u})) ds dt.$$

Since getting an exact solution for (3.1.4) is difficult in general, most of the methods considered for solving (3.1.4) are iterative. In an iterative method, one looks for a higher order of convergence with high efficiency index. Recall that, if there exist positive reals β_1, β_2, such that for all $n = 1, 2, 3, \ldots$, $\|u_n - x^*\| \leq \beta_1 e^{-\beta_2 p^n}$ (Kelley (1995)), for some $p > 1$, then a sequence (u_n) in X with $\lim_{n\to\infty} u_n = x^*$ is said to be convergent of order p. In the case of an ill-posed problem, the iteration method is used to obtain an approximation for the regularized solution and hence the order of optimality of the regularization method will not change with respect to the iterative method. However, one can find a fast/ efficient algorithm to obtain the required accurate solution . For this reason we consider the analogous of the fifth order method introduced by Singh et. al in Singh et al. (2016), defined for each $k = 0, 1, \cdots$, by

$$\begin{aligned} v_k &= u_k - (F'(u_k) + \alpha I)^{-1}(F(u_k) + \alpha(u_k - u_0) - y^\delta), \\ w_k &= v_k - (F'(u_k) + \alpha I)^{-1}(F(v_k) + \alpha(v_k - u_0) - y^\delta), \qquad (3.1.10) \\ u_{k+1} &= w_k - (F'(v_k) + \alpha I)^{-1}(F(w_k) + \alpha(w_k - u_0) - y^\delta), \end{aligned}$$

to obtain an approximation for u_α^δ. Using recurrence relations we prove that method (3.1.10) is of order five in Section 3.4. A numerical example is given in Section 3.5.

3.2 ERROR BOUNDS UNDER SOURCE CONDITION (3.1.9)

Let

$$r := \|\hat{u} - u_0\|.$$

We assume that the following conditions hold:

(i) $\overline{B(u_0, r)} \subseteq D(F)$,

(ii) F has positive self-adjoint Fréchet derivatives $F'(u)$ for every $u \in \overline{B(u_0, \rho)}$, for some $\rho > 0$.

Note that $\|u_\alpha - u_0\| \leq \|u_\alpha - \hat{u}\| + \|\hat{u} - u_0\| \leq 2r$. Throughout the chapter $B(x,a) = \{y \in E : \|x-y\| < a\}$ and $\bar{B}(x,a) = \{y \in E : \|x-y\| \leq a\}$. Next, we obtain an error estimate for $\|u_\alpha - \hat{u}\|$, under the a new source condition (3.1.9) using the assumption:

Assumption 3.2.1. *There exists a constant* $K > 0$ *such that for every* $u, v \in B(\hat{u}, \rho) \subset D(F)$ *for some* $\rho > 0$, *and* $w \in X$, *there exists an element* $\phi(u,v,w) \in X$ *satisfying* $[F'(u) - F'(v)]w = F'(v)\phi(u,v,w)$, $\| \phi(u,v,w) \| \leq K \| w \| \| u - v \|$.

Theorem 3.2.2. *Let* $3Kr < 2$, *Assumptions 3.2.1 hold with* $\rho = r$ *and (3.1.9) be satisfied. Then,*

$$\|\hat{u} - u_\alpha\| \leq \left(\frac{Kr+2}{2-3Kr}\right)\alpha^\nu \|z\|.$$

Proof. Since $F(\hat{u}) = y$ and $F(u_\alpha) + \alpha(u_\alpha - u_0) = y$, so that

$$F(u_\alpha) - F(\hat{u}) + \alpha(u_\alpha - u_0) = 0,$$

i.e.,

$$F(u_\alpha) - F(\hat{u}) + \alpha(u_\alpha - \hat{u}) = \alpha(u_0 - \hat{u}), \tag{3.2.1}$$

or

$$(M + \alpha I)(u_\alpha - \hat{u}) = \alpha(u_0 - \hat{u}), \tag{3.2.2}$$

where

$$M = \int_0^1 F'(\hat{u} + t(u_\alpha - \hat{u}))\, dt.$$

Again (3.2.2) can be written as

$$(A_0+\alpha I)(u_\alpha-\hat{u})=(A_0-M)(u_\alpha-\hat{u})+\alpha(u_0-\hat{u}),$$

where $A_0=F'(u_0)$. Thus, we have

$$\begin{aligned}
u_\alpha-\hat{u} &= -(A_0+\alpha I)^{-1}(M-A_0)(u_\alpha-\hat{u})+\alpha(A_0+\alpha I)^{-1}(u_0-\hat{u})\\
&= -(A_0+\alpha I)^{-1}\int_0^1[F'(\hat{u}+t(u_\alpha-\hat{u}))-A_0]dt(u_\alpha-\hat{u})\\
&\quad +\alpha(A_0+\alpha I)^{-1}(u_0-\hat{u})\\
&= -(A_0+\alpha I)^{-1}A_0\int_0^1\phi(\hat{u}+t(u_\alpha-\hat{u}),u_0,u_\alpha-\hat{u})\,dt\\
&\quad +\alpha(A_0+\alpha I)^{-1}(u_0-\hat{u})
\end{aligned}$$

and hence

$$\|u_\alpha-\hat{u}\| \leq K\frac{3r}{2}\|u_\alpha-\hat{u}\|+\|\alpha(A_0+\alpha I)^{-1}(u_0-\hat{u})\|$$

so,

$$\begin{aligned}
\left(1-\frac{3Kr}{2}\right)\|u_\alpha-\hat{u}\| &\leq \|\alpha(A_0+\alpha I)^{-1}(u_0-\hat{u})\|\\
&\leq \|\alpha[(A_0+\alpha I)^{-1}-(A+\alpha I)^{-1}](u_0-\hat{u})\| \qquad (3.2.3)\\
&\quad +\|\alpha(A+\alpha I)^{-1}(u_0-\hat{u})\|\\
&\leq \|(A_0+\alpha I)^{-1}[A-A_0]\alpha(A+\alpha I)^{-1}(u_0-\hat{u})\|\\
&\quad +\|\alpha(A+\alpha I)^{-1}(u_0-\hat{u})\|\\
&\leq \|(A_0+\alpha I)^{-1}A_0\int_0^1\phi(\hat{u}+t(u_0-\hat{u}),u_0,\alpha(A+\alpha I)^{-1}(u_0-\hat{u}))dt\|\\
&\quad +\|\alpha(A+\alpha I)^{-1}(u_0-\hat{u})\|\\
&\leq (\frac{Kr}{2}+1)\|\alpha(A+\alpha I)^{-1}(u_0-\hat{u})\|. \qquad (3.2.4)
\end{aligned}$$

Now, since $u_0-\hat{u}=A^\nu z$, by (3.2.4), we have

$$\|u_\alpha-\hat{u}\| \leq \left(\frac{Kr+2}{2-3Kr}\right)\|\alpha(A+\alpha I)^{-1}A^\nu z\| \qquad (3.2.5)$$

$$\leq \left(\frac{Kr+2}{2-3Kr}\right)\alpha^{\nu}\rho. \tag{3.2.6}$$

□

Combining, (3.1.5) and Theorem 3.2.2, we obtain the following Theorem.

Theorem 3.2.3. *Let $3Kr < 2$, Assumptions 3.2.1 hold with $\rho = r$ and (3.1.9) be satisfied. Then*

$$\|\hat{u} - u_{\alpha}^{\delta}\| \leq \max\left\{1, \left(\frac{Kr+2}{2-3Kr}\right)\rho\right\}(\frac{\delta}{\alpha} + \alpha^{\nu}).$$

Proof. Follow from (3.1.5), Theorem 3.2.2 and the triangle inequality,

$$\|\hat{u} - u_{\alpha}^{\delta}\| \leq \|\hat{u} - u_{\alpha}\| + \|u_{\alpha} - u_{\alpha}^{\delta}\|.$$

□

Remark 3.2.4. *The estimate $\frac{\delta}{\alpha} + \alpha^{\nu}$ in Theorem 3.4.2 is order optimal, when $\frac{\delta}{\alpha} = \alpha^{\nu}$ or $\alpha = \delta^{\frac{1}{\nu+1}}$ and in this case*

$$\|\hat{u} - u_{\alpha}^{\delta}\| = O(\delta^{\frac{\nu}{\nu+1}}). \tag{3.2.7}$$

But since ν is not known, such a choice of α (i.e., $\frac{\delta}{\alpha} = \alpha^{\nu}$) is not possible in practice, so one has to consider some parameter choice strategy for choosing the parameter $\alpha = \alpha(\delta)$ so as to obtain the estimate in (3.2.7).

Next, section, we introduce a new “apriori" parameter choice strategy depending on the linear operator $A_0 = F'(x_0)$.

3.3 APRIORI PARAMETER CHOICE STRATEGY

For $u \in X$, define

$$\Phi(\alpha, u) := \|\alpha^2(A_0 + \alpha I)^{-2}(F(u_0) - u)\|. \tag{3.3.1}$$

Theorem 3.3.1. *For each $u \in X$, the function $\alpha \to \Phi(\alpha, u)$ for $\alpha > 0$, defined in (3.3.1),* *is continuous, monotonically increasing and*

$$\lim_{\alpha\to 0}\Phi(\alpha,u)=0,\quad \lim_{\alpha\to\infty}\Phi(\alpha,u)=\|F(u_0)-u\|.$$

Proof. Analogous to the proof of Lemma 1 in George and Nair (1993).

□

In addition to (3.1.4), we assume that

$$c\delta\le\|F(u_0)-y^\delta\|, \tag{3.3.2}$$

for some $c>1$. Then by the intermediate value theorem, we have the following theorem.

Theorem 3.3.2. *If y^δ satisfies (3.1.4) and (3.3.2). Then, there exists a unique α such that*

$$\Phi(\alpha,y^\delta)=c\,\delta. \tag{3.3.3}$$

Next, we shall show that if $\alpha=\alpha(\delta)$ satisfies (3.3.3), then $\|\alpha(A+\alpha I)^{-1}A^\nu z\|=O(\delta^{\frac{\nu}{\nu+1}})$. Our proof is based on the following moment inequality

$$\|B^u x\|\le\|B^v x\|^{\frac{u}{v}}\,\|x\|^{1-\frac{u}{v}},\quad 0\le u\le v, \tag{3.3.4}$$

where B is positive self-adjoint operator.

Theorem 3.3.3. *Let Assumptions 3.2.1 hold with $\rho=r$ and (3.1.9) be satisfied and let $\alpha=\alpha(\delta)$ be the unique solution of (3.3.3). Then*

$$\|\hat{u}-u_\alpha\|\le O(\delta^{\frac{\nu}{\nu+1}}).$$

Proof. Let $u=\nu$, $v=1+\nu$, $B=\alpha(A+\alpha I)^{-1}A$ and $x=\alpha^{1-\nu}(A+\alpha I)^{-(1-\nu)}z$. Then, by (3.3.4), we have

$$\begin{aligned}\|\alpha(A+\alpha I)^{-1}A^\nu z\| &= \|B^\nu x\|\\ &\le \|B^{1+\nu}x\|^{\frac{\nu}{1+\nu}}\,\|x\|^{\frac{1}{1+\nu}}\end{aligned}$$

$$\begin{aligned}
&\le \|\alpha^2 (A+\alpha I)^{-2} A^{1+\nu} z\|^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}}\\
&\le \|\alpha^2 (A+\alpha I)^{-2} A (u_0-\hat{u})\|^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}}\\
&= \|\alpha^2 (A+\alpha I)^{-2} (F(u_0)-y)\|^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}}\\
&\le \Big(\|\alpha^2 (A+\alpha I)^{-2} (F(u_0)-y^\delta)\|\\
&\quad + \|\alpha^2 (A+\alpha I)^{-2} (y^\delta - y)\|\Big)^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}}\\
&= (\mathscr{B}_1+\delta)^{\frac{\nu}{1+\nu}} \|z\|^{\frac{1}{1+\nu}}, \qquad (3.3.5)
\end{aligned}$$

where $\mathscr{B}_1 = \|\alpha^2 (A+\alpha I)^{-2} (F(u_0)-y^\delta)\|$ and

$$\|\alpha^2 (A+\alpha I)^{-2} (y^\delta - y)\| \le \delta.$$

We have again by (3.3.3)

$$\begin{aligned}
\mathscr{B}_1 &= \|\alpha^2 (A+\alpha I)^{-2} (F(u_0)-y^\delta)\|\\
&= \|\alpha^2[(A+\alpha I)^{-2}-(A_0+\alpha I)^{-2}](F(u_0)-y^\delta)+\alpha^2(A_0+\alpha I)^{-2}(F(u_0)-y^\delta)\|\\
&\le \|\alpha^2[(A+\alpha I)^{-2}-(A_0+\alpha I)^{-2}](F(u_0)-y^\delta)\|+\|\alpha^2(A_0+\alpha I)^{-2}(F(u_0)-y^\delta)\|\\
&=: \mathscr{D}_1+c\delta, \qquad (3.3.6)
\end{aligned}$$

where $\mathscr{D}_1 = \|\alpha^2[(A+\alpha I)^{-2}-(A_0+\alpha I)^{-2}](F(u_0)-y^\delta)\|$. Let $w = \alpha^2(A_0+\alpha I)^{-2}(F(u_0)-y^\delta)$. Then, $\|w\| = c\delta$ and hence, we get

$$\begin{aligned}
\mathscr{D}_1 &= \|\alpha^2[(A+\alpha I)^{-2}-(A_0+\alpha I)^{-2}](F(u_0)-y^\delta)\|\\
&= \|(A+\alpha I)^{-2}[A^2-A_0^2+2\alpha(A-A_0)]w\|\\
&= \|(A+\alpha I)^{-2}[(A+A_0)+2\alpha I](A-A_0)w\|\\
&= \|(A+\alpha I)^{-2}[A-A_0+2A_0+2\alpha I](A-A_0)w\|\\
&= \|[(A+\alpha I)^{-1}(A-A_0)]^2 w+2(A+\alpha I)^{-1}(A-A_0)w\|\\
&\le (\Gamma^2+\Gamma)c\delta, \qquad (3.3.7)
\end{aligned}$$

where $\Gamma = \|(A+\alpha I)^{-1}(A-A_0)\|$. Note that

$$\begin{aligned}
\|(A+\alpha I)^{-1}(A-A_0)x\| &\leq \|[(A+\alpha I)^{-1}-(A_0+\alpha I)^{-1}](A-A_0)x\| \\
&\quad +\|(A_0+\alpha I)^{-1}(A-A_0)x\| \\
&\leq \|(A_0+\alpha I)^{-1}[A_0-A](A+\alpha I)^{-1}(A-A_0)x\| \\
&\quad +\|(A_0+\alpha I)^{-1}(A-A_0)x\| \\
&\leq \|-\int_0^1 \phi(\hat{u}+t(u_0-\hat{u}),u_0,(A+\alpha I)^{-1}(A-A_0)x)dt\| \\
&\quad +\|\int_0^1 \phi(\hat{u}+t(u_0-\hat{u}),u_0,x)dt\| \\
&\leq \frac{Kr}{2}\|(A+\alpha I)^{-1}(A-A_0)x)\| + \frac{Kr}{2}\|x\|,
\end{aligned}$$

i.e., $\|(A+\alpha I)^{-1}(A-A_0)x\| \leq \frac{Kr}{2-Kr}\|x\|$ and hence

$$\mathscr{D}_1 \leq \frac{Kr}{2-Kr}\left(\frac{Kr}{2-Kr}+1\right)c\delta. \tag{3.3.8}$$

Now, the result follows from (3.3.5)–(3.3.8). □

Theorem 3.3.4. *Suppose Assumption 3.2.1 hold with* $\rho = r$ *and if* $\alpha = \alpha(\delta)$ *is the solution of (3.3.3). Then* $\frac{\delta}{\alpha} = O\left(\delta^{\frac{\nu}{\nu+1}}\right)$.

Proof. By (3.3.3), we have

$$\begin{aligned}
c\delta &= \|\alpha^2(A_0+\alpha I)^{-2}(F(u_0)-y^\delta)\| \\
&\leq \|\alpha^2(A_0+\alpha I)^{-2}(F(u_0)-y)\| + \|\alpha^2(A_0+\alpha I)^{-2}(y-y^\delta))\| \\
&\leq \|\alpha^2(A_0+\alpha I)^{-2}(F(u_0)-y)\| + \|\alpha^2(A_0+\alpha I)^{-2}\|\,\|(y-y^\delta))\|. \\
&\leq \|\alpha^2(A_0+\alpha I)^{-2}(F(u_0)-y)\| + \delta,
\end{aligned}$$

so,

$$\begin{aligned}
(c-1)\delta &\leq \|\alpha^2(A_0+\alpha I)^{-2}(F(u_0)-y)\| \qquad (3.3.9)\\
&\leq \|\alpha^2\left[(A_0+\alpha I)^{-2}-(A+\alpha I)^{-2}\right](F(u_0)-y)\| \\
&\quad +\|\alpha^2(A+\alpha I)^{-2}(F(u_0)-y)\|
\end{aligned}$$

$$
\begin{aligned}
&= \|(A_0+\alpha I)^{-2}\left[(A+\alpha I)^2-(A_0+\alpha I)^2\right]\alpha^2(A+\alpha I)^{-2}(F(u_0)-y^\delta)\| \\
&\quad +\|\alpha^2(A+\alpha I)^{-2}(F(u_0)-y)\|. \qquad (3.3.10)
\end{aligned}
$$

Let $w_1=\alpha^2(A+\alpha I)^{-2}(F(u_0)-y)$. Then, as in (3.3.7), we have

$$
\begin{aligned}
(c-1)\delta &\leq \|(A_0+\alpha I)^{-2}\left[(A+\alpha I)^2-(A_0+\alpha I)^2\right]w_1\|+\|w_1\| \\
&\leq (\Gamma_1^2+2\Gamma_1+1)\|w_1\| \qquad (3.3.11)
\end{aligned}
$$

where $\Gamma_1=\|(A_0+\alpha I)^{-1}(A-A_0)\|$. Note that

$$
\begin{aligned}
\|(A_0+\alpha I)^{-1}(A-A_0)x\| &\leq \|(A_0+\alpha I)^{-1}\int_0^1(F'(\hat{u}+t(u_0-\hat{u}))-F'(u_0))x\| \\
&\leq \|(A_0+\alpha I)^{-1}A_0\int_0^1\varphi(u_0,\hat{u}+t(u_0-\hat{u}),x)\| \\
&\leq \frac{K}{2}r\|x\|,
\end{aligned}
$$

so

$$
\Gamma_1\leq\frac{K}{2}r. \qquad (3.3.12)
$$

Therefore by (3.3.11) and (3.3.12), we have

$$
\begin{aligned}
(c-1)\delta &\leq \left(\frac{K}{2}r(\frac{K}{2}r+1)+1\right)\|w_1\| \\
&= \left(\frac{K}{2}r(\frac{K}{2}r+1)+1\right)\|\alpha^2(A+\alpha I)^{-2}A(u_0-\hat{u})\| \\
&= \left(\frac{K}{2}r(\frac{K}{2}r+1)+1\right)\|\alpha^2(A+\alpha I)^{-2}A^{1+\nu}z\| \\
&\leq \left(\frac{K}{2}r(\frac{K}{2}r+1)+1\right)\|\alpha^2(A+\alpha I)^{-1}A^{\nu}z\| \\
&\leq \left(\frac{K}{2}r(\frac{K}{2}r+1)+1\right)\alpha^{1+\nu}\|z\|,
\end{aligned}
$$

i.e.,

$$
\alpha^{1+\nu}\geq\frac{c-1}{(\frac{K}{2}r(\frac{K}{2}r+1)+1)\|z\|}\delta:=C_{K,r}\delta. \qquad (3.3.13)
$$

Thus , we have

$$\frac{\delta}{\alpha} = \left(\frac{\delta}{\alpha^{1+\nu}}\right)^{\frac{1}{1+\nu}} \delta^{\frac{\nu}{1+\nu}} \leq \left(\frac{1}{C_{K,r}}\right)^{\frac{1}{1+\nu}} \delta^{\frac{\nu}{1+\nu}}.$$

$\square$

By combining Theorem 3.3.3 and Theorem 3.3.4, we obtain the following Theorem.

Theorem 3.3.5. *Suppose* $3Kr < 2$, *Assumption 3.2.1 hold with* $\rho = r$ *and if* $\alpha = \alpha(\delta)$ *is the solution of (3.3.3). Then*

$$\|u_\alpha^\delta - \hat{u}\| = O\left(\delta^{\frac{\nu}{\nu+1}}\right).$$

$\square$

Next, section, we prove that the iterative method (3.1.10) converges to u_α^δ with a convergence order five.

3.4 CONVERGENCE ANALYSIS OF (3.1.10)

We define some functions and parameters, used for proving the convergence of method (3.1.10).

Define functions $\xi, \Pi : [0,\infty) \longrightarrow [0,\infty)$ by

$$\xi(t) = \frac{K^2}{2}(\frac{K}{4}t + 1),$$

$$\Pi(t) = K\left[(\frac{3}{2} + \frac{K}{4}t)(\frac{K}{2}t + 1) + 1\right],$$

where K is as in Assumption 3.2.1. Let $h_1 : [0,\infty) \longrightarrow [0,\infty)$ be defined by

$$h_1(t) = K\Pi(t)t^2 - 1.$$

Then, $h_1(0) = -1 < 0$ and $h_1(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. So, by intermediate value theorem h_1 has a minimal zero r_1.

Define functions $\Theta, \psi, \gamma, \delta, \Upsilon : [0, r_1) \longrightarrow [0, \infty)$ by

$$\Theta(t) = \frac{K[\Pi(t) + \frac{1}{2}\xi(t)t]\xi(t)t^4}{1 - K\Pi(t)t^2},$$

$$\psi(t) = \left(1 + (KR_0 + 1)K\left[K\frac{t^2}{2} + \frac{\xi(t)t^3}{2}\right]\right)\xi(t),$$

$$\gamma(t) = K[\xi(t) + \frac{1}{2}\psi(t)]\psi(t)t^3 + (2 + KR_0)\xi(t),$$

where $R_0 > 0$ to be precised later,

$$\delta(t) = \psi(t)t + \frac{K}{2},$$

$$\Upsilon(t) = K[(\psi(t) + \frac{1}{2}\gamma(t))\gamma(t)t + \delta(t)\xi(t)].$$

Then, $\xi, \psi, \gamma, \delta, \Upsilon$ are monotonically increasing and continuous functions.

Let $h_2 : [0, r_1] \longrightarrow [0, \infty)$ be defined by

$$h_2(t) = \Theta(t) - 1.$$

Then, $h_2(0) = -1 < 0$ and $h_2(t) \longrightarrow \infty$ as $t \longrightarrow r_1^-$. By intermediate value theorem h_2 has a minimal zero r_0 in $(0, r_1)$. Then, for all $t \in [0, r_0]$, we have

$$0 < K\Pi(t)t^2 < 1 \tag{3.4.1}$$

$$0 < \Theta(t) < 1. \tag{3.4.2}$$

Let $\eta_k := \|u_k - v_k\|$. Note that

$$\begin{aligned}
\eta_0 &= \|(F'(u_0) + \alpha I)^{-1}(F(u_0) - y^\delta)\| \\
&= \|(F'(u_0) + \alpha I)^{-1}(F(u_0) - y + y - y^\delta)\| \\
&\leq \|(F'(u_0) + \alpha I)^{-1}\int_0^1 F'(\hat{u} + t(u_0 - \hat{u})(u_0 - \hat{u})dt\| \\
&\quad + \|(F'(u_0) + \alpha I)^{-1}(y - y^\delta)\| \\
&\leq \|(F'(u_0) + \alpha I)^{-1}\int_0^1 [F'(\hat{u} + t(u_0 - \hat{u}) - F'(u_0)](u_0 - \hat{u})dt\|
\end{aligned}$$

$$
\begin{aligned}
&+\|(F'(u_0)+\alpha I)^{-1}F'(u_0)(u_0-\hat{u})\|+\frac{\delta}{\alpha}\\
\leq\ & \frac{K}{2}r^2+r+1 \text{ (for small } \delta).
\end{aligned}
$$

Hence, we assume that $\frac{K}{2}r^2+r+1\leq r_0$. First we shall show using mathematical induction, that the sequence $\{\eta_k\}$ converges with order of convergence five.

Lemma 3.4.1. *Let u_n, v_n, w_n be as in (3.1.10) and Assumption 3.2.1 hold with $\rho = R_0$. Then, using the above notation, we have the following:*

(a) $\|w_k - v_k\| \leq \frac{K}{2}\eta_k^2$.

(b) $\|u_{k+1}-w_k\|\leq \psi(\eta_k)\eta_k^3$.

(c) $\|u_{k+1}-v_k\|\leq \Pi(\eta_k)\eta_k^2$.

(d) $\eta_k\leq \Theta(\eta_{k-1})\eta_{k-1}$.

(e) $\Theta(\eta_k)\leq \Theta(r)^{5^k}, \forall k\geq 0$.

(f) $\eta_k\leq \Theta(r)^{\frac{(5^k-1)}{4}}\eta_0$.

Proof. The proof is by induction. Let $G(u) = F(u)+\alpha(u-u_0)-f^{\delta}$. Then. $G'(u) = F'(u)+\alpha I$. Since F is monotone and $F'(.)$ is positive self adjoint $\|G'(u)^{-1}\|\leq\frac{1}{\alpha}$ and $\|G'(u)^{-1}F'(u)\|\leq 1$, for all $u\in D(F)$. By (3.1.10), we have

$$
\begin{aligned}
w_0-v_0 &= v_0-u_0-G'(u_0)^{-1}[F(v_0)-F(u_0)+\alpha(v_0-u_0)]\\
&= G'(u_0)^{-1}[F'(u_0)(v_0-u_0)-\int_0^1 F'(u_0+t(v_0-u_0))(v_0-u_0)dt\\
&= -G'(u_0)^{-1}\int_0^1[F'(u_0+t(v_0-u_0))-F'(u_0)](v_0-u_0)dt\\
&= -G'(u_0)^{-1}F'(u_0)\int_0^1 \varphi(u_0+t(v_0-u_0),u_0,v_0-u_0)dt.
\end{aligned}
$$

Hence, we get

$$
\|w_0-v_0\|\leq\frac{K}{2}\|u_0-v_0\|^2=\frac{K}{2}\eta_0^2. \tag{3.4.3}
$$

We obtain again by (3.1.10),

$$\begin{aligned}
u_1 - w_0 &= w_0 - v_0 - [G'(v_0)^{-1}G(w_0) - G'(u_0)^{-1}G(v_0)] \\
&= w_0 - v_0 - G'(v_0)^{-1}[G(w_0) - G(v_0)] + [G'(u_0)^{-1} - G'(v_0)^{-1}]G(v_0) \\
&= G'(v_0)^{-1}[F'(v_0)(w_0 - v_0) - (F(w_0) - F(v_0))] \\
&\quad + G'(v_0)^{-1}[G'(v_0) - F'(u_0)]G'(u_0)^{-1}G(v_0) \\
&= G'(v_0)^{-1}\int_0^1 [F'(v_0) - F'(v_0 + t(w_0 - v_0))](w_0 - v_0)dt \\
&\quad - G'(v_0)^{-1}[F'(v_0) - F'(u_0)](w_0 - v_0) \\
&= -G'(v_0)^{-1}F'(v_0)\int_0^1 \varphi(v_0 + t(w_0 - v_0), v_0, w_0 - v_0)dt \\
&\quad + G'(v_0)^{-1}F'(v_0)\varphi(u_0, v_0, w_0 - v_0).
\end{aligned}$$

So, by (3.4.3), we have

$$\begin{aligned}
\|u_1 - w_0\| &\le K\|w_0 - v_0\|\left(\frac{1}{2}\|w_0 - v_0\| + \|u_0 - v_0\|\right) \\
&\le \frac{K^3}{8}\eta_0^4 + \frac{K^2}{2}\eta_0^3 \\
&= \xi(\eta_0)\eta_0^3.
\end{aligned} \tag{3.4.4}$$

Further, we get

$$\begin{aligned}
u_1 - v_0 &= w_0 - u_0 - G'(v_0)^{-1}G(w_0) + G'(u_0)^{-1}G(u_0) \\
&= w_0 - u_0 - G'(v_0)^{-1}[G(w_0) - G(u_0)] + [G'(u_0)^{-1} - G'(v_0)^{-1}]G(u_0) \\
&= G'(v_0)^{-1}[F'(v_0)(w_0 - u_0) - \int_0^1 F'(u_0 + t(w_0 - u_0))(w_0 - u_0)dt] \\
&\quad + G'(v_0)^{-1}[F'(v_0) - F'(u_0)]G'(u_0)^{-1}G(u_0) \\
&= -G'(v_0)^{-1}F'(v_0)\int_0^1 \varphi(u_0 + t(w_0 - u_0), v_0, w_0 - u_0)dt \\
&\quad - G'(v_0)^{-1}F'(v_0)\varphi(u_0, v_0, u_0 - v_0).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|u_1 - v_0\| &\leq K\left[\|u_0 - v_0\| + \frac{1}{2}\|w_0 - u_0\|\right]\|w_0 - u_0\| + K\|u_0 - v_0\|^2 \\
&\leq K\left[\left(\|u_0 - v_0\| + \frac{1}{2}(\|w_0 - v_0\| + \|v_0 - u_0\|)\right)\right. \\
&\quad \left.\times(\|w_0 - v_0\| + \|u_0 - v_0\|) + \|u_0 - v_0\|^2\right] \\
&\leq K\left[(\frac{3}{2}\|u_0 - v_0\| + \frac{K}{4}\|u_0 - v_0\|^2)(\|u_0 - v_0\|\right. \qquad (3.4.5) \\
&\quad \left. + \frac{K}{2}\|u_0 - v_0\|^2) + \|u_0 - v_0\|^2\right] \\
&\leq \Pi(\eta_0)\eta_0^2. \qquad (3.4.6)
\end{aligned}$$

Next, we shall prove that $\eta_1 < \eta_0$. Note that,

$$\begin{aligned}
u_1 - v_1 &= w_0 - u_1 - G'(v_0)^{-1}G(w_0) + G'(u_1)^{-1}G(u_1) \\
&= w_0 - u_1 - G'(v_0)^{-1}[G(w_0) - G(u_1)] + [G'(u_1)^{-1} - G'(v_0)^{-1}]G(u_1) \\
&= G'(v_0)^{-1}[F'(v_0)(w_0 - u_1) - \int_0^1 F'(u_1 + t(w_0 - u_1))dt(w_0 - u_1) \\
&\quad + G'(v_0)^{-1}[F'(v_0) - F'(u_1)]G'(u_1)^{-1}G(u_1) \\
&= -G'(v_0)^{-1}F'(v_0)\int_0^1 \varphi(u_1 + t(w_0 - u_1), v_0, w_0 - u_1)dt \\
&\quad - G'(v_0)^{-1}F'(v_0)\varphi(u_1, v_0, u_1 - v_1),
\end{aligned}$$

so that

$$\begin{aligned}
\|u_1 - v_1\| &\leq K\left[\|u_1 - v_0\| + \frac{1}{2}\|u_1 - w_0\|\right]\|u_1 - w_0\| + K\|u_1 - v_0\|\|u_1 - v_1\| \\
&\leq K\left[\Pi(\eta_0)\eta_0^2 + \frac{1}{2}\xi(\eta_0)\eta_0^3\right]\xi(\eta_0)\eta_0^3 + K\Pi(\eta_0)\eta_0^2\|u_1 - v_1\| \\
&\leq K\left[\Pi(\eta_0)\eta_0^2 + \frac{1}{2}\xi(\eta_0)\eta_0^3\right]\xi(\eta_0)\eta_0^3 + K\Pi(\eta_0)\eta_0^2\|u_1 - v_1\|
\end{aligned}$$

$$(1 - K\Pi(\eta_0)\eta_0^2)\|u_1 - v_1\| \leq K\left[\Pi(\eta_0) + \frac{1}{2}\xi(\eta_0)\eta_0\right]\xi(\eta_0)\eta_0^5. \qquad (3.4.7)$$

Therefore, we have

$$\eta_1 = \|u_1 - v_1\| \leq \Theta(\eta_0)\eta_0 \leq \eta_0. \qquad (3.4.8)$$

The induction for (a), (b), (c) and (d), is completed, if we simply replace, u_0, v_0, w_0, u_1, v_1 by $u_k, v_k, w_k, u_{k+1}, v_{k+1}$ in the preceding arguments. To prove (e), we observe that

$$\Theta(\mu t) \leq \mu^4 \Theta(t) \quad \text{for} \quad 0 < \mu < 1$$

$$\Theta(\eta_k) \leq \Theta(\eta_{k-1})^5 \leq \cdots \leq \Theta(\eta_0)^{5^k} \leq \Theta(r)^{5^k}. \tag{3.4.9}$$

Notice that

$$\begin{aligned} \eta_k &\leq \Theta(\eta_{k-1})\eta_{k-1} \\ &\leq \Theta(\eta_0)^{5^{k-1}} \Theta(\eta_{k-2})\eta_{k-2} \\ &\leq \Theta(\eta_0)^{5^{k-1}+5^{k-2}+\cdots+1}\eta_0 \\ &\leq \Theta(r)^{\frac{5^k-1}{4}}\eta_0. \end{aligned} \tag{3.4.10}$$

□

Theorem 3.4.2. *Let $R_0 = r_0 + \max\{\frac{K}{2}r_0^2, \Pi(r_0)r_0^2\}$. Assumptions of Lemma 3.4.1 hold. Then, $v_k, w_k, u_{k+1} \in B(u_0, R_0)$ for all $K \geq 0$ and*

$$\|u_{k+2} - u_\alpha^\delta\| \leq C_0 \, e^{-\gamma_0 \, 5^k} \tag{3.4.11}$$

where $C_0 = \frac{\mu_0}{1-\Theta(r_0)^{5^k}} \Upsilon(r_0)$ and $\gamma_0 = -\log(\Theta(r_0))$.

Proof. Clearly, $u_0, v_0 \in B(u_0, r_0) \subset B(u_0, R_0)$. Observe that, by triangle inequality and (3.4.3), we have $\|w_0 - u_0\| \leq \|w_0 - v_0\| + \|v_0 - u_0\| \leq \frac{K}{2}\eta_0^2 + \eta_0 \leq R_0$, so $w_0 \in B(u_0, R_0)$. Also

$$\|u_1 - u_0\| \leq \|u_1 - v_0\| + \|v_0 - u_0\| \leq \Pi(\eta_0)\eta_0^2 + \eta_0 \leq R_0.$$

If we replace v_0, w_0, u_1 by v_k, w_k, u_{k+1} in the above argument, we obtain $v_k, w_k, u_{k+1} \in B(u_0, R_0)$ for all $k \geq 0$. Observe that, for all $u, v \in B(u_0, R_0)$,

$$\begin{aligned} \|G'(u)^{-1}F'(v)x\| &= \|G'(u)^{-1}[F'(v) - F'(u) + F'(u)]x\| \\ &\leq \|G'(u)^{-1}F'(u)\varphi(v,u,x)\| + \|G'(u)^{-1}F'(u)x\| \end{aligned}$$

$$
\begin{aligned}
&\le (K\|u-v\|+1)\|x\| \\
&\le (KR_0+1)\|x\|,
\end{aligned}
$$

i.e., $\|G'(u)^{-1}F'(v)\| \le KR_0+1$. Also, since

$$
G(u_1)-G(w_0)=\int_0^1 G'(w_0+t(u_1-w_0))dt(u_1-w_0),
$$

we have

$$
\begin{aligned}
v_1-w_0 &= u_1-w_0-G'(u_1)^{-1}G(u_1) \\
&= u_1-w_0-G'(u_1)^{-1}\left[G(w_0)+\int_0^1 G'(w_0+t(u_1-w_0))dt(u_1-w_0)\right] \\
&= u_1-w_0+G'(u_1)^{-1}\left[G(w_0)\right. \\
&\quad \left.+\int_0^1 F'(w_0+t(u_1-w_0))(u_1-w_0)dt+\alpha(u_1-w_0)\right] \\
&= u_1-w_0+G'(u_1)^{-1}[-G'(v_0)(u_1-w_0) \\
&\quad +\int_0^1 F'(w_0+t(u_1-w_0))(u_1-w_0)dt+\alpha(u_1-w_0)] \\
&= u_1-w_0+G'(u_1)^{-1}\int_0^1 [F'(w_0+t(u_1-w_0))-F'(v_0)](u_1-w_0)dt \\
&= u_1-w_0+G'(u_1)^{-1}F'(v_0)\int_0^1 \varphi(w_0+t(u_1-w_0),v_0,u_1-w_0)dt,
\end{aligned}
$$

so by (3.4.4), we get

$$
\begin{aligned}
\|v_1-w_0\| &\le \|u_1-w_0\|+\|G'(u_1)^{-1}F'(v_0)\|K(\|w_0-v_0\|+\frac{\|u_1-w_0\|}{2})\|u_1-w_0\| \\
&\le \xi(\eta_0)\eta_0^3+(KR_0+1)K\left[K\frac{\eta_0^2}{2}+\frac{\xi(\eta_0)\eta_0^3}{2}\right]\xi(\eta_0)\eta_0^3 \\
&= \psi(\eta_0)\eta_0^3. \qquad (3.4.12)
\end{aligned}
$$

$$
\begin{aligned}
w_1-w_0 &= v_1-w_0-G'(u_1)^{-1}G(v_1) \\
&= G'(u_1)^{-1}[G'(u_1)(v_1-w_0)-G(v_1)] \\
&= G'(u_1)^{-1}[G'(u_1)(v_1-w_0)-(G(v_1)-G(w_0))]-G'(u_1)^{-1}G(w_0)
\end{aligned}
$$

$$
\begin{aligned}
&= -G'(u_1)^{-1}\int_0^1 [F'(w_0+t(v_1-w_0))-F'(u_1)](v_1-w_0)dt \\
&\quad +G'(u_1)^{-1}G'(v_0)(u_1-w_0) \\
&= -G'(u_1)^{-1}F'(u_1)\int_0^1 \varphi(w_0+t(v_1-w_0),u_1,v_1-w_0)dt \\
&\quad +G'(u_1)^{-1}G'(v_0)(u_1-w_0).
\end{aligned}
$$

Again since

$$\|G'(u_1)^{-1}G'(v_0)\| = \|G'(u_1)^{-1}(F'(v_0)+\alpha I)\| \le KR_0+2,$$

we have, by (3.4.6)

$$
\begin{aligned}
\|w_1-w_0\| &= K[\|u_1-w_0\|+\frac{1}{2}\|v_1-w_0\|]\|v_1-w_0\|+(2+KR_0)\|u_1-w_0\| \\
&\le K[\xi(\eta_0)\eta_0^3+\frac{1}{2}\psi(\eta_0)\eta_0^3]\psi(\eta_0)\eta_0^3+(2+KR_0)\xi(\eta_0)\eta_0^3 \\
&= \gamma(\eta_0)\eta_0^3. \qquad (3.4.13)
\end{aligned}
$$

Further,

$$
\begin{aligned}
\|v_1-v_0\| &\le \|v_1-w_0\|+\|w_0-v_0\| \\
&\le \psi(\eta_0)\eta_0^3+\frac{K}{2}\eta_0^2 \\
&= \delta(\eta_0)\eta_0^2, \qquad (3.4.14)
\end{aligned}
$$

$$
\begin{aligned}
u_2-u_1 &= w_1-w_0-G'(v_1)^{-1}[G(w_1)-G(w_0)]-[G'(v_1)^{-1}-G'(v_0)^{-1}]G(w_0) \\
&= G'(v_1)^{-1}[F'(v_1)(w_1-w_0)-\int_0^1 F'(w_0+t(w_1-w_0))(w_1-w_0)dt] \\
&\quad -G'(v_1)^{-1}[G'(v_0)-G'(v_1)]G'(v_0)^{-1}G(w_0) \\
&= -G'(v_1)^{-1}F'(v_1)\int_0^1 \varphi(w_0+t(w_1-w_0),v_1,w_1-w_0)dt \\
&\quad -G'(v_1)^{-1}F'(v_1)\varphi(v_0,v_1,u_1-w_0),
\end{aligned}
$$

so, we obtain

$$\begin{aligned} \|u_2-u_1\| &\leq K\left[\|w_0-v_1\|+\frac{1}{2}\|w_1-w_0\|\right]\|w_1-w_0\|+K\|v_1-v_0\|\|u_1-w_0\| \\ &\leq K[\psi(\eta_0)\eta_0^3+\frac{1}{2}\gamma(\eta_0)\eta_0^3]\gamma(\eta_0)\eta_0^3+K\delta(\eta_0)\eta_0^2\xi(\eta_0)\eta_0^3 \\ &= \Upsilon(\eta_0)\eta_0^5. \qquad (3.4.15) \end{aligned}$$

Now we replace $v_0, v_1, w_0, w_1, u_1, u_2$ in the above argument by $v_k, v_{k+1}, w_k, w_{k+1}, u_{k+1}, u_{k+2}$. By induction, we obtain

$$\|u_{k+2}-u_{k+1}\| \leq \Upsilon(\eta_k)\eta_k^5 \leq \Upsilon(r)\eta_k^5. \qquad (3.4.16)$$

Hence,

$$\begin{aligned} \|u_{k+m}-u_{k+1}\| &\leq \sum_{i=0}^{m-2}\|u_{k+i+2}-u_{k+i+1}\| \\ &\leq \sum_{i=0}^{m-2}\Upsilon(r)\eta_{k+i}^5 \\ &\leq \Upsilon(r)[\eta_k^5+\eta_{k+1}^5+\ldots+\eta_{k+m-2}^5] \\ &\leq \Upsilon(r)[\Theta(r)^{\frac{5^k-1}{4}}\eta_0+\Theta(r)^{\frac{5^{k+1}-1}{4}}\eta_0+\ldots+\Theta(r)^{\frac{5^{k+m-2}-1}{4}}\eta_0] \\ &\leq \Upsilon(r)\Theta(r)^{\frac{5^k-1}{4}}\eta_0[1+\Theta(r)^{5^k}+\ldots+\Theta(r)^{5^k\frac{5^{m-2}-1}{4}}] \\ &\leq \Upsilon(r)\Theta(r)^{\frac{5^k-1}{4}}\eta_0\frac{1-\Theta(r)^{5^k\frac{5^{m-2}-1}{4}+1}}{1-\Theta(r)^{5^k}} \\ &\leq \frac{1}{1-\Theta(r)^{5^k}}\Upsilon(r)\Theta(r)^{\frac{5^k-1}{4}}\eta_0 \\ &\leq \frac{\eta_0}{1-\Theta(r)^{5^k}}\Upsilon(r)\Theta(r)^{5^k} \\ &\leq Ce^{-\gamma_0 5^k}. \qquad (3.4.17) \end{aligned}$$

Thus, sequence $\{u_n\}$ is a Cauchy's sequence in $B(u_0, R_0)$. Hence it converges, and from the first step of the method (3.1.10) and (d) of Lemma 3.4.1, we have $\{u_n\}$ converges to u_α^δ. Now by letting $m \longrightarrow \infty$ in (3.4.17), we obtain (3.4.11).

□

Theorem 3.4.3. *Suppose conditions in Theorem 3.4.2 and Assumption 3.2.1 hold and* *if* $\alpha = \alpha(\delta)$ *is the solution of 3.3.3. Let*

$$n = \min\{k : C_0 e^{-\gamma_0 5^k} \leq \frac{\delta}{\alpha}\}.$$

Then

$$\|u_{n+2}^{\delta} - \hat{u}\| = O\left(\delta^{\frac{\nu}{\nu+1}}\right).$$

Proof. The proof follows from Theorem 3.3.3, Theorem 3.3.5, Theorem 3.4.2 and the triangle inequality

$$\|u_{n+2}^{\delta} - \hat{u}\| \leq \|u_{n+2}^{\delta} - u_{\alpha}^{\delta}\| + \|u_{\alpha}^{\delta} - \hat{u}\|.$$

□

3.4.1 IMPLEMENTATION OF THE METHOD

We use the following algorithm for implementing the method.

Algorithm

1. Choose α satisfying (3.3.3)
2. Choose $n = \min\{k : C_0 e^{-\gamma_0 5^k} \leq \frac{\delta}{\alpha}\}$.
3. Solve u_n^{δ} by using the iteration (3.1.10).

3.5 NUMERICAL EXAMPLES

The results in this chapter were obtained under assumption that the Fréchet derivative of F is positive self adjoint. In fact, in the proof of these results only the property

$$\|(F'(u) + \alpha I)^{-1}\| \leq \frac{1}{\alpha},\ u \in D(F)$$

and

$$\|(F'(u) + \alpha I)^{-1} F'(u)\| \leq 1,\ u \in D(F)$$

are used. To satisfy these conditions, for example, it is sufficient to require the conditions (see Nair and Ravishankar (2008); Vasin and George (2014)): the spectrum of

$F'(u)$; $\sigma(F'(u)) \subseteq [0,\infty)$ and $F'(u)^* = F'(u)$ for all $u \in B(u_0,r)$ for some $r > 0$. Or $F'(u)$ is weakly sectorial in the sense that $\|(F'(u)+\alpha I)^{-1}\| \leq \frac{c}{\alpha}$ for $\alpha > 0$ and for some $c > 0$. In this case $\|(F'(u)+\alpha I)^{-1}F'(u)\| \leq \|I-\alpha(F'(u)+\alpha I)^{-1}\| \leq 1+c$. All the results in this chapter hold under the above assumption up to a constant. Therefore, we apply the result to a simple one dimensional example studied in Groetsch (1993); Hofmann and Scherzer (1994); Nair and Ravishankar (2008); Tautenhahn (2002). Note that the Fréchet derivative of the operator in the following example is not positive self adjoint but is positive type Nair and Ravishankar (2008).

Example 3.5.1. *Let $c > 0$ be a constant. Consider the inverse problem of identifying the distributed growth law $x(t), t \in (0,1)$, in the initial value problem*

$$\frac{dy}{dt} = x(t)y(t),\ y(0) = c,\ t \in (0,1) \tag{3.5.1}$$

from the noisy data $y^\delta(t) \in L^2(0,1)$. One can reformulate the above problems as an ill-posed operator equation $F(x) = y$ with

$$[F(x)](t) = ce^{\int_0^t x(\theta)d\theta},\ x \in L^2(0,1),\ t \in (0,1). \tag{3.5.2}$$

The Fréchet derivative of F is given by

$$[F'(x)h](t) = [F(x)](t)\int_0^t h(\theta)d\theta. \tag{3.5.3}$$

It is proved in Nair and Ravishankar (2008), that F' is positive type (sectorial) and spectrum of $F'(x)$ is the singleton set $\{0\}$. As in Tautenhahn (2002), one can see that $\phi(u,v,w)$ in Assumption 3.2.1 must satisfy the equation

$$\int_0^t \phi(u,v,w)d\tau = \left[exp\left(\int_0^t [u-v]d\tau\right) - 1\right]\int_0^t w d\tau.$$

So,

$$\phi(u,v,w) = exp\left(\int_0^t [u-v]d\tau\right)(u-v)\int_0^t w d\tau + \left[exp\left(\int_0^1 [u-v]d\tau\right) - 1\right]w.$$

Then, using $\|f.g\| \leq \|f\|_\infty \|g\|$, $\| \int_0^t w d\tau \|_\infty \leq \|w\|$, $|e^\lambda - 1| \leq \|\lambda\| e^{|\lambda|}$ and triangle inequality, we obtain $\|\phi(u,v,w)\| \leq 2exp(\|u-v\|)\|u-v\|\|w\|$. Hence, Assumption 3.2.1 is satisfied with $K = 2\|u-v\|exp(\|u-v\|) \leq 2re^r$.

Further it is proved in Tautenhahn (2002) that $\hat{u} - u_0 \in R(F'(\hat{u}))$ provided $u* := \hat{u} - u_0 \in H^1(0,1)$ and $u*(0) = 0$. Now since $\hat{u} - u_0 = F'(\hat{u})w$, we have

$$\begin{aligned}
[\hat{u} - u_0](t) &= [F(\hat{u})](t) \int_0^t w(\theta) d\theta \\
&= c e^{\int_0^t \hat{u}(\theta) d\theta} \int_0^t w(\theta) d\theta \\
&= \frac{\int_0^1 c e^{\int_0^t [\hat{u} + \tau(u_0 - \hat{u})](\theta) d\theta} d\tau \int_0^t w(\theta) d\theta}{\int_0^1 e^{\int_0^t [\tau(u_0 - \hat{u})](\theta) d\theta} d\tau} \\
&= [A\bar{w}](t),
\end{aligned}$$

where $\bar{w} = \frac{w}{\int_0^1 e^{\int_0^t [\tau(u_0 - \hat{u})](\theta) d\theta} d\tau}$. This shows the source that condition (3.1.9) is satisfied.

In Table 3.1, we present the relative error $E_\alpha = \frac{\|CS - \hat{u}\|}{\|\hat{u}\|}$, where CS is the computed solution for three similar methods when α is chosen according to the discrepancy principle (3.3.3), for different values of δ. The numerical results demonstrate the superiority of the method (3.1.10).

Table 3.1 Comparison of Relative errors.

Method	α & E_α	$\delta = 0.1$	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.0001$
	α	1.234457e-01	3.721868e-02	1.147985e-02	3.602498e-03
George and Nair (2017)	E_α	4.422503e-01	9.058463e-02	3.130262e-02	1.036917e-02
George and Elmahdy (2012), Mahale and Nair (2009)	E_α	4.516886e-01	9.062572e-02	3.130270e-02	1.036917e-02
method(3.1.10)	E_α	9.011146e-02	3.678658e-02	3.129978e-02	1.036914e-02

3.6 CONCLUSION

In this chapter we introduced a new source condition and a new “apriori" parameter choice strategy for choosing the regularization parameter α in Lavrentiev regularization method for nonlinear ill-posed equation. The "apriori” parameter choice strategy is so general, that it can be applied to other methods for approximately solving the equation

(3.1.4). The "apriori" parameter gives the optimal order with respect to the new source condition and does not require the knowledge of unknown ν in the source condition (3.1.9).

CHAPTER 4

ON THE CONVERGENCE OF HOMEIER METHOD AND ITS EXTENSIONS

4.1 INTRODUCTION

For solving the equation $\mathscr{T}(t) = 0$, where $\mathscr{T} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a vector function, Homeier (2004) considered the iteration defined by

$$t_{n+1} = \psi_{\mathscr{T}}(t_n). \tag{4.1.1}$$

Here $\psi_{\mathscr{T}}(t) = t - A_{\mathscr{T}}(t)(t - \frac{1}{2}A_{\mathscr{T}}(t)\mathscr{T}(t))\mathscr{T}(t)$, where $A_{\mathscr{T}}(t) = [J_{\mathscr{T}}(t)]^{-1}$ is the inverse of the Jacobian of $\mathscr{T}$. Note that the iteration (4.1.1) can be written as the two-step iteration

$$\begin{aligned} r_n &= t_n - \frac{1}{2}A_{\mathscr{T}}(t_n)\mathscr{T}(t_n) \\ t_{n+1} &= t_n - A_{\mathscr{T}}(r_n)\mathscr{T}(t_n). \end{aligned}$$

The convergence order three for the iteration (4.1.1) was proved in Homeier (2004), but using assumptions on the derivatives of $\mathscr{T}$ up to order four.

In this study, we consider a more general setting of the iteration (4.1.1) for solving the non-linear equation

$$\mathscr{T}(t) = 0, \tag{4.1.2}$$

where $\mathscr{T} : \Delta \subset \mathscr{X} \longrightarrow \mathscr{Y}$ is a nonlinear operator, $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces and $\Delta \neq \emptyset$. This equation has been studied extensively due to its applications in various fields (see, Argyros (2008, 2007); Argyros and Regmi (2019)). Throughout this Chapter $B(t_0, \delta) = \{t \in \mathscr{X} : \|t - t_0\| < \delta\}$ and $B[t_0, \delta] = \{t \in \mathscr{X} : \|t - t_0\| \leq \delta\}$ for some $\delta > 0$.

We consider the following form of the Homeier method

$$\begin{aligned} r_k &= t_k - \frac{1}{2}\mathscr{T}'(t_k)^{-1}\mathscr{T}(t_k) \\ t_{k+1} &= t_k - \mathscr{T}'(r_k)^{-1}\mathscr{T}(t_k), k = 0, 1, 2, \ldots. \end{aligned} \tag{4.1.3}$$

For convenience we use the following definition of order of convergence (see, Ortega and Rheinboldt (1970)), i.e, a sequence $\{\alpha_n\}$ converging to α is said to have an order of convergence $q \geq 1$ and rate of convergence (asymptotic error constant) C if there exist $N \in \mathbb{N}$ such that

$$\|\alpha_{n+1} - \alpha_n\| \leq C\|\alpha_n - \alpha_{n-1}\|^q, \ \forall n > N.$$

In literature, convergence analysis using Taylor series expansion and assumptions on the higher-order derivative is prevalent (Abad et al. (2013); Cordero et al. (2010, 2012, 2015a); Grau-Sánchez et al. (2011); Regmi (2020); Sharma and Gupta (2014); Sharma et al. (2015)). Our analysis is not based on Taylor expansion, so we do not need assumptions on the derivatives of the operator higher than two.

The assumptions involving higher-order derivatives reduce the applicability of (4.1.3); for example,

let $\mathscr{X} = \mathscr{Y} = \mathbb{R}, \Delta = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Δ by

$$f(u) = \begin{cases} u^3 \log u^2 + u^5 - u^4 & if\ u \neq 0 \\ 0 & if\ u = 0. \end{cases}$$

Then,

$$f'''(u) = 6 \log u^2 + 60u^2 - 24u + 22.$$

Clearly, $f'''(u)$ is not bounded on Δ. So, the convergence of the Homeier method is not assured by the analysis in Homeier (2004).

We assume the following in the convergence analysis.

Assumption 4.1.1. *There exists* $K_0 > 0$ *such that* $\forall\, t, r \in D(\mathscr{T})$,

$$\|\mathscr{T}'(t)^{-1}(\mathscr{T}'(r) - \mathscr{T}'(t))\| \leq K_0\|r - t\|,$$

and there exists $\rho > 0$ *and* $K_1, c > 0$ *such that for all* $r,t,s \in B(t^*, \rho)$,

$$\begin{aligned} \|\mathscr{T}'(r)^{-1}\mathscr{T}''(t)\| &\leq c, \\ \|\mathscr{T}'(r)^{-1}(\mathscr{T}''(t) - \mathscr{T}''(s))\| &\leq K_1\|t - s\|. \end{aligned} \tag{4.1.4}$$

Convergence analysis of the Homeier method is given in Section 4.2, and the convergence analysis of its extensions is given in Section 4.3. The Section also contains a comparison of the method and its extensions using various efficiency indices. In Section 4.4, we studied two numerical examples and provided their radii of convergence. Section 4.5 gives a table of ACOC of the methods with various examples. It also contains some graphic images of basins of attractions of some examples.

4.2 CONVERGENCE ANALYSIS OF THE METHOD (4.1.3)

Convergence analysis for (4.1.3) is given next.
Let $\bar{\tau} : [0, \infty) \longrightarrow [0, \infty)$ be defined as

$$\bar{\tau}(u) = \frac{K_1}{16}(3K_0u + 16) + \frac{c^2}{4}, \tag{4.2.1}$$

and

$$\tau(u) = \tau(\bar{u})u^2 - 1 \tag{4.2.2}$$

then by intermediate value theorem, $\tau(u)$ has a minimal postive zero $\acute{r}$, as $\tau(0) = -1$ and $\tau(u) \to \infty$ as $u \to \infty$. Take

$$r = \min\{\acute{r}, 1\}. \tag{4.2.3}$$

Then,

$$0 < \tau(t) < 1 \quad \forall t \in (0,r). \tag{4.2.4}$$

Theorem 4.2.1. *The sequence* $\{t_n\}$ *defined by (4.1.3) with* $t_0 \in B(t^*,r) - \{t^*\}$, *converges to* t^*, *where r is as in (4.2.3), such that*

$$\|t_{n+1} - t^*\| \leq \bar{\tau}(r)\|t_n - t^*\|^3, \tag{4.2.5}$$

where $\bar{\tau}$ *is as in (4.2.1).*

Proof. We shall prove

$$\|t_{n+1} - t^*\| \leq \bar{\tau}(r)\|t_n - t^*\|^3, \tag{4.2.6}$$

using induction. Let $t_0 \in B(t^*,r)$. Observe that

$$\begin{aligned}
t_1 - t^* &= t_0 - t^* - \mathscr{T}'(r_0)^{-1}[\mathscr{T}(t_0) - \mathscr{T}(t^*)] \\
&= \mathscr{T}'(r_0)^{-1}[\mathscr{T}'(r_0) - \int_0^1 \mathscr{T}'(t^* + u(t_0 - t^*))du](t_0 - t^*) \\
&= \mathscr{T}'(r_0)^{-1}\int_0^1 [\mathscr{T}'(r_0) - \mathscr{T}'(t^* + u(t_0 - t^*))du](t_0 - t^*) \\
&= \mathscr{T}'(r_0)^{-1}\int_0^1\int_0^1 (\mathscr{T}''(t^* + u(t_0 - t^*) + \theta(r_0 - (t^* + u(t_0 - t^*)))d\theta \\
&\quad \times (r_0 - (t^* + u(t_0 - t^*)))du(t_0 - t^*).
\end{aligned}$$

Let $\xi = t^* + u(t_0 - t^*) + \theta(r_0 - (t^* + u(t_0 - t^*))$. Also from (4.1.3), we have

$$\begin{aligned}
r_0 - (t^* + u(t_0 - t^*)) &= \left(\frac{1}{2} - u\right)(t_0 - t^*) + \frac{1}{2}[t_0 - t^* - \mathscr{T}'(t_0)^{-1}\mathscr{T}(t_0)] \\
&= \left(\frac{1}{2} - u\right)(t_0 - t^*) + \frac{\mathscr{T}'(t_0)^{-1}}{2}\left[\int_0^1 \mathscr{T}'(t_0)(t_0 - t^*)\right. \\
&\left. - \mathscr{T}'(t^* + v(t_0 - t^*))(t_0 - t^*)dv\right].
\end{aligned} \tag{4.2.7}$$

Hence,

$$t_1 - t^* = \mathscr{T}'(r_0)^{-1}\int_0^1\int_0^1 \mathscr{T}''(\xi)\left[\left(\frac{1}{2} - u\right)(t_0 - t^*) + \frac{1}{2}[t_0 - t^* - \mathscr{T}'(t_0)^{-1}\mathscr{T}(t_0)]\right]$$

$$\times d\theta du(t_0 - t^*).$$

Let $t_1 - t^* = \Delta_1 + \Delta_2$, where $\Delta_1 = \mathscr{T}'(r_0)^{-1}\int_0^1\int_0^1 \mathscr{T}''(\xi)\left(\frac{1}{2}-u\right)(t_0-t^*)^2 d\theta du$ and $\Delta_2 = \mathscr{T}'(r_0)^{-1}\int_0^1\int_0^1 \mathscr{T}''(\xi)\frac{1}{2}[t_0 - t^* - \mathscr{T}'(t_0)^{-1}\mathscr{T}(t_0)](t_0-t^*)d\theta du$. Then,

$$\begin{aligned}
\|\Delta_1\| &= \|\mathscr{T}'(r_0)^{-1}\int_0^1\int_0^1 (\mathscr{T}''(\xi) - \mathscr{T}''(t^*) + \mathscr{T}''(t^*))\left(\frac{1}{2}-u\right)(t_0-t^*)^2 d\theta du\| \\
&= \|\mathscr{T}'(r_0)^{-1}\int_0^1\int_0^1 \left(\mathscr{T}''(\xi) - \mathscr{T}''(t^*)\right)\left(\frac{1}{2}-u\right)(t_0-t^*)^2 d\theta du\| \\
&+ \|\mathscr{T}'(r_0)^{-1}\int_0^1\int_0^1 \mathscr{T}''(t^*)\left(\frac{1}{2}-u\right)(t_0-t^*)^2 d\theta du\| \\
&\leq \int_0^1\int_0^1 \|K_1(\xi - t^*)\||\frac{1}{2}-u|\|t_0-t^*\|^2 d\theta du \\
&\leq \frac{K_1}{2}\int_0^1\int_0^1 \|u(t_0-t^*) + \theta(r_0 - (t^* + u(t_0-t^*))\||1-2u|d\theta du\|t_0-t^*\|^2 \\
&\leq \frac{K_1}{2}\int_0^1\int_0^1 u\|(t_0-t^*)\| + \theta\|(r_0 - (t^* + u(t_0-t^*)))\||1-2u|d\theta du\|t_0-t^*\|^2 \\
&\leq \frac{K_1}{2}\int_0^1 u\|(t_0-t^*)\| + \frac{\|(r_0 - (t^* + u(t_0-t^*)))\|}{2}|1-2u|du\|t_0-t^*\|^2.
\end{aligned}$$

From (4.2.7) and using Assumption (4.1.1), we have

$$\|r_0 - (t^* + u(t_0-t^*))\| = |(\frac{1}{2}-u)|\|t_0-t^*\| + \frac{K_0}{4}\|t_0-t^*\|.$$

So,

$$\begin{aligned}
\|\Delta_1\| &\leq \frac{K_1}{2}\int_0^1 u\|t_0-t^*\| + \frac{|\frac{1}{2}-u|\|t_0-t^*\| + \frac{K_0}{4}\|t_0-t^*\|^2}{2}|1-2u|du\|t_0-t^*\|^2 \\
&\leq \frac{K_1}{96}(16 + 3K_0\|t_0-t^*\|)\|t_0-t^*\|^3
\end{aligned}$$

whereas,

$$\begin{aligned}
\|\Delta_2\| &= \|\mathscr{T}'(r_0)^{-1}\int_0^1\int_0^1 \mathscr{T}''(\xi)\frac{1}{2}[t_0 - t^* - \mathscr{T}'(t_0)^{-1}\mathscr{T}(t_0)](t_0-t^*)d\theta du\| \\
&\leq \frac{c}{2}\int_0^1\int_0^1 \|t_0 - t^* - \mathscr{T}'(t_0)^{-1}\mathscr{T}(t_0)\|\|t_0-t^*\|d\theta du \\
&\leq \frac{c}{2}\int_0^1\int_0^1 \|\mathscr{T}'(t_0)^{-1}\left(\mathscr{T}'(t_0)(t_0-t^*) - \int_0^1 \mathscr{T}'(t^* + v(t_0-t^*))dv(t_0-t^*)\right)\|
\end{aligned}$$

$$
\begin{aligned}
&\times \|t_0-t^*\|d\theta du\\
&\leq \frac{c}{2}\int_0^1\int_0^1 \|\mathscr{T}'(t_0)^{-1}\int_0^1 \left[\mathscr{T}'(t_0)-\mathscr{T}'(t^*+v(t_0-t^*))\right](t_0-t^*)dv\|\|t_0-t^*\|d\theta du\\
&\leq \frac{c}{2}\int_0^1\int_0^1 \|F'(t_0)^{-1}\int_0^1\int_0^1 \mathscr{T}''(t^*+v(t_0-t^*)+w(t_0-t^*-v(t_0-t^*)))\\
&\times (t_0-t^*-v(t_0-t^*))dwdv\|\|t_0-t^*\|^2 d\theta du\\
&\leq \frac{c^2}{4}\|t_0-t^*\|^3.
\end{aligned}
$$

Hence, from (4.2.4)

$$
\begin{aligned}
\|t_1-t^*\| &\leq \left[\frac{K_1}{16}(3K_0\|t_0-t^*\|+16)+\frac{c^2}{4}\right]\|t_0-t^*\|^3\\
&\leq \bar{\tau}(\|t_0-t^*\|)\|t_0-t^*\|^3 \qquad (4.2.8)\\
&\leq \|t_0-t^*\|\\
&\leq r.
\end{aligned}
$$

Thus, $t_1 \in B(t^*,r)$. Also from (4.2.8), we have

$$
\|t_1-t^*\| \leq \bar{\tau}(r)\|t_0-t^*\|^3,
$$

The induction (4.2.6), is complete, if we simply replace, t_0, r_0, t_1 by t_n, r_n, t_{n+1} in the preceding arguments. □

Proposition 4.2.2. *Suppose Assumption 4.1.1 holds and t^* is a simple solution of equation $\mathscr{T}(t)=0$. Then, $\mathscr{T}(t)=0$ has a unique solution t^* in $S=\Delta\cap B[t^*,\bar{r}]$ provided*

$$
K_0\bar{r} < 2. \qquad (4.2.9)
$$

Proof. Let $p\in S$ be such that $\mathscr{T}(p)=0$. Define $B=\int_0^1 \mathscr{T}'(t^*+u(p-t^*))du$. Then, by Assumption 4.1.1 and (4.2.9), we have, in turn

$$
\begin{aligned}
\|\mathscr{T}'(t^*)^{-1}(B-\mathscr{T}'(t^*))\| &\leq K_0\int_0^1 \|t^*+u(p-t^*)-t^*\|du\\
&\leq K_0\int_0^1 u\|p-t^*\|du
\end{aligned}
$$

$$\leq \frac{K_0 \bar{r}}{2} < 1.$$

This implies B is invertible, and hence $p = t^*$ by the identity $0 = \mathscr{T}(p) - \mathscr{T}(t^*) = B(p - t^*)$. $\square$

4.3 EXTENSIONS OF THE METHOD (4.1.3)

We consider two extensions of (4.1.3) given by

$$\begin{aligned} r_k &= t_k - \frac{1}{2}\mathscr{T}'(t_k)^{-1}\mathscr{T}(t_k) \\ s_k &= t_k - \mathscr{T}'(r_k)^{-1}\mathscr{T}(t_k) \\ t_{k+1} &= s_k - \mathscr{T}'(2r_k - t_k)^{-1}\mathscr{T}(s_k), k = 0, 1, 2, \ldots \end{aligned} \tag{4.3.1}$$

and

$$\begin{aligned} r_k &= t_k - \frac{1}{2}\mathscr{T}'(t_k)^{-1}\mathscr{T}(t_k) \\ s_k &= t_k - \mathscr{T}'(r_k)^{-1}\mathscr{T}(t_k) \\ t_{k+1} &= s_k - \mathscr{T}'(s_k)^{-1}\mathscr{T}(s_k) k = 0, 1, 2, \ldots. \end{aligned} \tag{4.3.2}$$

We shall prove that method (4.3.1) and (4.3.2) are of convergence order five and six, respectively.

Let $\phi : [0, \infty) \longrightarrow [0, \infty)$ be defined by

$$\phi(u) = \frac{K_0}{2}\bar{\tau}(u)(K_0 + \bar{\tau}(u)), \tag{4.3.3}$$

and

$$\psi(u) = \phi(u)u^4 - 1. \tag{4.3.4}$$

Observe that ψ is continuous, $\psi(0) = -1$ and $\psi(u) \longrightarrow \infty$ as $t \longrightarrow \infty$, so by intermediate value theorem ψ has a minimal zero $\hat{r} > 0$. Let

$$r_1 = \min\{\frac{2}{K_0},\ r\ ,\hat{r}\}, \tag{4.3.5}$$

then, $\forall t \in (0, r_1), 0 < \phi(t) < 1$.

Theorem 4.3.1. *Let r_1 be as in (4.3.5). The sequence $\{t_n\}$ defined by (4.3.1) with $t_0 \in B(t^*, r_1) - \{t^*\}$, converges to t^* such that*

$$\|t_{n+1} - t^*\| \leq \phi(r_1)\|t_n - t^*\|^5, \tag{4.3.6}$$

where ϕ is as in (4.3.3).

Proof. From Theorem 4.2.1, by taking $t_{n+1} = s_n$, we obtain

$$\|s_n - t^*\| \leq \bar{\tau}(\|t_0 - t^*\|)\|t_n - t^*\|^3. \tag{4.3.7}$$

Observe that

$$\begin{aligned} \|2r_0 - t_0 - t^*\| &= \|t_0 - t^* + \mathscr{T}'(t_0)^{-1}\int_0^1 [\mathscr{T}'(t_0) - \mathscr{T}'(t^* + u(t_0 - t^*))]du \\ &\quad \times (t_0 - t^*) - (t_0 - t^*)\| \\ &\leq \frac{K_0}{2}\|t_0 - t^*\|^2 \qquad (4.3.8) \\ &\leq r_1, \end{aligned}$$

so $2r_0 - t_0 \in B(t^*, r_1)$. Now, by the third step of (4.3.1) for $k = 0$,

$$\begin{aligned} t_1 - t^* &= s_0 - t^* - \mathscr{T}'(2r_0 - t_0)^{-1}\mathscr{T}(s_0) \\ &= \mathscr{T}'(2r_0 - t_0)^{-1}\left[\int_0^1 \mathscr{T}'(2r_0 - t_0) - \mathscr{T}'(t^* + u(s_0 - t^*))du\right](s_0 - t^*), \end{aligned}$$

and hence by (4.3.7) and (4.3.8), we have

$$\begin{aligned} \|t_1 - t^*\| &\leq K_0\int_0^1 \|2r_0 - t_0 - t^* - u(s_0 - t^*)\|du\|s_0 - t^*\| \\ &\leq K_0(\|2r_0 - t_0 - t^*\| + \frac{1}{2}\|s_0 - t^*\|)\|s_0 - t^*\| \end{aligned}$$

$$\begin{aligned}
&\leq K_0\left(\frac{K_0}{2}\|t_0-t^*\|^2+\frac{\bar{\tau}(\|t_0-t^*\|)}{2}\|t_0-t^*\|^3\right)\bar{\tau}(\|t_0-t^*\|)\|t_0-t^*\|^3\\
&\leq \phi(\|t_0-t^*\|)\|t_0-t^*\|^5 \qquad (4.3.9)\\
&\leq \|t_0-t^*\|\\
&< r_1.
\end{aligned}$$

Hence, $t_1 \in B(t^*, r_1)$ and Also from (4.3.9), we have

$$\|t_1-t^*\| \leq \phi(r_1)\|t_0-t^*\|^5.$$

So, (4.3.6) holds for $n = 0$. The proof is completed using induction, if we replace t_0, r_0, s_0, t_1 by t_n, r_n, s_n, t_{n+1} respectively in the preceeding estimates. □

Let $\phi_1 : [0,\infty) \longrightarrow [0,\infty)$ be defined by

$$\phi_1(u) = \frac{\bar{\tau}^2(u)}{2}$$

and

$$\psi_1(u) = \frac{\bar{\tau}^2(u)}{2}u^5 - 1.$$

ψ is a continuous function and $\psi(0) = -1$ and $\psi \to \infty$ as $u \to \infty$. So by intermediate value theorem, ψ has a minimal positive root $\bar{r}$.

$$r_2 = \min\left\{r\ , \frac{2}{K_0}, \bar{r}\right\}. \qquad (4.3.10)$$

Theorem 4.3.2. *Let r_2 be as in (4.3.10). The sequence $\{t_n\}$ defined by (4.3.2) with $t_0 \in B(t^*, r_2) - \{t^*\}$, converges to t^* such that*

$$\|t_{n+1}-t^*\| \leq \phi_1(r_2)\|t_n-x^*\|^6, \qquad (4.3.11)$$

Proof. Note that, by the third step of (4.3.2) for $k = 0$, we have

$$\begin{aligned} t_1 - t^* &= s_0 - t^* - \mathscr{T}'(s_0)^{-1}\mathscr{T}(s_0) \\ &= \mathscr{T}'(s_0)^{-1}\left[\int_0^1 \mathscr{T}'(s_0) - \mathscr{T}'(t^* + u(s_0 - t^*))du\right](s_0 - t^*), \end{aligned}$$

and hence we have,

$$\begin{aligned} \|t_1 - t^*\| &\leq \frac{K_0}{2}\|s_0 - t^*\|^2 \\ &\leq \frac{\bar{\tau}^2(\|t_0 - t^*\|)}{2}\|t_0 - t^*\|^6 \qquad (4.3.12) \\ &\leq \|t_0 - t^*\| \\ &< r_2. \end{aligned}$$

Hence $t_1 \in B(t^*, r_2)$ and from (4.3.12), we have

$$\|t_1 - t^*\| \leq \phi_1(r_1)\|t_0 - t^*\|.$$

So, (4.3.11) holds for $n = 0$. The rest of the proof follows as in Theorem 4.3.1.

□

Remark 4.3.3. *We obtained the informational efficiency I and computational efficiency E of Homeier method (4.1.3) as* $3/3 = 1$ *and* $3^{1/3} = 1.44225$, *respectively. The method (4.3.1) has* $I = 5/5 = 1$ *and* $E = 5^{1/5} = 1.37973$ *and method (4.3.2) has* $I = 6/5 = 1.2$ *and* $E = 6^{1/5} = 1.43097$.

4.4 NUMERICAL EXAMPLES

We compute radius of convergence of the following two examples.

Example 4.4.1. *Let* $\mathscr{X} = \mathscr{Y} = \mathbb{R}, t_0 = 1, \Delta = [t_0 - (1-q), t_0 + (1-q)], q \in (2 - \sqrt{2}, 1)$ *and* $\mathscr{T} : \Delta \to \mathscr{Y}$ *be defined by*

$$\mathscr{T}(t) = t^3 - q.$$

We have, $\|\mathscr{T}'(t_0)^{-1}\| = \frac{1}{3}$. *We can obtain the constant* K_0 *and the radius of convergence* *r as follows;*

$$\begin{aligned}\|\mathscr{T}'(t_0)^{-1}(\mathscr{T}'(t)-\mathscr{T}'(t_0))\| &= \frac{1}{3}\|(3t^2-3)\|\\ &\leq \|t+1\|\|t-1\|\\ &= (3-q)(1-q).\end{aligned}$$

By using Banach Lemma on invertible operators (Argyros (2008)),

$$\begin{aligned}\|\mathscr{T}'(t)^{-1}\| &\leq \frac{\|\mathscr{T}'(t_0)^{-1}\|}{1-\|\mathscr{T}'(t_0)^{-1}\mathscr{T}'(t)-I\|}\\ &= \frac{1}{3(1-(3-q)(1-q))}.\end{aligned}$$

Thus, we have,

$$\begin{aligned}\|\mathscr{T}'(t)^{-1}(\mathscr{T}'(r)-\mathscr{T}'(t))\| &\leq \|\mathscr{T}'(t)^{-1}\|\|3r^2-3t^2\|\\ &\leq \frac{3(r+t)(r-t)}{3(1-(3-q)(1-q))}\\ &= \frac{2(2-q)}{(1-(3-q)(1-q))}\|r-t\|.\end{aligned}$$

Therefore, $K_0 = \frac{2(2-q)}{(1-(3-q)(1-q))}$. *Take* $q = 0.75$, *then we get,* $K_0 = c \approx 5.714285714$ *and* $\hat{r} \approx 0.27118$. *Radius of convergence of method (4.1.3),* $r = 0.35$. *Radii of convergence of method (4.3.1),* $r_1 = \min\{\frac{2}{K_0},\ r\ ,\hat{r}\} = 0.27118$, *and that of method (4.3.2),* $r_2 = \min\left\{r\ , \frac{2}{K_0}, \bar{r}\right\} = 0.35$.

Example 4.4.2. *Let* $\mathscr{X} = \mathscr{Y} = \mathbb{R}^3$, $\Delta = B[0,1]$, $t_0 = (0,0,0)^T$. *Define function* $\mathscr{T}$ *on* Δ *for* $w = (t,r,s)^T$ *by*

$$\mathscr{T}(w) = \left(e^t-1, r^2\frac{e-1}{2}+r, s\right)^T.$$

Then, we have, $\mathscr{T}'(w) = \begin{pmatrix} e^t & 0 & 0\\ 0 & r(e\text{-}1)+1 & 0\\ 0 & 0 & 1\end{pmatrix}$. *Thus,* $K_0 = e-1, c = e$ *and* $\hat{r} \approx 0.784319$.

The radius of convergence of method (4.1.3), $r \approx 0.92541$. *Radius of convergence of* *extension (4.3.1),* $r_1 = r_1 = \min\{\frac{2}{K_0},\ r\ ,\hat{r}\} \approx 0.784319$ *and radius of convergence of (4.3.2),* $r_2 = \min\left\{r\ ,\frac{2}{K_0},\bar{r}\right\} \approx 0.92541.$

Remark 4.4.3. *We have observed that the method (4.3.2) has larger radius of convergence and better informational efficiency and computational efficency than the method (4.3.1).*

4.5 COMPUTATIONAL ORDER AND GRAPHICAL ILLUSTRATION

To verify that the theoretical order is attained in practice, we computed the ACOC of the following examples.We used the stopping criterion $\|t_n - t_{n-1}\| + \|\mathscr{T}(t_n)\| \leq 10^{-100}$. Some of the examples used appeared in Cordero et al. (2012); Behl et al. (2019). The obtained results are given in Table 4.1.

1. $\mathscr{T}(t_1,t_2,t_3) = \left(e^{t_1} - 1, \frac{e-1}{2}t_2^2 + t_2, t_3\right), \alpha = (0,0,0), \beta \approx (0, 1.16395, 0).$
2. $\mathscr{T}(t_1,t_2) = (t_1^2 - 4t_2 + t_2^2, 2t_1 - t_2^2 - 2), \alpha \approx (0.3542, 1.1364), \beta \approx (0.3542, -1.1364).$
3. $\mathscr{T}(t_1,t_2) = (t_1^3 - t_2, t_2^3 - t_2), \alpha = (-1,-1), \beta = (0,0), \gamma = (1,1).$
4. $\mathscr{T}(t_1,t_2) = (t_1^2 + t_2^2 - 1, t_1^2 - t_2^2 + 0.5), \alpha = (\frac{1}{2}, \frac{\sqrt{3}}{2}), \beta = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}).$
5. $\mathscr{T}(t_1,t_2) = (3t_1^2 t_2 - t_2^3, t_1^3 - 3t_1 t_2^2 - 1), \alpha = (\frac{-1}{2}, \frac{-\sqrt{3}}{2}), \beta = (\frac{-1}{2}, \frac{\sqrt{3}}{2}), \gamma = (1,0).$

4.5.1 BASINS OF ATTRACTION

We study the basins of attractions of the methods (4.1.3), (4.3.1) and (4.3.2) using the following examples.

Example 4.5.1. $\begin{cases} u^3 - v = 0 \\ v^3 - u = 0 \end{cases}$
with solutions $\{(-1,-1),(0,0),(1,1)\}$.

$\mathscr{T}(t)$	t^*	t_0	ACOC HM	ACOC HME1	ACOC HME2
1	$(0,0,0)$	$(1,0.03,0.03)$	3	4.7	5.8
2	$(0.3542,1.1364)$	$(0.35,0.9)$	2.9	4.3	5.3
3	$(1,1)$	$(1.5,1.5)$	3	4.3	5.4
4	$(\frac{1}{2},\frac{\sqrt{3}}{2})$	$(1,0.8)$	2.8	4.5	5.1
5	$\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$	$(-\frac{1}{2},\frac{1}{2})$	3	5.4	6.3

Table 4.1 ACOC for nonlinear systems using Homeier method (4.1.3) denoted by HM, first extension (4.3.1) denoted by HME1 and second extension (4.3.5) denoted by HME2; t^* and t_0 denote root and intial values, respectively.

Example 4.5.2. $\begin{cases} u^2+v^2-4=0 \\ 3u^2+7v^2-16=0 \end{cases}$
with solutions $\{(\sqrt{3},1),(-\sqrt{3},1),(\sqrt{3},-1),(-\sqrt{3},-1)\}$.

Basins of attraction for the roots of a system of nonlinear equations are generated on a 401×401 equidistant grid within a rectangular domain $Z=\{(u,v)\in\mathbb{R}^2:\ -2\le u\le 2,-2\le v\le 2\}$. Each point is assigned a color based on the corresponding root to which the iterative method converges, and marked black if convergence fails or diverges. Computations are performed using MATLAB on a PC with an Intel(R) Core(TM) i7-3770 processor and 8 GB RAM running Ubuntu 20.04.4 LTS. The resulting basins of attraction are shown in Figures 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6, with a tolerance of 10^{-8} and a maximum of 100 iterations allowed.

Algorithm :

1. Create an equidistant grid of the rectangular domain (here 401×401). Assign colors for each root. Set a maximum number of iteration say n and a tolerance limit.

2. For each point in equidistant grid, take the point as initial point and run the iteration n times.

3. If the error between the calculated value after n iteration is less than any of the roots, mark it with the corresponding root color. Else mark it black.

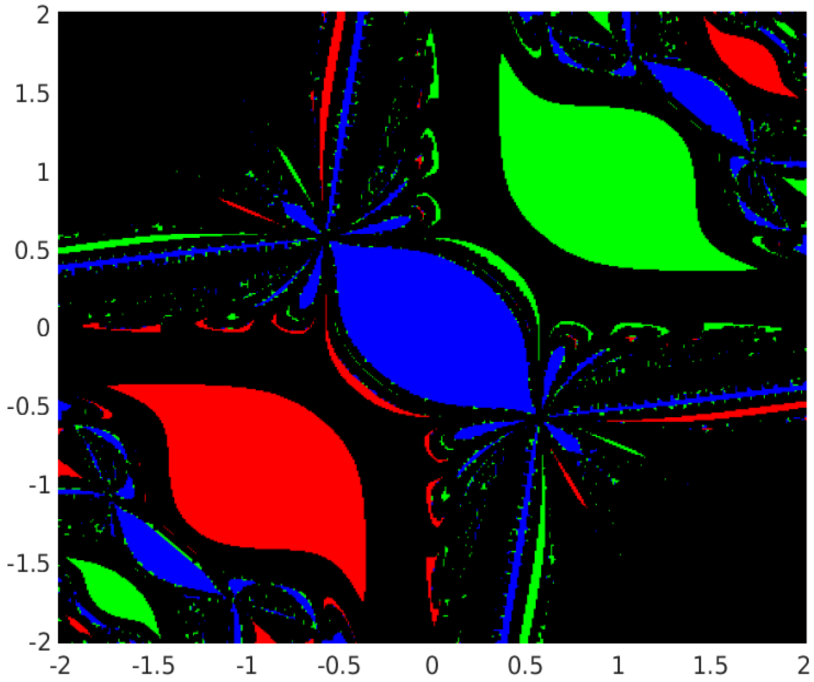

Figure 4.1 Basin of Attraction of Example 4.5.1 using Method (4.1.3)

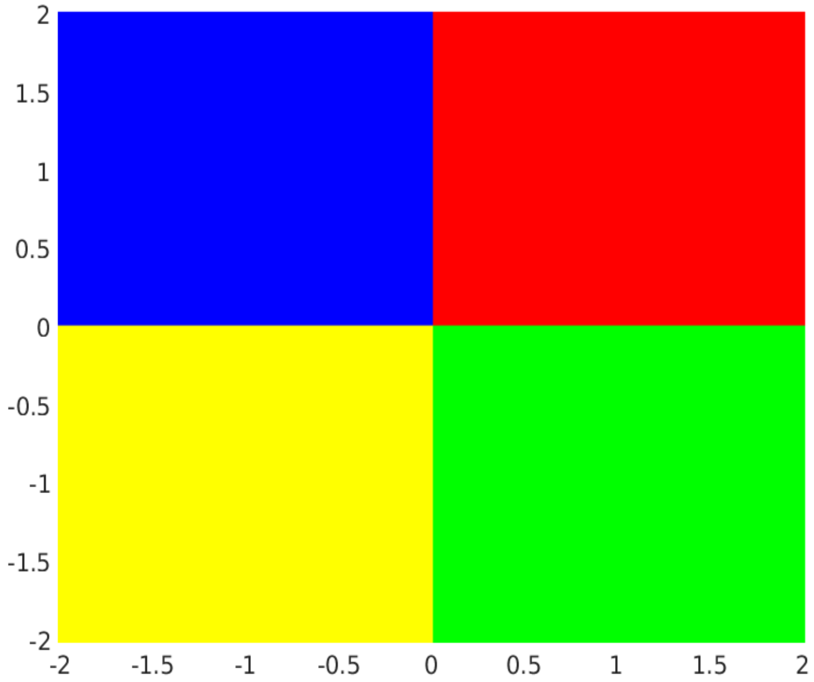

Figure 4.2 Basin of Attraction of Example 4.5.2 using Method (4.1.3)

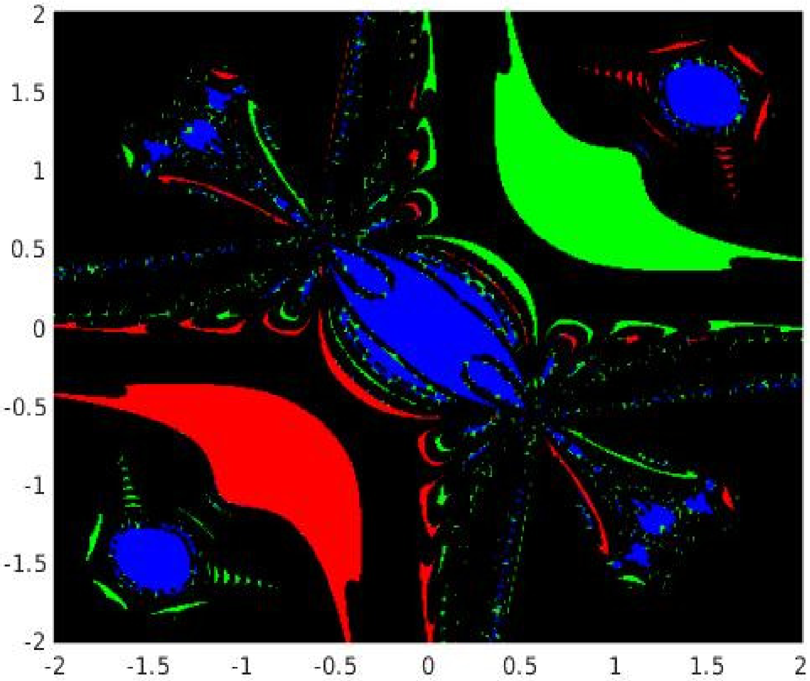

Figure 4.3 Basin of Attraction of Example 4.5.1 using Method (4.3.1)

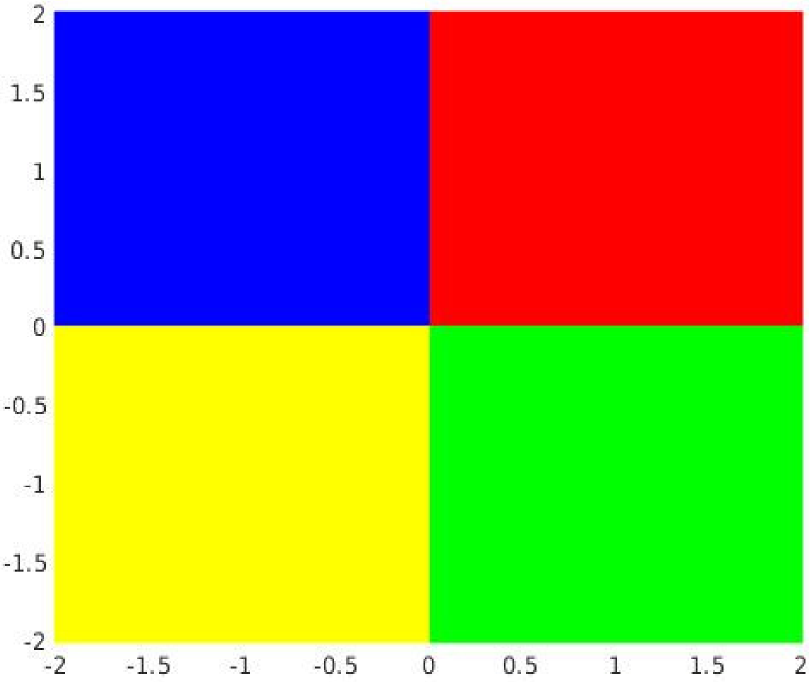

Figure 4.4 Basin of Attraction of Example 4.5.2 using Method (4.3.1)

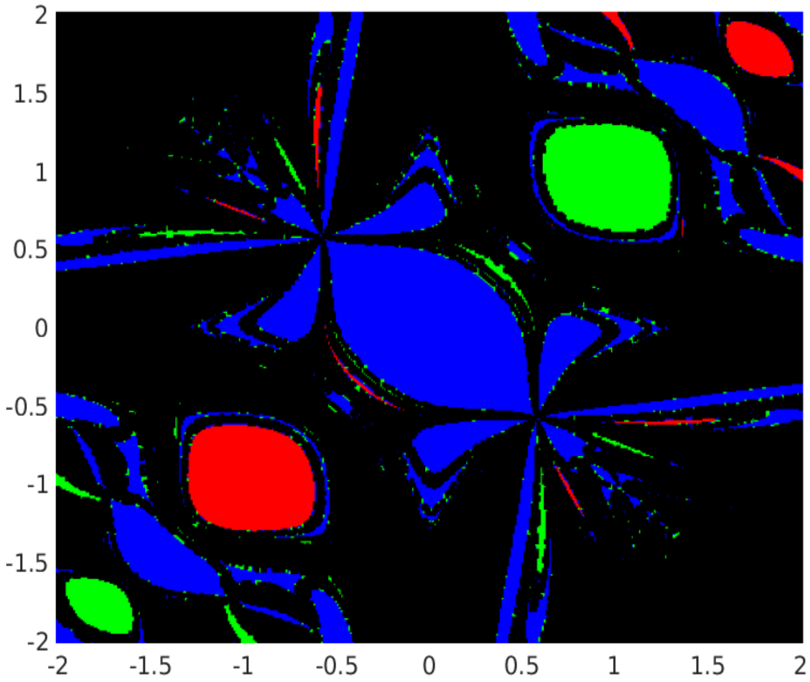

Figure 4.5 Basin of Attraction of Example 4.5.1 using Method (4.3.2)

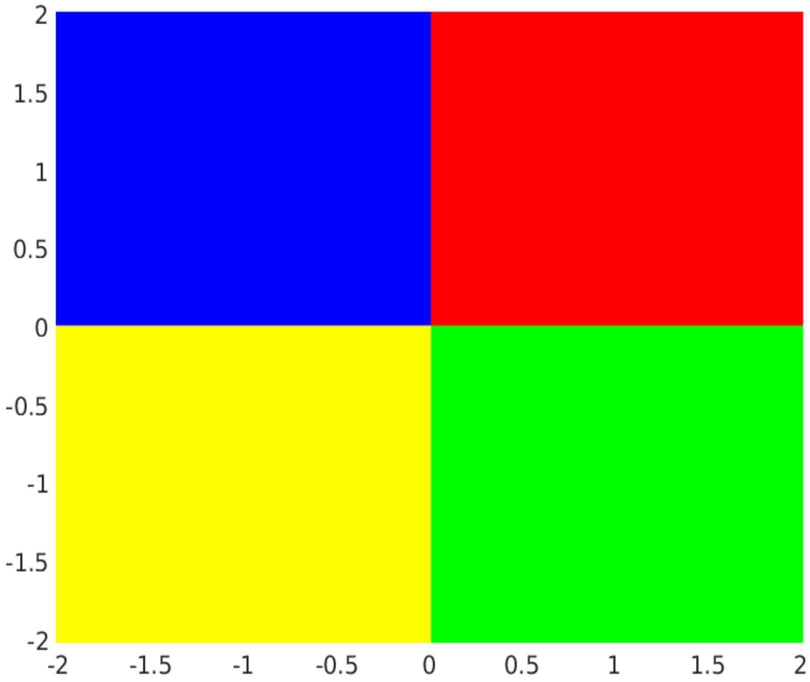

Figure 4.6 Basin of Attraction of Example 4.5.2 using Method (4.3.2)

Remark 4.5.3. *The black region (Julia set) contains all the initial points which do not converge to any root within 100 iterations.*

4.6 CONCLUSION

This chapter presented the Homeier method in a Banach space, assuming Fr’echet derivative up to order two. Our novel approach avoids Taylor series expansion. Two extensions of the method are proposed, with convergence orders of five and six.

CHAPTER 5

LOCAL CONVERGENCE OF TRAUB'S METHOD AND ITS EXTENSIONS

5.1 INTRODUCTION

Many problems in engineering and natural sciences can be modeled into an equation of the form

$$F(x) = 0, \tag{5.1.1}$$

where $F : \Omega \subseteq X \longrightarrow Y$ is a Fréchet differentiable function on a convex subset $\Omega \subseteq X; X$ and Y are Banach spaces.

We are interested in finding the local unique solution of (5.1.1). Typically, no analytical or closed-form solution exists in general. Therefore, we turn to iterative methods. Newton's method, defined as

$$x_{n+1} = x_n - A_n^{-1} F(x_n), \quad n = 0, 1, 2, \ldots,$$

where $A_n = F'(x_n)$, is very popular. Almost all methods in the literature are some modification or extension of this method (Ortega and Rheinboldt (1970)). Many authors have considered multi-step methods in order to increase efficiency, as well as the order of convergence (Behl and Arora (2022); Traub (1964); Petković (2013); Shakhno (2009)).

Among the multi-step methods, the Mean Newton methods are well studied (see

Özban (2004)). Traub introduced a modification of Newton's method in Traub (1964) (see also Petković and Petković (2007)), defined as follows:

$$
\begin{aligned}
y_n &= x_n - A_n^{-1}F(x_n), \\
x_{n+1} &= x_n - 2\Delta_n^{-1}F(x_n), \quad n = 0,1,2\ldots,
\end{aligned}
\tag{5.1.2}
$$

where $\Delta_n = F'(x_n) + F'(y_n)$. This method, also called the Arithmetic-Mean Newton's method, has an order of convergence of three. Later, Weerakoon and Fernando (2000) approached the method using a trapezoidal approximation of the interpolatory quadrature formula. Frontini and Sormani (2003), showed that this method is one of the most efficient variants of Newton's method, which uses the quadrature formula. Later, Cordero et al. (2012) extended the method defined for $n = 1,2,\ldots$ by

$$
\begin{aligned}
y_n &= x_n - A_n^{-1}F(x_n), \\
z_n &= x_n - 2\Delta_n^{-1}F(x_n), \\
x_{n+1} &= z_n - F'(y_n)^{-1}F(z_n),
\end{aligned}
\tag{5.1.3}
$$

where $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$. However, the method used Taylor series expansion and assumed the existence of derivatives up to order six. Sharma and Parhi (2020) removed these assumptions and studied this method in Banach spaces. Nevertheless, they were unable to obtain the desired order. Many local convergences, as well as semi-local convergence studies, have been conducted on this method (Argyros et al. (2016); Nishani et al. (2018); Parhi and Gupta (2008); Özban (2004)).

In this paper, we consider methods (5.1.2), (5.1.3) and an extension defined for $n = 1,2,\ldots$ by

$$
\begin{aligned}
y_n &= x_n - A_n^{-1}F(x_n), \\
z_n &= x_n - 2\Delta_n^{-1}F(x_n), \\
x_{n+1} &= z_n - F'(z_n)^{-1}F(z_n).
\end{aligned}
\tag{5.1.4}
$$

This method has an order of six. Parhi and Gupta (2008) has used the above extension to obtain an efficient sixth-order method using a linear interpolation of $F'(z_n)$.

This chapter is divided as follows. In Section 5.2, each method's preliminary functions, definitions, and auxiliary results are given in order. Some numerical examples to show the radii of convergence, approximate computational order of convergence (ACOC), and an example to illustrate the basins of attractions are given in Section 5.3. Section 5.3 also contains a representation of the number of iterations as a heatmap and tables that compare the iterates of methods (5.1.2)–(5.1.4) with corresponding methods in George et al. (2022b). Finally, the Chapter ends with conclusions in Section 5.4.

5.2 MAIN RESULTS

Firstly, we introduce some functions required in the proofs, along with necessary notations.

Throughout this Chapter, the open and closed balls centered at $\tilde{u}$ with radius δ are denoted as

$$B(\tilde{u},\delta)=\{u\in X:\|u-\tilde{u}\|<\delta\} \text{ and } B[\tilde{u},\delta]=\{u\in X:\|u-\tilde{u}\|\leq\delta\},$$

respectively. Furthermore, $\|x_n-x^*\|$ is denoted by ε_{x_n}, $\|y_n-x^*\|$ is denoted by ε_{y_n}, and $\|z_n-x^*\|$ is denoted by ε_{z_n}. Let K_1, K_2 and M be positive constants in $\mathbb{R}$; our proofs are subject to the following conditions.

Assumption 5.2.1. *Assume*

1. *x^* is the root of $F(x)=0$ and A_*^{-1} exists, where $A_*=F'(x^*)$.*

2. *There exists $K_1>0$, such that $\|A_*^{-1}(F'(x)-F'(y))\|\leq K_1\|x-y\|$ for all $x,y\in\Omega$.*

3. *There exists $M>0$, such that $\|A_*^{-1}F''(x)\|\leq M$.*

4. *There exists $K_2>0$, such that $\|A_*^{-1}(F''(x)-F''(y))\|\leq K_2\|x-y\|$ for all $x,y\in\Omega$.*

First, we define function $\zeta_1 : [0, \frac{1}{K_1}) \longrightarrow \mathbb{R}$ as

$$\zeta_1(t) = \frac{K_1}{2(1-K_1 t)}. \tag{5.2.1}$$

Let $\rho_1 = \frac{2}{3K_1}$. Note that $h_1(t) := \zeta_1(t)t$ is an increasing function in the interval $[0, \frac{1}{K_1})$. Furthermore, $h_1(0) = 0$ and $h_1(\rho_1) = 1$. That is,

$$0 \leq \zeta_1(t)t < 1 \quad \forall t \in [0, \rho_1). \tag{5.2.2}$$

Define $\zeta_2 : [0, \rho_1) \longrightarrow \mathbb{R}$ as below.

$$\zeta_2(t) = \frac{K_1}{2}(1 + \zeta_1(t)t)t. \tag{5.2.3}$$

Let

$$h_2(t) = \zeta_2(t) - 1.$$

Clearly, since $h_2(0) = -1$ and $h_2(t) \longrightarrow \infty$ as $t \longrightarrow \frac{1}{K_1}^{-}$, by the intermediate value theorem, there exist a smallest positive root for $h_2(t)$, say ρ_2, in the interval $[0, \frac{1}{K_1})$. So,

$$0 \leq \zeta_2(t)t < 1 \quad \forall t \in [0, \rho_2). \tag{5.2.4}$$

Consider, $\zeta_3 : [0, \rho_2) \longrightarrow \mathbb{R}$,

$$\zeta_3(t) = \frac{1}{2(1-\zeta_2(t))}\left[\frac{3K_2}{8} + M\zeta_1(t) + \frac{K_2}{12}\left(1 + \frac{K_1}{2}t\right)\right]. \tag{5.2.5}$$

Let

$$h_3(t) = \zeta_3(t)t^2 - 1.$$

Note that $h_3(0) = -1$ and $h_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^{-}$. Intermediate value theorem guarantees a smallest positive root ρ_3 in $[0, \rho_2)$ such that

$$0 \leq \zeta_3(t) < 1 \quad \forall t \in [0, \rho_3). \tag{5.2.6}$$

Let

$$R = \min\{\rho_1, \rho_3, 1\}, \tag{5.2.7}$$

then

$$0 \leq \zeta_1(t)t < 1,\ 0 \leq \zeta_2(t) < 1 \ \text{ and } \ 0 \leq \zeta_3(t)t^2 < 1 \quad \forall t \in [0, R). \tag{5.2.8}$$

Furthermore, one can see that $\zeta_3(t)$ is an increasing function in $[0, \rho_2)$ and hence

$$\zeta_3(t) \leq \zeta_3(R) \quad \forall t \in [0, R). \tag{5.2.9}$$

Theorem 5.2.2. *Let Assumption* 5.2.1 *hold and let R be as in (5.2.7); then the sequence* $\{x_n\}$ *defined by* (5.1.2) *with* $x_0 \in B(x^*, R) - \{x^*\}$, *converges to* x^* *such that*

$$\varepsilon_{x_{n+1}} \leq \zeta_3(R)\varepsilon_{x_n}^3, \tag{5.2.10}$$

where ζ_3 *is as defined in* (5.2.5).

Proof. First, we will show that $F'(x_0)^{-1}$ is bounded. Using Assumption 5.2.1 and (5.2.7) ,

$$\|A_*^{-1}(F'(x_0) - A_*)\| \leq K_1\varepsilon_{x_0} \leq K_1 R < 1.$$

Hence, by Banach's lemma on invertible operators (Argyros (2022)), $F'(x_0)^{-1}$ is invertible and

$$\|F'(x_0)^{-1}A_*\| \leq \frac{1}{1 - K_1\varepsilon_{x_0}}. \tag{5.2.11}$$

From (5.1.2), we have

$$
\begin{aligned}
y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\
&= x_0 - x^* + F'(x_0)^{-1}\left(F(x_0) - F(x^*)\right) \\
&= F'(x_0)^{-1}A_*\left[A_*^{-1}\int_0^1 (F'(x_0) - F'(x^* + t(x_0 - x^*)))dt(x_0 - x^*)\right],
\end{aligned}
$$

hence, by Assumption 5.2.1 , Equations (5.2.8) and (5.2.11),

$$
\begin{aligned}
\varepsilon_{y_0} &\leq \left\| F'(x_0)^{-1}A_*\left[\int_0^1 A_*^{-1}(F'(x_0) - F'(x^* + t(x_0 - x^*)))dt(x_0 - x^*)\right]\right\| \\
&\leq \|F'(x_0)^{-1}A_*\| \int_0^1 K_1\|x_0 - x^* - t(x_0 - x^*)\|dt\varepsilon_{x_0} \\
&\leq \frac{K_1}{2(1 - K_1\varepsilon_{x_0})}\varepsilon_{x_0}^2 \\
&\leq \zeta_1(\varepsilon_{x_0})\varepsilon_{x_0}^2 \qquad (5.2.12) \\
&< \varepsilon_{x_0} < R,
\end{aligned}
$$

so, iterate $y_0 \in B(x^*, R)$. Next, we employ Banach's lemma of invertible operators (Argyros (2022)), Equation (5.2.8), and Assumption 5.2.1 to show that Δ_0^{-1} is bounded..

$$
\begin{aligned}
\|(2A_*)^{-1}(\Delta_0 - 2A_*)\| &\leq \frac{1}{2}\|A_*^{-1}(F'(x_0) - A_* + F'(y_0) - A_*)\| \\
&\leq \frac{1}{2}(K_1\varepsilon_{x_0} + K_1\varepsilon_{y_0}) \\
&\leq \frac{K_1}{2}(1 + \zeta_1(\varepsilon_{x_0})\varepsilon_{x_0})\varepsilon_{x_0} \\
&= \zeta_2(\varepsilon_{x_0}) < 1.
\end{aligned}
$$

So, Δ_0^{-1} is bounded and

$$
\|\Delta_0^{-1}A_*\| \leq \frac{1}{2(1 - \zeta_2(\varepsilon_{x_0}))}. \qquad (5.2.13)
$$

From (5.1.2),

$$
x_1 - x^* = x_0 - x^* - 2\Delta_0^{-1}F(x_0)
$$

$$
\begin{aligned}
&= \Delta_0^{-1}\left[(F'(x_0)+F'(y_0))(x_0-x^*)-2(F(x_0)-F(x^*))\right]\\
&= \Delta_0^{-1}\left[\int_0^1 (F'(x_0)-F'(x^*+t(x_0-x^*)))dt(x_0-x^*)+\right.\\
&\qquad \left.\int_0^1 (F'(y_0)-F'(x^*+t(x_0-x^*)))dt(x_0-x^*)\right]\\
&= \Delta_0^{-1}\left[\int_0^1\int_0^1 F''(x^*+t(x_0-x^*)-\theta(x_0-x^*-t(x_0-x^*)))d\theta\right.\\
&\quad \times(x_0-x^*-t(x_0-x^*))dt(x_0-x^*)+\int_0^1\int_0^1 F''(x^*+t(x_0-x^*)\\
&\quad \left.-\theta(y_0-x^*-t(x_0-x^*)))d\theta(y_0-x^*-t(x_0-x^*))dt\times(x_0-x^*)\right].
\end{aligned}
$$

Let $\theta_{x_0}=\theta(1-t)(x_0-x^*)$ and $\theta_{y_0}=\theta(y_0-x^*-t(x_0-x^*))$. Now, we split up x_1-x^* as follows.

$$
x_1-x^*=\Delta_0^{-1}A_*[B_1+B_2+B_3], \tag{5.2.14}
$$

where

$$
\begin{aligned}
B_1 &= A_*^{-1}\int_0^1\int_0^1 F''(x^*+t(x_0-x^*)-\theta_{x_0})d\theta\\
&\quad \times(1-2t)(x_0-x^*)dt(x_0-x^*),\\
B_2 &= A_*^{-1}\int_0^1\int_0^1 F''(x^*+t(y_0-x^*)-\theta_{y_0})d\theta(y_0-x^*)dt(x_0-x^*),\\
B_3 &= A_*^{-1}\left[\int_0^1\int_0^1 F''(x^*+t(x_0-x^*)-\theta_{x_0})d\theta t dt\right.\\
&\quad \left.-\int_0^1\int_0^1 F''(x^*+t(y_0-x^*)-\theta_{y_0})d\theta t dt\right](x_0-x^*)^2.
\end{aligned}
$$

Using Assumption 5.2.1, we have

$$
\begin{aligned}
\|B_1\| &\le \left\|\int_0^1\int_0^1 A_*-1F''(x^*+t(x_0-x^*)-\theta_{x_0})d\theta\right.\\
&\qquad \times(x_0-x^*-2t(x_0-x^*))dt(x_0-x^*)\|\\
&\le \|A_*^{-1}\int_0^1\int_0^1\left[F''(x^*+t(x_0-x^*)-\theta_{x_0})-F''(x^*)\right]d\theta\\
&\quad \times(1-2t)(x_0-x^*)dt(x_0-x^*)
\end{aligned}
$$

$$
\begin{aligned}
&+A_*^{-1}\int_0^1\int_0^1 F''(x^*)d\theta(1-2t)(x_0-x^*)dt(x_0-x^*)\|\\
&\leq \|A_*^{-1}\int_0^1\int_0^1 \left[F''(x^*+t(x_0-x^*)-\theta_{x_0})-F''(x^*)\right]d\theta\\
&\times(1-2t)(x_0-x^*)dt(x_0-x^*)\|\\
&\leq K_2\int_0^1\int_0^1 \|t(x_0-x^*)-\theta_{x_0}\|d\theta|(1-2t)|dt\|(x_0-x^*)\|^2\\
&\leq K_2\int_0^1\int_0^1 (|t|+\theta|1-t|)d\theta|1-2t|dt\|x_0-x^*\|^3\\
&\leq \frac{3K_2}{8}\varepsilon_{x_0}^3. \qquad (5.2.15)
\end{aligned}
$$

In addition, by Assumptions 5.2.1 and (5.2.12), we have

$$
\begin{aligned}
\|B_2\| &= \left\|\int_0^1\int_0^1 A_*^{-1}F''(x^*+t(y_0-x^*)-\theta_{y_0})d\theta(y_0-x^*)dt(x_0-x^*)\right\|\\
&\leq M\varepsilon_{y_0}\varepsilon_{x_0}\leq M\zeta_1(\varepsilon_{x_0})\varepsilon_{x_0}^3, \qquad (5.2.16)
\end{aligned}
$$

and

$$
\begin{aligned}
\|B_3\| &= \left\|\int_0^1\int_0^1 A_*^{-1}[F''(x^*+t(x_0-x^*)-\theta_{x_0})d\theta\right.\\
&\left.-F''(x^*+t(y_0-x^*)-\theta_{y_0})d\theta]tdt(x_0-x^*)^2\right\|\\
&\leq K_2\int_0^1\int_0^1 t(t-\theta)\|x_0-y_0\|d\theta dt\varepsilon_{x_0}^2\\
&\leq \frac{K_2}{12}\|x_0-y_0\|\varepsilon_{x_0}^2\\
&\leq \frac{K_2}{12}(\varepsilon_{x_0}+\varepsilon_{y_0})\varepsilon_{x_0}^2\\
&\leq \frac{K_2}{12}\left(1+\zeta_1(\varepsilon_{x_0})\varepsilon_{x_0}\right)\varepsilon_{x_0}^3. \qquad (5.2.17)
\end{aligned}
$$

So, using (5.2.14), (5.2.15), (5.2.16), (5.2.17), and (5.2.8),

$$
\begin{aligned}
\varepsilon_{x_1} &\leq \|\Delta_0^{-1}A_*\|\left[\frac{3K_2}{8}\varepsilon_{x_0}^3+M\zeta_1(\varepsilon_{x_0})\varepsilon_{x_0}^3+\frac{K_2}{12}(1+\zeta_1(\varepsilon_{x_0})\varepsilon_{x_0})\varepsilon_{x_0}^3\right]\\
&\leq \frac{1}{2(1-\zeta_2(\varepsilon_{x_0}))}\left[\frac{3K_2}{8}+M\zeta_1(\varepsilon_{x_0})+\frac{K_2}{12}\left(1+\frac{K_1}{2}\varepsilon_{x_0}\right)\right]\varepsilon_{x_0}^3\\
&\leq \zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3 \qquad (5.2.18)
\end{aligned}
$$

$$\leq \zeta_3(R)R^2\varepsilon_{x_0} < \varepsilon_{x_0} < R.$$

Hence, $x_1 \in B(x^*,R)$ and (by (5.2.18)) (5.2.10) holds for $n = 1$. By induction, the proof is complete if x_0, y_0, x_1 are replaced by x_n, y_n, x_{n+1}, respectively. □

Next, to provide the convergence analysis of method (5.1.3), we define some functions and parameters below.

Define $\zeta_4 : [0, \frac{1}{K_1}) \longrightarrow \mathbb{R}$ by

$$\zeta_4(t) = K_1\zeta_1(t)t^2. \tag{5.2.19}$$

and $h_4 : [0, \frac{1}{K_1}) \longrightarrow \mathbb{R}$

$$h_4(t) = \zeta_4(t) - 1.$$

Then, $h_4(0) = -1$ and $h_4(t) \longrightarrow \infty$ as $t \longrightarrow \frac{1}{K_1}^-$. Hence, by the intermediate value theorem, h_4 has a smallest zero $\rho_4 \in [0, \frac{1}{K_1})$, and

$$0 \leq \zeta_4(t) < 1 \quad \forall t \in [0, \rho_4). \tag{5.2.20}$$

Define $\zeta_5 : [0, \rho_4) \longrightarrow \mathbb{R}$, by

$$\zeta_5(t) = \frac{K_1}{1 - \zeta_4(t)} \left(\zeta_1(t) + \frac{1}{2}\zeta_3(t)t \right) \zeta_3(t), \tag{5.2.21}$$

and $h_5 : [0, \rho_4) \longrightarrow \mathbb{R}$, by

$$h_5(t) = \zeta_5(t)t^4 - 1.$$

Observe that $h_5(0) = -1$ and $h_5(t) \longrightarrow \infty$ as $t \longrightarrow \rho_4^-$. Using the intermediate value theorem, h_5 has a smallest zero ρ_5 in the interval $[0, \rho_4)$. Let

$$R_1 = \min\{R, \rho_5\}, \tag{5.2.22}$$

then

$$0 \leq \zeta_4(t) < 1 \text{ and } 0 \leq \zeta_5(t)t^4 < 1 \quad \forall t \in [0, R_1). \tag{5.2.23}$$

It is clear that ζ_5 is an increasing function in $[0, \rho_4)$. In particular,

$$\zeta_5(t) \leq \zeta_5(R_1) \quad \forall t \in [0, R_1). \tag{5.2.24}$$

Theorem 5.2.3. *Let Assumption* 5.2.1 *hold. The sequence* $\{x_n\}$ *is as in (5.1.3). If* $x_0 \in B(x^*, R_1) - \{x^*\}$, R_1 *is as in (5.2.22). Then, the sequence* $\{x_n\}$ *converges to* x^* *and*

$$\varepsilon_{x_{n+1}} \leq \zeta_5(R_1)\varepsilon_{x_n}^5, \tag{5.2.25}$$

where ζ_5 *is as defined in (5.2.21).*

Proof. We use induction to prove the theorem. Clearly, one can mimic the proof of Theorem 5.2.2 to obtain

$$\varepsilon_{z_0} \leq \zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3. \tag{5.2.26}$$

Now, we will show that $F'(y_0)^{-1}$ is bounded using Assumption 5.2.1 and (5.2.23),

$$\begin{aligned} \|A_*^{-1}(F(y_0) - F(x^*))\| &\leq K_1 \varepsilon_{y_0} \\ &\leq K_1 \zeta_1(\varepsilon_{x_0})\varepsilon_{x_0}^2 \\ &= \zeta_4(\varepsilon_{x_0}) \leq \zeta_4(R_1) < 1. \end{aligned}$$

Hence, $F'(y_0)^{-1}$ is invertible and

$$\|F'(y_0)^{-1}A_*\| \leq \frac{1}{1 - \zeta_4(\varepsilon_{x_0})}. \tag{5.2.27}$$

$$\begin{aligned} x_1 - x^* &= z_0 - x^* - F'(y_0)^{-1}F(z_0) \\ &= z_0 - x^* - F'(y_0)^{-1}[F(z_0) - F(x^*)] \end{aligned}$$

$$
\begin{aligned}
&= F'(y_0)^{-1}\left[F'(y_0)(z_0 - x^*) - \int_0^1 F'(x^* + t(z_0 - x^*))(z_0 - x^*)dt\right]\\
&= F'(y_0)^{-1}A_*\left[\int_0^1 A_*^{-1}((F'(y_0) - F'(x^* + t(z_0 - x^*))))dt\right](z_0 - x^*).
\end{aligned}
$$

Hence, by using Assumption 5.2.1, (5.2.23) and (5.2.27), we obtain

$$
\begin{aligned}
\varepsilon_{x_1} &\le \|F'(y_0)^{-1}A_*\| \int_0^1 K_1\|y_0 - x^* - t(z_0 - x^*)\|dt\varepsilon_{z_0}\\
&\le \frac{1}{1-\zeta_4(\varepsilon_{x_0})}\int_0^1 K_1(\varepsilon_{y_0} + t\varepsilon_{z_0})dt\varepsilon_{z_0}\\
&\le \frac{K_1}{1-\zeta_4(\varepsilon_{x_0})}\left(\varepsilon_{y_0} + \frac{\varepsilon_{z_0}}{2}\right)\varepsilon_{z_0}\\
&\le \frac{K_1}{1-\zeta_4(\varepsilon_{x_0})}\left(\zeta_1(\varepsilon_{x_0})\varepsilon_{x_0}^2 + \frac{1}{2}\zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3\right) \times \zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3 \qquad (5.2.28)\\
&\le \frac{K_1}{1-\zeta_4(\varepsilon_{x_0})}\left(\zeta_1(\varepsilon_{x_0}) + \frac{1}{2}\zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}\right)\zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^5\\
&\le \zeta_5(\varepsilon_{x_0})\varepsilon_{x_0}^5\\
&< \varepsilon_{x_0} < R_1.
\end{aligned}
$$

That is, $x_1 \in B(x^*, R_1)$, and from (5.2.28) and (5.2.24),

$$
\begin{aligned}
\varepsilon_{x_1} &\le \frac{K_1}{1-\zeta_4(\varepsilon_{x_0})}\left(\zeta_1(\varepsilon_{x_0})\varepsilon_{x_0}^2 + \frac{1}{2}\zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3\right)\zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3\\
&\le \frac{K_1}{1-\zeta_4(\varepsilon_{x_0})}\left(\zeta_1(\varepsilon_{x_0}) + \frac{1}{2}\zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}\right)\zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^5\\
&\le \zeta_5(\varepsilon_{x_0})\varepsilon_{x_0}^5 \le \zeta_5(R_1)\varepsilon_{x_0}^5. \qquad (5.2.29)
\end{aligned}
$$

The rest of the proof follows as in Theorem 5.2.2. □

To prove the convergence of method (5.1.4), we introduce some more functions and parameters. Let $\zeta_6 : [0, \rho_2) \longrightarrow \mathbb{R}$ be defined as

$$\zeta_6(t) = K_1\zeta_3(t)t^3 \qquad (5.2.30)$$

and $h_6 : [0, \rho_2) \longrightarrow \mathbb{R}$ as

$$h_6(t) = \zeta_6(t) - 1.$$

Then, $h_6(0) = -1$ and $h_6(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. By the intermediate value theorem there exists a smallest zero ρ_6 in the interval $[0, \rho_2)$ such that

$$0 \leq \zeta_6(t) < 1 \quad \forall t \in [0, \rho_6). \tag{5.2.31}$$

Lastly, we define functions $\zeta_7 : [0, \rho_6) \longrightarrow \mathbb{R}$ by

$$\zeta_7(t) = \frac{K_1}{2(1 - \zeta_6(t))} \zeta_3^2(t) \tag{5.2.32}$$

and $h_7 : [0, \rho_6) \longrightarrow \mathbb{R}$ by

$$h_7(t) = \zeta_7(t) t^5 - 1.$$

Then, $h_7(0) = -1$ and $h_7(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. The intermediate value theorem gives a smallest root ρ_7 in $[0, \rho_6)$ such that

$$0 \leq \zeta_7(t) t^5 \leq 1 \quad \forall t \in [0, \rho_7). \tag{5.2.33}$$

If

$$R_2 = \min\{R, \rho_7\}, \tag{5.2.34}$$

then

$$0 \leq \zeta_6(t) < 1 \text{ and } 0 \leq \zeta_7(t) t^5 < 1 \quad \forall t \in [0, R_2). \tag{5.2.35}$$

Furthermore, observe that ζ_7 is an increasing function in $[0, \rho_6)$. Specifically,

$$\zeta_7(t) \leq \zeta_7(R_2) \quad \forall t \in [0, R_2). \tag{5.2.36}$$

Theorem 5.2.4. *Let Assumption* 5.2.1 *hold, and the sequence* $\{x_n\}$ *defined by (5.1.4) with* $x_0 \in B(x^*, R_2) - \{x^*\}$, *where* R_2, *as in (*5.2.34*), converges to* x^* *such that*

$$\varepsilon_{x_{n+1}} \leq \zeta_7(R_2)\varepsilon_{x_n}^6, \tag{5.2.37}$$

where ζ_7 *is as in (5.2.32).*

Proof. In extension (5.1.4), only the last step is different from extension (5.1.3). So, we can easily repeat the proof to obtain

$$\varepsilon_{z_0} \leq \zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3. \tag{5.2.38}$$

Now, as in previous case, we will show that $F'(z_0)^{-1}$ exists using Assumptions 5.2.1 and (5.2.35).

$$\begin{aligned}\|A_*^{-1}(F'(z_0) - A_*)\| &\leq K_1\varepsilon_{z_0}\\ &\leq K_1\zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3\\ &\leq \zeta_6(\varepsilon_{x_0}) < \zeta_6(R_2) < 1.\end{aligned}$$

Hence, $F'(z_0)^{-1}$ is invertible and

$$\|F'(z_0)^{-1}A_*\| \leq \frac{1}{1 - \zeta_6(\varepsilon_{x_0})}. \tag{5.2.39}$$

$$\begin{aligned}x_1 - x^* &= z_0 - x^* - F'(z_0)^{-1}F(z_0)\\ &= F'(z_0)^{-1}\left(F'(z_0)(z_0 - x^*) - \int_0^1 F'(x^* + t(z_0 - x^*))(z_0 - x^*)dt\right)\\ &= F'(z_0)^{-1}A_*\left(\int_0^1 A_*(F'(z_0) - F'(x^* + t(z_0 - x^*)))dt\right)(z_0 - x^*).\end{aligned}$$

Hence, by using (5.2.38), Assumptions 5.2.1 and (5.2.35), it follows that

$$\varepsilon_{x_1} \leq \|F'(z_0)^{-1}F'(x*)\|\left\|\left(\int_0^1 A_*^{-1}(F'(z_0) - F'(x^* + t(z_0 - x^*)))dt\right)\right\|\varepsilon_{z_0}$$

$$
\begin{aligned}
&\leq \frac{1}{1-\zeta_6(\varepsilon_{x_0})}\int_0^1 K_1|(1-t)|\varepsilon_{z_0}dt\varepsilon_{z_0}\\
&\leq \frac{K_1}{2(1-\zeta_6(\varepsilon_{x_0}))}\zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3 \times \zeta_3(\varepsilon_{x_0})\varepsilon_{x_0}^3\\
&\leq \zeta_7(\varepsilon_{x_0})\varepsilon_{x_0}^6 \qquad (5.2.40)\\
&< \varepsilon_{x_0} < R_2,
\end{aligned}
$$

so, $x_1 \in B(x^*, R_2)$. Furthermore, from (5.2.36) and (5.2.40), we have

$$
\varepsilon_{x_1} \leq \zeta_7(R_2)\varepsilon_{x_0}^6. \tag{5.2.41}
$$

Hence, (5.2.37) is satisfied for $n = 1$. Now, the proof follows as in Theorem 5.2.3. $\square$

The conditions that guarantee a unique solution are given in the following lemma.

Lemma 5.2.5. *Suppose Assumption 5.2.1 holds and x^* is a simple solution of the equation $F(x) = 0$. Then, $F(x) = 0$ has a unique solution x^* in $E := \Omega \cap B[x^*, \bar{r}]$ provided*

$$
K_1\bar{r} < 2. \tag{5.2.42}
$$

Proof. Let $p \in E$ be such that $F(p) = 0$. Define $J = \int_0^1 F'(x^* + u(p - x^*))du$. Then, by Assumption 5.2.1, we have, in turn

$$
\begin{aligned}
\|A_*^{-1}(J - A_*)\| &\leq K_1\int_0^1 \|x^* + u(p - x^*) - x^*\|du\\
&\leq K_1\int_0^1 u\|p - x^*\|du\\
&\leq \frac{K_1\bar{r}}{2} < 1.
\end{aligned}
$$

It follows that J is invertible, and hence $p = x^*$ by the identity $0 = F(p) - F(x^*) = J(p - x^*)$. $\square$

5.3 ILLUSTRATIONS AND NUMERICAL EXAMPLES

In this section, we will illustrate our results using numerical examples. In the first three examples, we compute the radii of convergence. The next example compares the iterations of methods (5.1.2)–(5.1.4) with the corresponding methods in George et al. (2022b). We also compute the ACOC for Examples 5.3.2 and 5.3.4 (the iterations of Examples 5.3.1 and 5.3.3 converge within three iterations on almost all initial points, so we have not computed ACOC for these examples). An illustration of the basins of attraction and a representation of the number of iterates as a heatmap follows.

The values of $\rho_i, i \in \{1,2,\ldots,7\}$ for the examples (Examples 5.3.1–5.3.3) are given in Table 5.1, and the ACOC of Examples 5.3.2 and 5.3.4 are given in Table 5.2.

Example 5.3.1. *Let* $X = Y = \mathbb{R}, \Omega = [r, 2-r], r \in (2-\sqrt{2}, 1)$ *and* $F : \Omega \to Y$ *be defined by*

$$F(x) = x^3 - r.$$

Here, $x^* = r^{1/3}$. $M = \frac{2(2-r)}{r^{2/3}}, K_1 = \frac{2(2-r)}{r^{2/3}}$ *and* $K_2 = \frac{2}{r^{2/3}}$. *For instance, if we take* $r = 1$, *from Table 5.1, we obtain the values of* R, R_1, *and* R_2 *as* $0.33196, 0.30365$, *and* 0.331963, *respectively.*

Example 5.3.2. *Let* $X = Y = \mathbb{R}^3,\ \Omega = B[0,1]$. *Define function* $F(w)$ *on* Ω *for* $w = (a_1, a_2, a_3)^T$ *by*

$$F(w) = \left(e^{a_1} - 1, a_2^2\frac{e-1}{2} + a_2, a_3\right)^T.$$

Here, $x^* = (0,0,0)^T$. *We have* $F'(w) = \begin{pmatrix} e^{a_1} & 0 & 0 \\ 0 & (e-1)a_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *Furthermore,* $M = e, K_1 = \frac{e-1}{e}$ *and* $K_2 = e - 1$. *Similar to the previous case, we obtain* $R = 0.36315$, $R_1 = 0.33616$, *and* $R_2 = 0.36314$ *(see Table 5.1).*

Example 5.3.3. *Define* F *on* $\Omega = [-1,1]$ *as*

$$F(x) = \sin(x)$$

$x^* = 0$. *We obtain* $M = 1, K_1 = 1$ *and* $K_2 = 1$. *Consequently,* $R = 0.66119, R_1 = 0.618403$ *and* $R_2 = 0.66119$ *(see Table 5.1).*

Table 5.1 The parameters ρ_i of the Examples 5.3.1–5.3.3.

Example	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
5.3.1	0.3333	0.38197	0.33196	0.36603	0.30365	0.34469	0.33207
5.3.2	0.3880	0.44459	0.36315	0.42604	0.33616	0.38418	0.36559
5.3.3	0.6667	0.76393	0.66119	0.73205	0.61840	0.68749	0.66163

Table 5.2 ACOC of Examples 5.3.2 and 5.3.4.

Example	**Root**	x_0	**Traub's Method**	**Extension** (5.1.3)	**Extension** (5.1.4)
5.3.2	$(0,0,0)$	$(1,0.03,0.03)$	2.98	4.64	5.70
		$(0.5,0.5,0.5)$	2.90	4.24	5.32
5.3.4	$(0.9,0.3)$	$(2,-1)$	2.91	4.60	4.43
		$(1.3,0.4)$	3.01	4.48	5.60

In the next example, we compare the performance of the methods (5.1.2)–(5.1.4) with that of Noor–Waseem-type methods studied in George et al. (2022b).

Example 5.3.4. *The system of equations*

$$\begin{aligned} 3t_1^2 t_2 + t_2^2 &= 1 \\ t_1^4 + t_1 t_2^3 &= 1, \end{aligned}$$

has solutions $(-1,0.2), (-0.4,-1.3)$, *and* $(0.9,0.3)$. *The solution* $(0.9,0.3)$ *is considered for approximating using the methods* (5.1.2)*–*(5.1.4) *and the corresponding methods studied in George et al. (2022b). We use the initial point* $(2,-1)$ *in our computation. Tables 5.3–5.5 provide the obtained results.*

Table 5.3 Traub's Method (5.1.2) and the Noor–Waseem Method in George et al. (2022b).

k	**Traub's Method** (5.1.2) $x_k = (t_1^k, t_2^k)$	**Noor-Waseem Method in** George et al. (2022b) $x_k = (t_1^k, t_2^k)$
0	$(2.0000000000000000, -1.0000000000000000)$	$(2.0000000000000000, -1.0000000000000000)$
1	$(1.02074824149820786, 0.25352907082513398)$	$(1.01962359355810994, 0.26538605472406479)$
2	$(0.99287801967429134, 0.30629644813087153)$	$(0.99285365860566110, 0.30634643384624071)$
3	$(0.99277999485546530, 0.30644044650526403)$	$(0.99277999485264400, 0.30644044650915097)$
4	$(0.99277999485112322, 0.30644044651102042)$	$(0.99285365860566110, 0.30634643384624071)$
5	$(0.99277999485112322, 0.30644044651102042)$	$(0.99277999485264400, 0.30644044650915097)$

Table 5.4 Fifth-Order Method (5.1.3) and the Noor–Waseem Fifth-order Extension Method in George et al. (2022b).

k	**Fifth Order Method** (5.1.3) $x_k = (t_1^k, t_2^k)$	**Method** (5.1.3) **in** George et al. (2022b) $x_k = (t_1^k, t_2^k)$
0	$(2.0000000000000000, -1.0000000000000000)$	$(2.0000000000000000, -1.0000000000000000)$
1	$(0.99339266265870362, 0.30563908458855637)$	$(0.97999747117802393, 0.31079296183420979)$
2	$(0.99277999485112611, 0.30644044651101687)$	$(0.99252009675815366, 0.30661919359513767)$
3	$(0.99277999485112322, 0.30644044651102042)$	$(0.99277988170910103, 0.30644055554978738)$
4	$(0.99277999485112322, 0.30644044651102042)$	$(0.99277999485110035, 0.30644044651104612)$

Table 5.5 The Sixth-Order Method (5.1.4) and the Noor–Waseem Sixth-order Extension Method in George et al. (2022b).

k	**Sixth Order Method** (5.1.4) $x_k = (t_1^k, t_2^k)$	**Method** (5.1.4) **in** George et al. (2022b) $x_k = (t_1^k, t_2^k)$
0	$(2.0000000000000000, -1.0000000000000000)$	$(2.0000000000000000, -1.0000000000000000)$
1	$(0.99278598580223975, 0.30643277796171902)$	$(1.03759994297628344, 0.26149549469920185)$
2	$(0.99277999485112322, 0.30644044651102042)$	$(0.99619799193796287, 0.30257508692302936)$
3	$(0.99277999485112322, 0.30644044651102042)$	$(0.99277999575683006, 0.30644044541552573)$

In the next example, we compare basins of attraction for each of the discussed methods using an example in two dimensions.

Example 5.3.5. *Define* F *on* $\mathbb{R}^2$ *by*

$$F(x,y) = (x^3 - y, y^3 - x)$$

with roots $r_1 = (-1,-1), r_2 = (0,0)$ *and* $r_3 = (1,1)$.

Figures 5.1, 5.2, 5.3 and 5.4 are generated using 400×400 *equally spaced grid points from the rectangular region* $D = \{(x,y) : x,y \in [-2,2]\}$ *as initial points for the iterations. The points that converge to* r_1, r_2 *and* r_3 *are colored cyan, magenta, and yellow, respectively. The points that do not converge to any roots after 50 iterations are marked black. The stopping criterion used is* $\|x_n - x^*\| < 10^{-8}$. *The algorithm used is the same as in Chicharro et al. (2013). Figures 5.5, 5.6, 5.7 and 5.8 are generated with the same grid for the corresponding methods representing the number of iterations required to converge by each point of the grid. It represents the number of iterations required to converge on each grid point. In black, the initial points that did not converge within 50 iterations are represented. The technique used can be found in Ardelean (2011).*

The algorithm used for basin of attraction is same as in Chapter 4. Algorithm for heat map is as follows.

Algorithm: Generating 'heat map'

1. **Initialization:**
    - Create an equidistant grid for the rectangular domain (e.g., 401×401).
    - Fix a tolerance limit for convergence checking.
    - Set the maximum number of iterations, denoted as n (e.g., $n = 50$).
2. **Iterative Method:**
    - For each point in the equidistant grid:
        - Use the point as the initial guess for the iterative method.
        - Run the iterative method up to a maximum of n iterations.
        - Check if the iteration has converged within the tolerance limit to any root.
        - If convergence fails within the specified iterations, mark the point black.
3. **Coloring Scheme:**
    - If convergence is achieved within a fast threshold (e.g., within 5 iterations):

  - Assign a light shade of a fixed color (e.g., blue) to the point.
  - Progressively increase the shade of the color for points that converge faster.

We used a PC with Intel Core i7 processor running Ubuntu 22.4.1 LTS. The programs were executed using MATLAB programming language with version code R2022b.

5.4 CONCLUSIONS

Traub's method (also known as Arithmetic-Mean Newton's Method and Weerakoon and Fernando method) and its two extensions were studied in this thesis using assumptions on the derivatives of the operator up to the order two. The theoretical parameters are verified using examples. The dynamics of the methods are also included in this study.

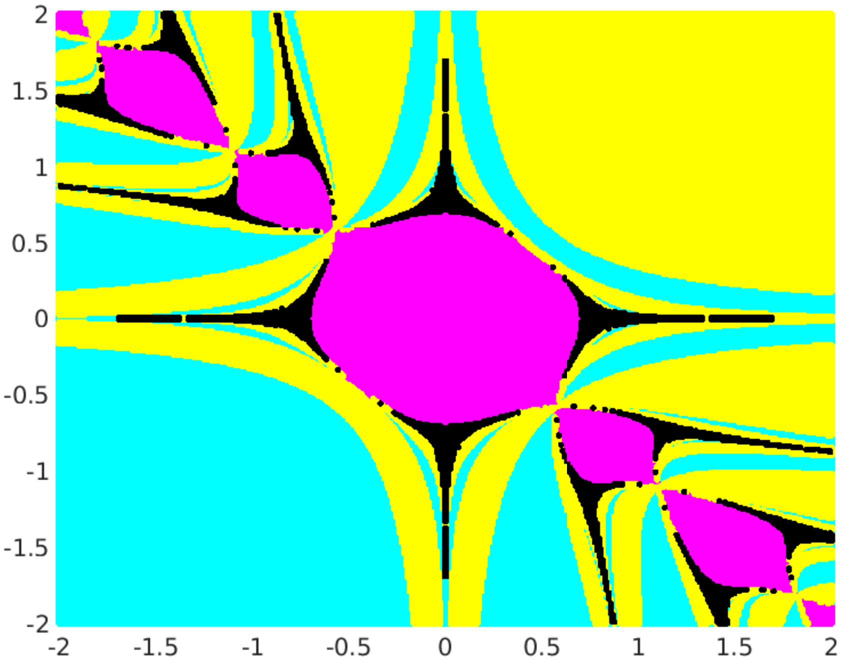

Figure 5.1 Basin of Attraction of Example 5.3.5 using Newton’s Method

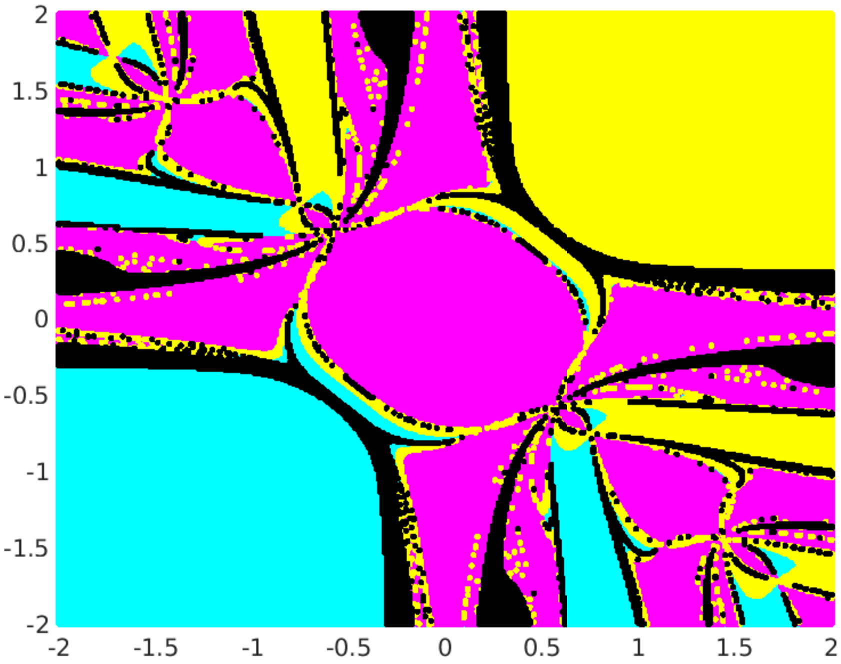

Figure 5.2 Basin of Attraction of Example 5.3.5 using Method (5.1.2)

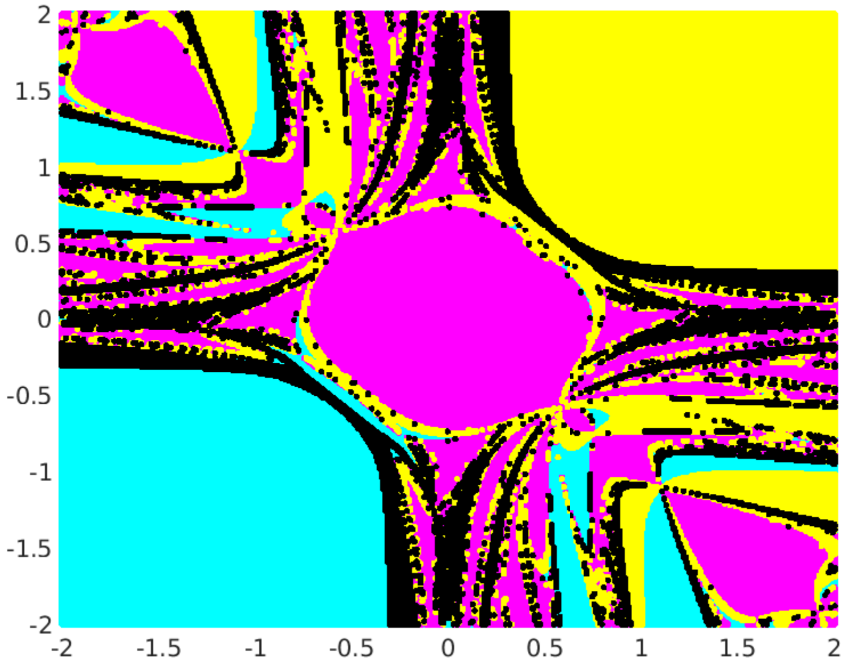

Figure 5.3 Basin of Attraction of Example 5.3.5 using Method (5.1.3)

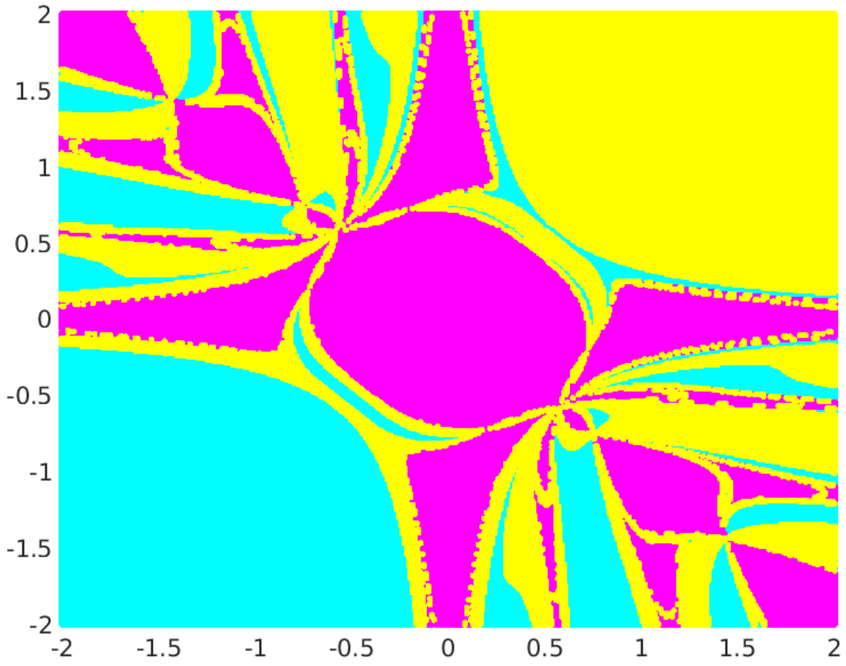

Figure 5.4 Basin of Attraction of Example 5.3.5 using Method (5.1.4)

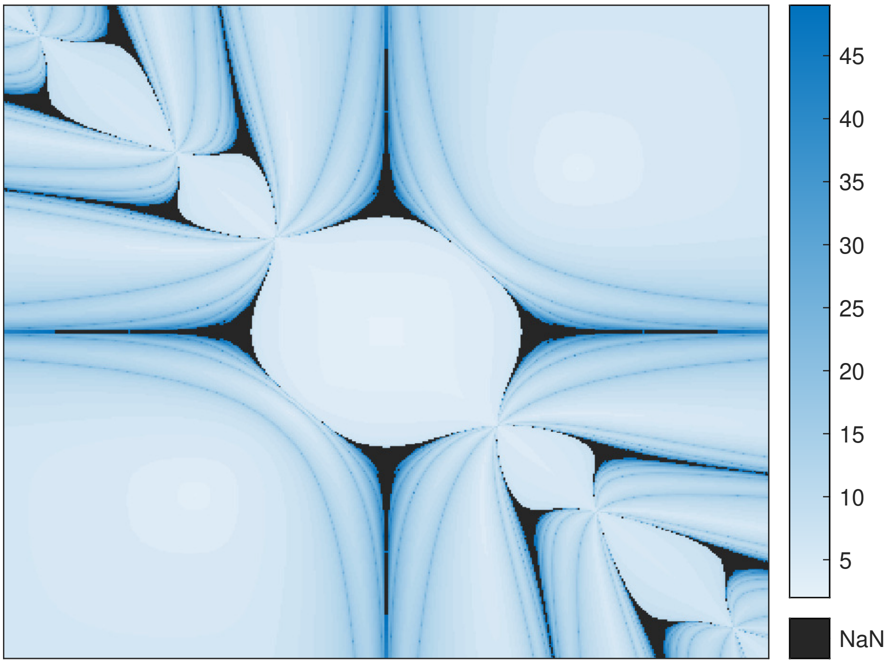

Figure 5.5 Heat map of Example 5.3.5 using Newton's Method

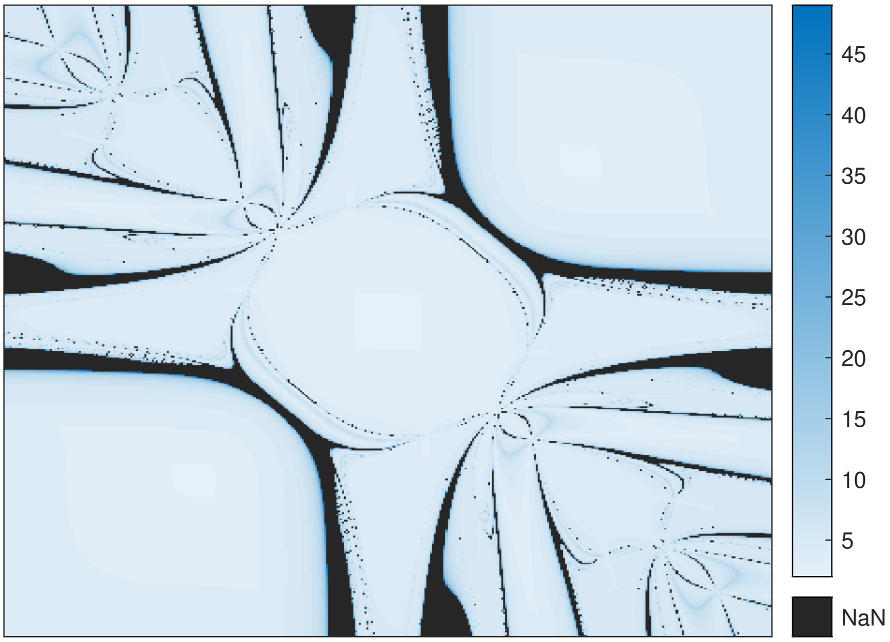

Figure 5.6 Heat map of Example 5.3.5 using Method (5.1.2)

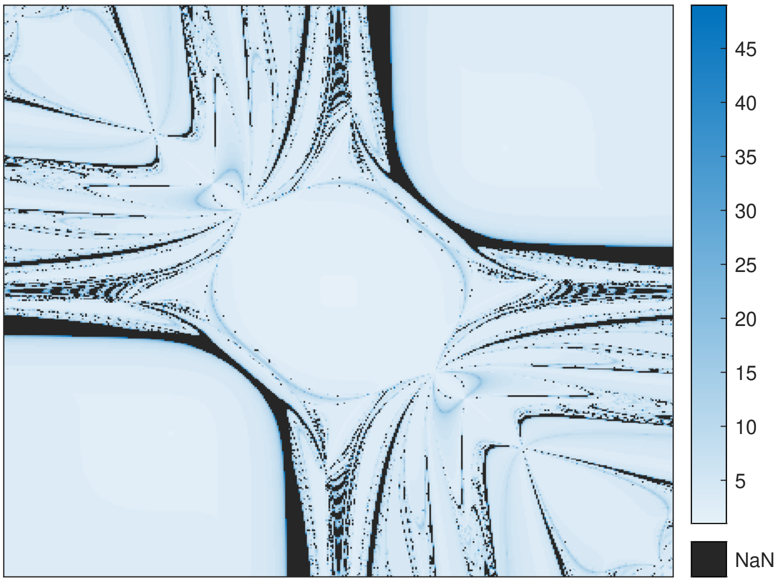

Figure 5.7 Heat map of Example 5.3.5 using Method (5.1.3)

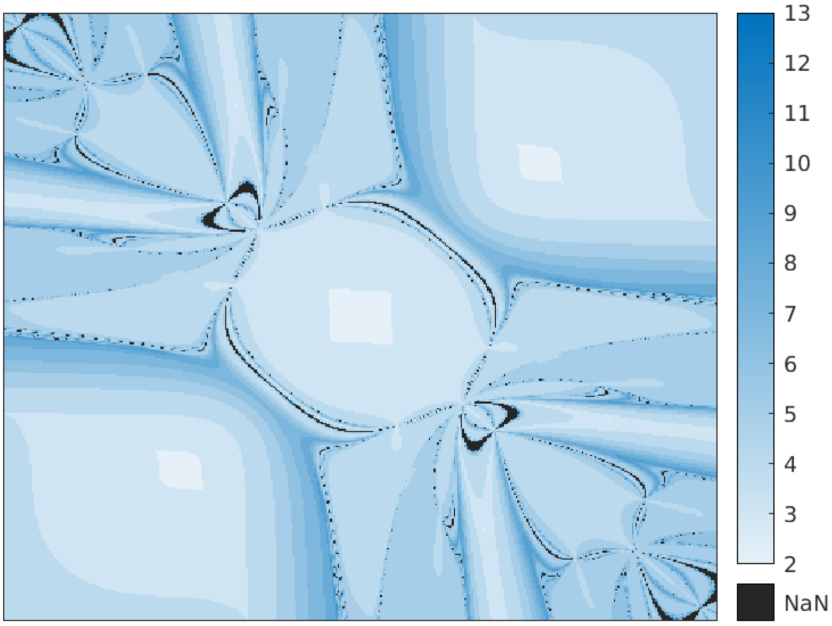

Figure 5.8 Heat map of Example 5.3.5 using Method (5.1.4)

CHAPTER 6

CONCLUSION AND FUTURE SCOPE

This thesis has dealt with iterative methods for solving non-linear equations in Banach Space. We have studied local and semi-local convergence of iterative schemes using only assumptions on the first or second Fréchet derivative instead of using Taylor series expansion as in former works. This increased the applicability of the methods. Another highlight of our work is obtaining the desired convergence orders. Computable error estimate and stability illustration using the basin of attraction is another salient feature.

Also, in the case of a non-linear ill-posed equation, we used a modified iterative scheme for solving the regularized equation in a Hilbert space. The new source condition and parameter choice strategy introduced reduces computation.

In Chapter 2, we studied an iterative method introduced by Singh et al. (2016). We established fifth-order convergence of the method using only assumptions on the first Fréchet derivative by the recurrence relations technique. It increased the applicability of the method.

Chapter 3 adopts the same method for solving regularized equations corresponding to a Laurentiev regularization method for non-linear ill-posed equations in a Hilbert space setting. We have introduced a new source condition and a parameter choice strategy which enabled us to obtain optimal order and did not demand the knowledge of unknown ν in the source condition.

Chapter 4 deals with local convergence analysis of the method introduced by Homeier (2004). We only used assumptions on the first and second Fréchet derivatives and obtained the desired three-order. We also gave two extensions of order five and six. We computed the radii of convergence and provided basins of attractions for some examples.

Lastly, in Chapter 5, we dealt with an iterative method introduced by Traub (1964). It is also well-known in literature as the Weerakoon method or Arithmetic-mean Newton's method. We studied the local convergence of this method and established the third order of convergence using assumptions only on the first and second Fréchet derivatives. Computable radii of convergence, the basins of attraction, and a comparison of iterations of similar iterative methods are other highlights.

6.1 FUTURE SCOPE OF WORK

Our work is on iterative methods with integral order. One can consider the same questions in the case of iterative methods with non-integral or even irrational convergence order as in the secant method and its extensions.

In the case of solving regularized equations, attempts can be in the direction of non-monotone operators. Also, working in Banach spaces instead of Hilbert spaces would be an improvement.

Our assumptions while dealing with local convergence analysis was Lipschitz type or central Lipschitz type. One can think of Hölder type assumptions or ω continuity assumptions, further increasing the method's applicability.

Cordero et al. (2012) have given a theorem in the setting of Real numbers. It can be paraphrased as "given a Newton method-based iterative scheme of order p, we can get an extension of order $p+2$ with just one more function evaluation". Their proof used Taylor series expansion. There are some works in literature which are particular cases of this theorem in the setting of Banach spaces without using Taylor series expansion. One can think of a generalization.

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BIODATA

Name : Muhammed Saeed K
Email : saeedkchelari@gmail.com
Date of Birth : May 26, 1995
Permanent address :
Muhammed Saeed K,
S/o Kunhahammed K,
Muttayil Thodi House, Chenakkalangadi PO,
Thenhippalam, Malappuram(District),
Kerala-673636.
Mobile - +91 8281929946

Educational Qualifications :

Degree	**Year**	**Institution / University**
M.Sc. Mathematics	2018	School of Mathematics and Statistics, University of Hyderabad.
B.Sc. Mathematics	2016	SS college Areekode, University of Calicut.

Other Publications :

1. Remesh, K., Argyros, I. K., Saeed K, M., George, S., and Padikkal, J. (2022). Extending the Applicability of Cordero Type Iterative Method. Symmetry, 14(12), 2495.
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